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STRICTLY QUADRATIC FUNCTIONAL EQUATIONS ON QUASIGROUPS *I*

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In a series of works, of which this is a first part, we shall investigate so called stricly quadratic functional equations on quasigroups (of various arities). Those equations are generalization of balanced functional equations which are solved in [1].

Complexity of problem forces us to begin with a very narrow class of equations, as it was done in the case of balanced equations ([1], [3], [4], [5], [7], [8], [2]). For this class we give the general solution (Th5).

Notions and notation used here are standard in quasigroup theory and can be find for example in [6].

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We say that a functional equation $t = t'$ is balanced if any variable from $t = t'$ occurs exactly once in every term t, t'.

Functional equation $t = t'$ is strictly quadratic if any variable from $t = t'$ occurs exactly twice in $t = t'$.

To make things easier, all equations we are dealing with, are denoted in a special way:

– we use only variables x_i, $y_i (i = 1, 2, \dots)$ are call them variables of type x, type y respectively.

– variables of type x occur exactly once in every term of a given equation

– variables of type y occur exactly twice one of two terms of given equation.

This paper is a part of my Ph. D. thesis, which I have done under the guidance of Dr. B. Alimpić

Definition 1. Let term t be given. A block is any subterm of t with variables of the same type, so we can call them x-blocks, y-blocks. If term t contains a constant symbol c, we call subterm c of t an emply block.

Definition 2. A block is closed iff it is either empty or any variable in it occurs exactly twice. A block is open iff any variable in it occurs exactly once. If v is a variable, then we call the block v travial.

Definition 3. If a block is of the form $A(\dots)$ then A is the main operation of this block. If $A(t', \mathcal{B})$ (or $A\ (\mathcal{B}, t')$) is a subterm of t and a $\mathcal{B}$ a block, then A is a connecting operation of $\mathcal{B}$.

It is usual to represent terms by trees, i.e. to associate terms with the appropriate ordering of the set of their operational and individual variables (constants). In this ordering individual variables (constants) are maximal elements while main operation is the least element.

To avoid confusion we will not use terms in which some operations appear more that once, and in the case where some individual variable y appears twice in the term, we will denote appearances by y' and y'' and take care of which appearance we are dealing with, although we will often write only y.

Let A be an n-groupoid, $A(x_1, \dots, x_n) = x_0$, $i_1, \dots, i_n \le n$ and $a_1, \dots, a_n \in S$. We define:

$$A_{i_1} \dots, i_n(x_{i_1}, \dots, x_{i_k}) = A(a_1, \dots, a_{i_1'-1}, x_{i_1}, a_{i_1+1}, \dots \\ \dots, a_{i_k-1}, x_{i_k}, a_{i_k+1}, \dots, a_n)$$

Operation $A_{i_1}, \dots, {}_{i_k}$ depends on the choise of $a_1, \dots, a_n$ so we write $A_{i_1}, \dots, {}_{i_k}$ only when $a_1, \dots, a_n$ are given and there is no doubt what is exact meaning of $A_{i_1}, \dots {}_{i_k}$ is a retract of A. Specially, unary retracts A_{i_1} are called translations of A.

In this paper, we will make retracts of operations, always by substituting a_i for variables x_i, $y_i (i = 1, 2, \dots)$.

Let t be a term (and no operational variable occurs more than once in it) and let all variables of t are among $x_1, \dots, x_n$ $(y_1, \dots, y_n)$. For $i_1 \dots, i_k \le n$ (and fixed $a_1, \dots, a_n \in S$) we define $t_{i_1} \dots {}_{i_k}$ as term obtained from t by substituting a_i for those x_i (y_i) for which $i \ne i_1, \dots, i_k$. In this process we also substitute some operations occuring in t by their apropriate retracts. Also operation defined by t is replaced by its retract designed by $t_{i_1 \dots i_k}$.

If equality $t = t'$ is given and all variables occuring in it are among $x_1, \dots,$ x_n $(y_1, \dots, y_n)$ and $i_1, \dots, i_k \le n$, then we call equality $t_{i_1 \dots i_k} = t'_{i_1 \dots i_k}$ $i_1, \dots, i_k$-consequence of $t = t'$.

Functional equation is generalized if any operational variable ocuurs exactly once in this equation.

In this work we will consider only generalized functional equations.

DEFINITION 4. Let t be a term, v a variable and A operational variable. If $t = B(t^1, \dots, t^n)$, where B is operational variable and $t^1, \dots, t^n$ terms, and $v(A)$ occurs in t^i, then $\bar{v}(t) = B_i \bar{v}(t^i)$ $\overline{A}(t) = \overline{B_i} A(t^i))$. In all other cases $\bar{v}(t) = \varepsilon$ and $\overline{A}(t) = \varepsilon$.

Those other cases are $\overline{A}(A(\dots)) = \varepsilon$ and $\overline{A}(B(\dots)) = \varepsilon$ and A does not occur in term $B(\dots)$.

Instead of $\bar{v}(t)$ and $\overline{A}(t)$ we usualy write only $\bar{v}$ and $\overline{A}$, while in the case where variable y occurs twice in t, we define $\bar{y}'$ and $\bar{y}''$ because $\bar{y}$ is not well defined.

DEFINITION 5. Let t be a term, v a variable, A and B operational variables. If $t' = A(t^1, \dots, t^n)$ is a substerm of t and $B(v)$ does not occur in t^i; them $\overline{AB} = A_i^{-1} \overline{A}^{-1} \overline{B}(\overline{Av} = A_i^{-1} \overline{A}^{-1} \bar{v})$. In all other cases $\overline{AB} = \varepsilon$ and $\overline{Av} = \varepsilon$.

If in some term t variable y occurs twice, the we define $\overline{Ay'}$ and $\overline{Ay''}$ instead of $\overline{Ay}$.

D4 and D5 can be used in the case of rectracts too, as in the following example: let X, Y be operations, restacts or variables which occur in a subterm t^k of the term $B_{i_1 \dots i_m}(t^1, \dots, t^m)$. Then $\overline{X}(B_{i_1 \dots i_m}(t^1, \dots t^m)) = B_{i_k} \overline{X}(t^k)$ and similarly in the case of defining $\overline{XY}(t^k)$.

DEFINITION 6. Equation $t = t'$ is of the first kind if the following conditions are satisfied:

– all variables of type x appear in t and t' in the same order

– variables of type y with the same index apear one after another without any other variable between them.

DEFINITION 7. Let A and B be two quasigroups from $t^1 = t^2$. $A \leftrightarrow B$ iff there are i and j such that $t^1_{ij} = t^2_{ij}$ has the following properties:

– A and B apear in $t^1_{ij} = t^2_{ij}$

– if in t^1_{ij} or t^2_{ij} there is an y-block, then this block is equal to t^1_{ij} or t^2_{ij} (i.e. in $t^1_{ij} = t^2_{ij}$ there is no y-block of the form $Q(y,y)$)

– it t^1_{ij} or t^2_{ij} is an y-block, then neither A nor B is the main operation of this block.

DEFINITION 8. Relation $\sim$ is reflexive and transitive closure od $\leftrightarrow$ in the set of all operations appearing in equation $t^1 = t^2$.

D7 becomes clearer if we look at all functional equations with at most two variables:

$$P(y_1, y_1) = e \tag{1}$$

$$A(x_1, x_2) = B(x_1, x_2) \tag{2}$$

$$P(y_1, y_1) = Q(y_2, y_2) \tag{3}$$

(4) $$P(x_1, Q(y_2, y_2)) = x_1$$

(5) $$A(B(x_1, y_2), y_2 = x_1$$

(6) $$P(y_1, A(y_2, B(y_1, y_2))) = e$$

(7) $$P(y_1, Q(y_1, R(y_2, y_2))) = e$$

(8) $$P(Q(y_1, y_1), R(y_2, y_2)) = e$$

(9) $$P(A(y_1, y_2), B(y_1, y_2)) = e.$$

Operations A and B are always $\leftrightarrow$-related while pairs (A, P), (B, P), (P, Q), (P, R), (Q, R) never are. If $A \sim B$ for some operation A, B then A and B are isostrophic.

Equations (1)–(9) are called almost trivial strictly quadratic functional equations.

THEOREM 1. *Functional equation in which every variable occurs at most twice and which is not strictly quadratic, has only trivial solution.*

PROOF: It $t^1 = t^2$ is not strictly quadratic, then there is a variable v which appears only once in one of t^1, t^2 for example t^1. 1-consequence of $t^1 = t^2$ is

(10) $$\bar{v}v = e$$

where we replace all functional variables occuring more that once in $t^1 = t^2$, with some new functional variables in such a way that equation $t^1 = t^2$ becomes generalized.

It follows from (10) that $v = \bar{v}^{-1}e$, so the set S where the all opperations are defined, has only one element.

On the other hand, the following theorem holds.

THEOREM 2. *Let $t^1 = t^2$ be a strictly quadratic functional equation and S either infinite or with 2^n elements. Then there is a solution of $t^1 = t^2$ on S.*

PROOF: There is a boolean group (i.e. group satisfying $x + x = e$) on such a set. Let us define an operation A from $t^1 = t^2$ by:

$$A(x, y) = x + y$$

Then, since any boolean group is commutative, we can reorder variables in $t^1 = t^2$ according to indices, for example in an increasing order. Hence all variables of type y are immediately one after another and can be omitted. The resulting equation contains only variables of the type x, once on each side of it and in the same order, so $t^1 = t^2$ is an identity.

Neither in Th1 nor in Th2 we need not have equations generalized.

Basic equations which we will regularly make use of when considering strictly quadratic functional equations, are the equation of generalized associativity and transitivity.

THEOREM 3. *The general solution (on a set S) of the generalized associativity equation:*

$$A(x_1, B(x_2, x_3)) = C(D(x_1, x_2), x_3) \tag{11}$$

is given by:

$$\begin{aligned} A(x,y) &= A_1x \cdot A_2y \\ B(x,y) &= A_2^{-1}(A_2B_1x \cdot A_2B_2y) \\ C(x,y) &= C_1x \cdot C_2y \\ D(x,y) &= C_1^{-1}(C_1D_1x \cdot C_1D_2y) \end{aligned} \tag{12}$$

where $\cdot$ is an arbitrary group on S and A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2 arbitrary permutations on S such that:

$$\begin{aligned} A_1 &= C_1D_1 \\ A_2B_1 &= C_1D_2 \\ A_2B_2 &= C_2 \end{aligned} \tag{13}$$

THEOREM 4. *The general solution (on a set S) of the generalized transitivity equation:*

$$A(B(x_1, y_2), C(y_2, x_3)) = D(x_1, x_3) \tag{14}$$

is given by:

$$\begin{aligned} A(x,y) &= A_1x \cdot A_2y \\ B(x,y) &= A_1^{-1}(A_1B_1x \cdot A_1B_2y) \\ C(x,y) &= A_2^{-1}(A_2C_1x \cdot A_2C_2y) \\ D(x,y) &= D_1x \cdot D_2y \end{aligned} \tag{15}$$

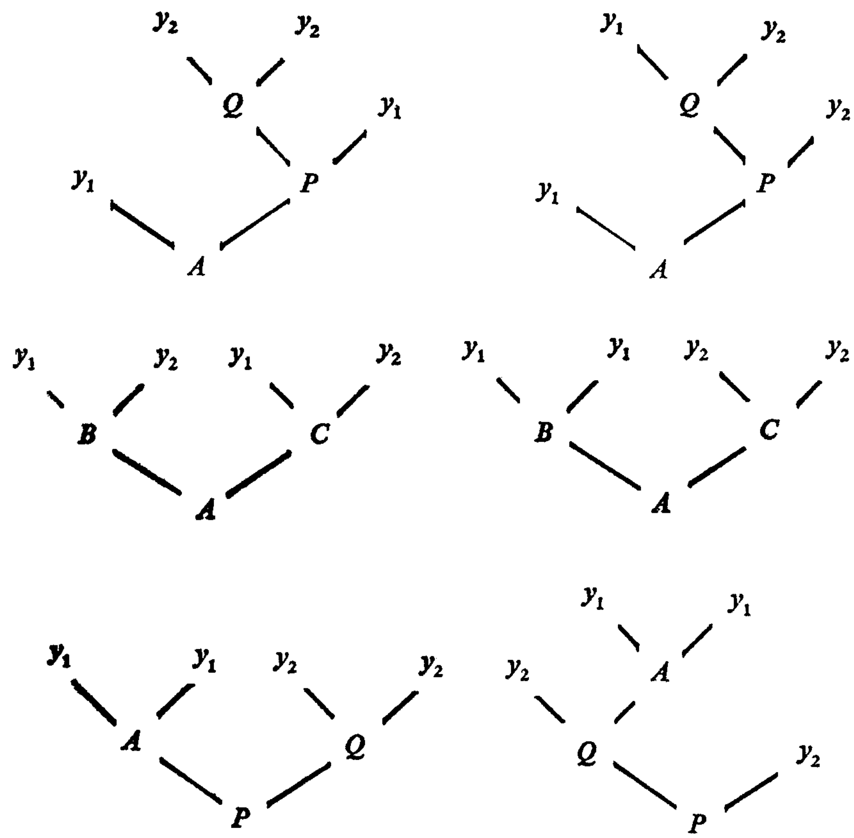
where $\cdot$ is an arbitrary group on S and $A1$, A_2, B_1, B_2, C_1, C_2, D_1, D_2 arbitrary permutations on S such that:

$$\begin{aligned} A_1B_1 &= D_1 \\ A_1B_2x \cdot A_2C_1x &= e \\ B_2C_2 &= C_2 \end{aligned} \tag{16}$$

where e is the unit of $\cdot$.

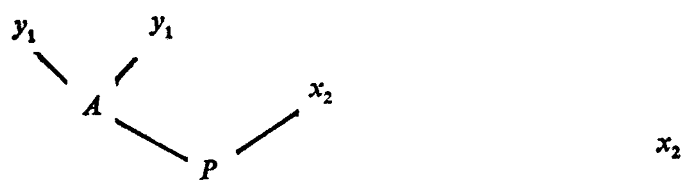
LEMMA 1. *Let $t^1 = t^2$ be (generalized) strictly quadratic functional equation and A an operation occuring in it. $|A^{\sim}| = 1$ iff A is either main or connecting operation of a closed block.*

PROOF: Let A be a main operation of a closed block. All i, j-consequences of $t^1 = t^2$ in which A occurs are given by following schemes:



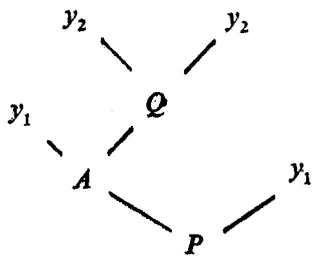
where only the terms from one side of equation are represented. The other side is the term e.

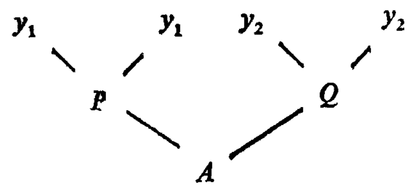
Beside those, the following case is also possible:



In each of those seven cases, operation A is not $\leftrightarrow$-related to some other operation, so $|A^{\sim}| = 1$.

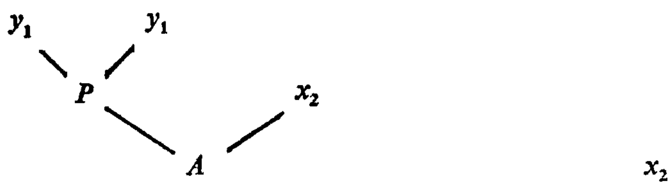
(b) Let A be a connecting operation of some closed block. All i, j-consequences of $t^1 = t^2$ in which A occurs are given by following schemes:





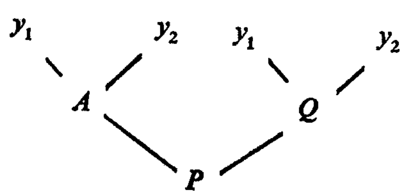
where only the terms from one side of equation are represented. The other side is the term e.

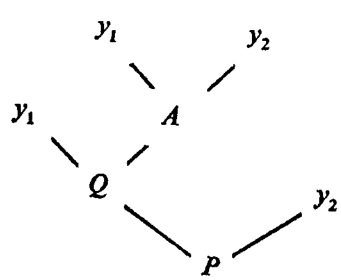
Beside those, the following case is also possible:



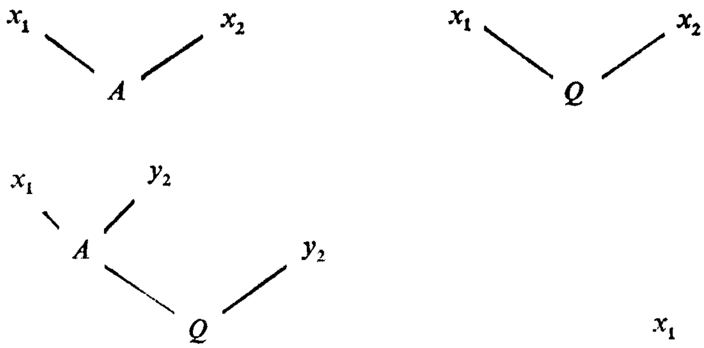
In eash of those three cases, operation A is not $\leftrightarrow$-related to some other operation, so $|A^{\sim}| = 1$.

(c) Let A be neither main nor connecting operation of some closed block. Then $A(t^3, t^4)$ is a subterm of one of t^1, t^2 and one of t^3, t^4 is a closed block: Possible i, j-consequences are given by following schemes:





(only one side of equation is given) and:



In any case $A \leftrightarrow Q$ so $|A^{\sim}| > 1$.

L 1 shows that solving strictly quadratic functional equations is more complicated than solving balanced equations. Also some results true for balanced equations are not valid for strictly quadratic equations.

So in the rest of the paper we consider only those strictly quadratic equations in which all operations are in the same class of $\sim$. For these, the folowing theorem holds:

THEOREM 5. *Let $t^1 = t^2$ be a generalized strictly guadratic equation which is not almost trivial and such that all operations occuring in it belong to the same clas of $\sim$. The general solution (on the set S) of $t^1 = t^2$ is given by:*

$$A(x,y) = \overline{A}^{-1}(\overline{A}A_1 x \circ \overline{A}A^2 y) \tag{17}$$

where $x \cdot y = x \circ y$ or $x \circ y = x * y = y \cdot x$ and:

(a) $\cdot$ is an arbitrary group on S iff by replacing some operations from $f^1 = t^2$ by their dual operations, $t^1 = t^2$ is transformed into an equation of the tirst kind

(b) $\cdot$ is an arbitrary Abelian group on S iff no replacement of some operations from $t^1 = t^2$ by their dual operations, transforms $t^1 = t^2$ into an equation of the first kind

(c) in the case (a), depending on definition of $\cdot$, it is uniquely determined whether $\circ$ is $\cdot$ or $*$, in the case (b) choice is free

(d) $\ldots, A_1, A_2, \ldots$

are arbitrary purmutations on S such that:

$$\overline{x_i}' = \overline{x_i}'' \quad \overline{y_j}' x \circ \overline{y_j}'' x = e$$

for all variables x_i, y_j of $t^1 = t^2$, where e is the unit of $\cdot$.

PROOF: (i) Since $t^1 = t^2$ is not almost trivial, in contains at least three different variables. According to L1, at least one of them is of type x, for example x_1.

At least in one of t^1, t^2 for example t^1, there are two variables: Let A be the main operation of t^1 and:

$$x \cdot y = A(A_1^{-1}x, A_2^{-1}y) \tag{18}$$

x_1 occurs in some agrument of A, say first. The following cases are possible:

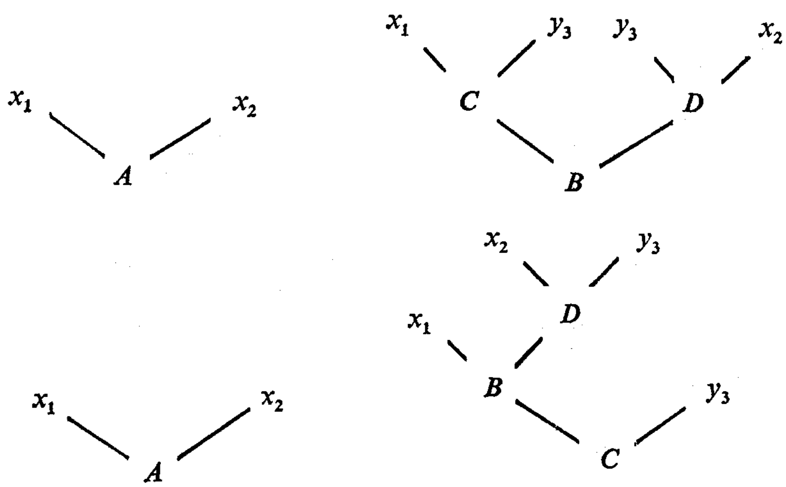
(i') In the second agrument of A there is a variable of type x, for example x_2. 1, 2-consequence of $t^1 = t^2$ is given by the scheme:

If there is a variable of type x occuring in agruments of A, B each time with other variable, then, according to Th3, $\cdot$ must be a group.

So, let us suppose that all variables of type x occur in the same agruments of A, B as some of the variables x_1, x_2.

Since all operations are in the same class of $\sim$, there is at least one variable of type y. It is not possible that every variable of tupe y occurs twice in the same agrument of some A, B.

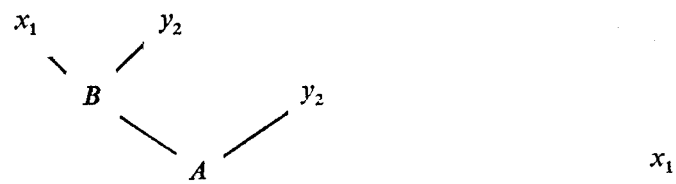
The following two cases ere posible:



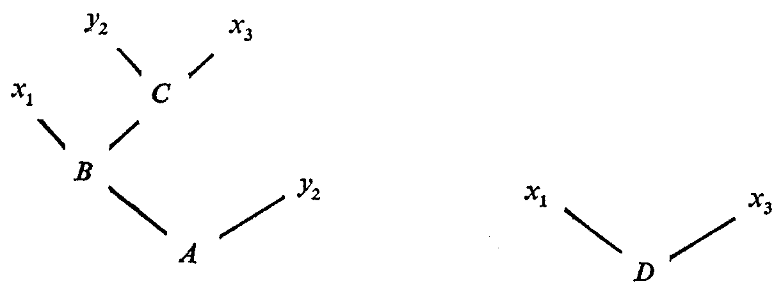
In the first case we have transitivity equation and according to Th4 $\cdot$ is a group. In the second case from $y_4 = D(x_2, y_3)$ and replacing y_3 by $D^{-1}(x_2, y_4)$ we obtain a transitivity equation.

It folows that $\cdot$ must be a group.

(i'') In the second argument of A there is no variable of type x. Then there must be some variable occuring also in the first agrument of A and we have the following scheme:



It is posible that in the first argument of A there is a variable of type x. Also it is not possible that all of them occur in the same argument of B as x_1, since A and B must be $\sim$-related to other operations. The opposite case is given by the following scheme:

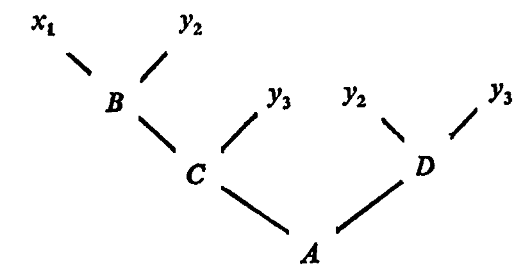


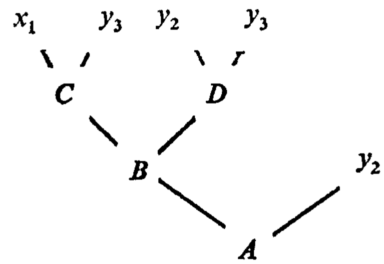
Replacing x_4 by $C(y_2, x_3)$ we get:

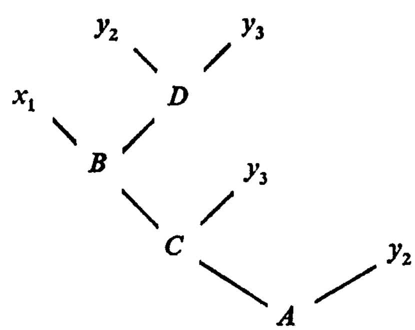
and $\cdot$ is a group.

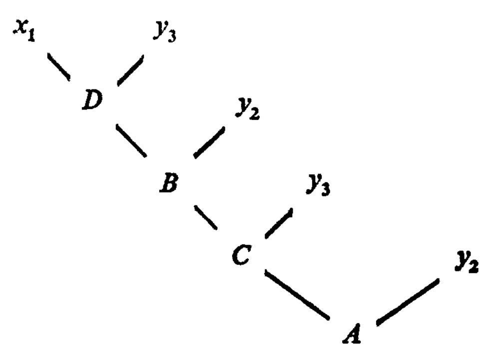
It is not possible that there are no operations in t^1 except A and B since A and B must be $\sim$-related to other operations.

So, if in t^1 there is no other variable of type x except x_1, there must be some variable of type y. For some of them, for example y_3, one of the following cases is true:

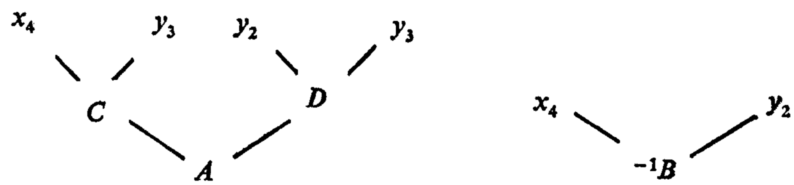

x_1
y_2
B
y_3
y_2
y_3
C
D
A
x_1


x_1
y_3
y_2
y_3
C
D
B
y_2
A
x


y_2
y_3
x_1
D
B
y_3
C
y_2
A
x_1


x_1
y_3
D
y_2
B
y_3
C
y_2
A
x_1

In the first case from $x_4 = B(x_1, y_2)$ we get:



which is a transitivity equation and consequently $\cdot$ is a group.

So we prowed that the main operation of at least one of t^1, t^2 is diisotopic to a group.

(ii) The operation $\cdot$ we defined by (18) using the main operation of one of the terms t^1, t^2 and proved that it is a group. We will prove that (17) is true for any operation A from $t^1 = t^2$.

Since (17) holds for some operation from $t^1 = t^2$ and $\sim$ is transitive closure of $\leftrightarrow$, it is enough to prove that if (17) holds for some A and $A \leftrightarrow B$, then the similar equality holds for B.

– Let

(19) $$\overline{A}A(\overline{Ax_1}x_1, \overline{Ax_2}x2) = \overline{B}B(\overline{Bx_1}x_1, \overline{Bx_2}x_2)$$

This case is proved in [1 (I)].

– If

(20) $$\overline{A}A(\overline{AB}B(\overline{Bx_1}x_1, \overline{By_2'}y_2), \overline{Ay_2''}y_2) = \overline{x_1''}x_1$$

then:

$$\overline{A}A_1\overline{AB}B(\overline{Bx_1}x_1\overline{By_2'}y_2) \circ \overline{A}A_2\overline{Ay_2''}y_2 = \overline{x_1'}, x_1$$
$$\overline{B}B(\overline{Bx_1'}x_1, \overline{By_2'}y_2) = \overline{B}B_1\overline{Bx_1}x_1 \circ I\overline{A}A_2\overline{Ay_2''}y_2$$
$$\overline{B}B_2\overline{By_2'}y_2 \cdot \overline{A}A_2\overline{Ay_2''}y_2 = e$$

where $Ix = x^{-1}$ is the inverse of x in a group $\cdot$. So:

(21) $$B(x, y) = \overline{B}^{-1}(\overline{B}B_1x \cdot \overline{B}B_2y)$$

– If

(22) $$\overline{B}B(\overline{BA}A(\overline{Ax_1}x_1, \overline{Ay_2{}'}y_2), \overline{By_2''}y_2) = \overline{x_1''}x_1$$

then:

$$
\begin{aligned}
{}^{-1}B(\overline{B}^{-1}\overline{x_1''}x_1, \overline{By_2''}y_2) &= \overline{BA}(\overline{Ax_1}x_1, \overline{By_2{}'}y_2)\\
\overline{B}B^{-1}B(\overline{B}^{-1}\overline{x_1{}'}x_1, \overline{By_2''}y_2) &= \overline{A}A_1\overline{Ax_1}x_1 \circ \overline{A}A_2\overline{Ay_2{}'}y_2\\
\overline{B}B_1{}^{-1}B(\overline{B}^{-1}\overline{x_1{}'}x_1, \overline{By_2{}''}y_2) &= \overline{x_1{}'}x_1\overline{y_2{}'}y_2 = z\\
B((\overline{B}B_1)^{-1}z, \overline{By_2''}y_2) &= \overline{B}^{-1}\overline{x_1{}'}x_1,\\
\overline{x_1{}'}x_1 &= z \circ \overline{Iy_2{}'}y_2.\\
B(x, \overline{By_2''}y_2) &= \overline{B}^{-1}(\overline{B}B_1x \circ \overline{Iy_2{}'}y_2)\\
\overline{y_2''}y_2 = \overline{B}B_2\overline{By_2''}y_2 &= \overline{B}B(\overline{BA}A_1\overline{Ax_1}a_1, \overline{By_2''}y_2) =\\
&= \overline{B}B_1\overline{Bx_1}a_1 \circ \overline{Iy_2{}'}y_2 = e \circ I\overline{y_2{}'}y_2 = I\overline{y_2}y_2\\
B(x, \overline{By_2''}y_2) &= \overline{B}^{-1}(\overline{B}B_1x \circ \overline{B}B_2\overline{By_2''}y_2)
\end{aligned}
$$

and finally we get (21).

– If

$$
\overline{P}P(\overline{Py_1{}'}y_1, \overline{PA}A(\overline{Ay_2{}'}y_2, \overline{AB}B(\overline{By_1''}y_1, \overline{By_2''}y_2))) = e \tag{23}
$$

then since P is a qusigroup:

$$
\begin{aligned}
\overline{P}P(\overline{Py_1{}'}y_1, \overline{Py_1{}''}y_1) &= e\\
\overline{P}P_2\overline{PA}A(\overline{Ay_2'}, (\overline{AB}B(\overline{By_1''}y_1, \overline{By_2''}y_2)) &= \overline{P}P_2\overline{Py_1''}y_1
\end{aligned} \tag{24}
$$

which is analogous to (20).

– If

$$
\overline{P}P(\overline{Py_1{}'}y_1, \overline{PB}B(\overline{By_2{}'}y_2, \overline{BA}A(\overline{Ay_1''}y_1, \overline{Ay_2''}y_2))) = e \tag{25}
$$

than, using (24) we get:

$$
\overline{P}P_2\overline{PB}B(\overline{By_2{}'}y_2, \overline{BA}A(\overline{Ay_1''}y_1, \overline{Ay_2''}y_2)) = \overline{P}P_2\overline{Py_1''}y_1
$$

which is analogous to (22)

– If

$$
\overline{P}P(\overline{PA}A(\overline{Ay_1'}y_1, \overline{Ay_2'}y_2, \overline{PB}B(\overline{By_1''}y_1, \overline{By_2''}y_2)) = e \tag{26}
$$

then, it is not possible that subterm $P(\dots)$ does not contain a variable of type x or a variable of type y occuring also outside of $P(\dots)$. Otherwise it canot be $A \sim P$. In both cases it must be

$$
P(x, y) = \overline{P}^{-1}(\overline{P}P_1x \circ \overline{P}P_2y)
$$

Then, from (26) it follows:

$$\overline{P}P_1\overline{PA}A(\overline{Ay_1'}y_1, \overline{Ay_2'}y_2) \circ \overline{P}P_2\overline{PB}B(\overline{By_1''}y_1, \overline{By_2''}y_2) = e$$
$$\overline{B}B(\overline{By_1''}y_1, \overline{By_2''}y_2) = I\overline{y_2'}y_2 \circ I\overline{y_1'}y_1$$
$$\overline{y_1{}'}y_1 \circ \overline{y_1{}''}y_1 = e$$
$$\overline{y_2{}'}y_2 \circ \overline{y_2{}''}y_2 = e$$
$$B(\overline{By_1''}y_1, \overline{By_2''}y_2 = \overline{B}^{-1}(\overline{B}B_2\overline{By_2''}y_2 \circ \overline{B}B_1\overline{By_1''}y_1)$$

and (21) also holds.

(iii) If i is index of a variable of type x then i-consequence of $t^1 = t^2$ is:

$$\overline{x_i{}'}x_i = \overline{x_i{}''}x_i$$

If i is index af a variable of type y then i-consequence of $t^1 = t^2$ is:

$$\overline{A}A(\overline{Ay_i{}'}y_i, \overline{Ay_i{}''}y_i) = e$$

Using (17) we get

$$\overline{y_i{}'}y_i \circ \overline{y_i{}''}y_i = e$$

so (d) is completely proved.

(iv) The operation $\cdot$ is a group. If we suppose that no replacement ot some operations from $t^1 = t^2$ by their dual operations, transforms $t^1 = t^2$ into an equation of the first kind, then there are variables with indices i, j such that i, j-consequence of $t^1 = t^2$ is of the one ot the following forms:

$$\begin{aligned} &\overline{x_i{}'}x_i \cdot \overline{x_j{}'}x_j = \overline{x_j{}''}x_j \cdot \overline{x_i{}''}x_i \\ &\overline{y_i{}'}y_i \cdot \overline{x_j{}'}x_j \cdot \overline{y_j{}''}y_j = \overline{x_j{}''}x_j \\ &\overline{y_i{}'}y_i \cdot \overline{y_j{}'}y_j \cdot \overline{y_i{}''}y_i' \cdot \overline{y_j{}''}y_j = e, \end{aligned} \tag{27}$$

Using equations from (d) we obtain that $\cdot$ is an abelian group. Converse is trivial since any abelian group satisfies all conditions (27) (with (d)).

So we proved (a) and (b). (c) also easily follows.

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