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NOTE ON DISPERSION OF X^α

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Summary. Some inequalities for moments of X^α, $0 < \alpha \leq 1$, X nonneg. r. v., are presented, for example $DX^\alpha \leq (DX)^\alpha$, $DX^\alpha \leq (DX)/(EX)^{2(1-\alpha)}$,

$$(EX)^\alpha - EX^\alpha \leq (1-\alpha)(DX)/(EX)^{2-\alpha}.$$

It is proved that $nD\overline{X}_n^\alpha \to \alpha^2(DX)/(EX)^{2(1-\alpha)}$, $n \to \infty$, where $X_1, X_2, \dots, X_n$ are i. i d. r. v. and $\overline{X}_n = (X_1 + X_2 + \cdots + X_n)/n$. The estimation of $\sqrt{EX}$ is considered, and for binominal case some numerical evaluations are given.

1. Introduction. Let X be a r.v. with $DX < \infty$ and $f(x)$ be a differentiable function such that $f'(EX) \neq 0$. By approximation formula (see [**1**, vol. 1]) we have

$$Df(X) \approx [f'(EX)]^2 DX. \tag{1}$$

Let $X_1, X_2, \dots$ be the sequence of independent r.v. distributed as X and $\overline{X}_n = (X_1 + dots + X_n)/n$. Then by (1)

$$D\sqrt{n} f(\overline{X}_n) \approx [f'(EX)]^2 DX. \tag{2}$$

On the other hand (see [**2**]):

$$\sqrt{n}(f(\overline{X}_n) - f(EX) \xrightarrow{d} Z : N(0; [f'(EX)]^2 DX). \tag{3}$$

It seems, by (2) and (3), that

$$\lim_{n\to\infty} D\sqrt{n} f(\overline{X}_n) = (f'(EX))^2 DX, \tag{4}$$

but it is not necessary that $D\sqrt{n} f(\overline{X}_n)$ exists, and (4) does not have to be true.

In this paper we deal with r.v. $X \geq 0$, and $f(x) = x^\alpha$, $0 < \alpha \leq 1$, and some properties of EX^α, DX^α, $E\overline{X}_n^\alpha$, $D\overline{X}_n^\alpha$. Specially we consider $\alpha = 1/2$ and the

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estimation of $\sqrt{EX}$. Statistic $(\overline{X}_n)^{1/2}$ is the bias estimation for $\sqrt{EX}$, and our aim is to improve this estimation. We give some details and numerical evaluations in the case of binomial distribution $B(n;p)$ and estimation of $\sqrt{p}$.

2. Some elementary inequalities. We give a few easy provable inequalities.

(a) Let $0 < \alpha \le \beta$, $x \ge 0$. Then

$$|x^\alpha - 1| \le |x^\beta - 1|.$$

(b) Let $0 < \alpha \le 1$. Then

$$|x^\alpha - 1| \le |x - 1|^\alpha, \quad x \ge 0, \text{ or } |(1+z)^\alpha - 1| \le |z|^\alpha, \quad z \ge -1.$$

(c) Let $O < \alpha \le 1$. Then

$$\begin{aligned}
&(1+z)^\alpha - 1 \le \alpha z, \quad z \ge -1,\\
&(1+z)^\alpha - 1 \ge \alpha z/(1+(1-\alpha)z/2), \quad z \ge 0,\\
&(1+z)^\alpha - 1 \ge \alpha z/(1+z), \quad -1 < z \le 0,\\
&\alpha|z|/(1+|z|) \le (1+z)^\alpha - 1| \le \alpha|z|/(1-|z|), \quad |z| < 1.
\end{aligned}$$

(d) Let $0 < \alpha \le 1$. Then

$$\begin{aligned}
&(1+z)^\alpha - 1 \ge \alpha z - [\alpha(1-\alpha)/2]z^2, \quad z \ge 0,\\
&(1+z)^\alpha - 1 \ge \alpha z - (1-\alpha)z^2, \quad -1 \le z \le 0,\\
&\alpha z - (1-\alpha)z^2 \le (1+z)^\alpha - 1 \le \alpha z, z \ge -1,\\
&\alpha|z| - (1-\alpha)z^2 \le |(1+z)^\alpha - 1| \le \alpha|z| + (1-\alpha)z^2, \quad z \ge -1.
\end{aligned}$$

3. Some inequalities with moments of X^α. In what follows X is nonnegative r.v. with $DX < \infty$ and $EX = m$.

LEMMA 1. *Let $0 < \alpha\beta$. Then $E(X^\alpha - m^\alpha)^2 \le (E(X^\beta - m^\beta)^2]/m^2(\beta - \alpha)$.*

Proof. From the ineq. (a) it follows that

$$\begin{aligned}
E(X^\alpha - m^\alpha)^2 =& m^{2\alpha}E((X/m)^\alpha - 1)^2 \le m^{2\alpha}E((X/m)^\beta - 1)^2\\
&= (E(X^\beta - m^\beta)^2]/m^{2(\beta-\alpha)}.
\end{aligned}$$

LEMMA 2. (i) $E(X^\alpha - m^\alpha)^2 \le (DX)^\alpha$, $0 < \alpha 1$,
(ii) $DX^\alpha \left\{ \le (DX)^\alpha,\ 0 < \alpha \le 1 \ge (DX)^\alpha,\ \alpha \ge 1. \right.$

Proof. (i) From the ineq. (b) and Jensen's ineq. it follows that

$$\begin{aligned}
E(X^\alpha - m^\alpha)^2 &= m^{2\alpha}E((X/m)^\alpha - 1)^2 \le m^{2\alpha}E(X/m - 1)^{2\alpha}\\
&\le m^{2\alpha}(E(X/m - 1)^2)^\alpha = (DX)^\alpha,
\end{aligned}$$

(ii) Let $0 < \alpha \le 1$. Then $DX^\alpha = E(X^\alpha - EX^\alpha)^2 \le E(X^\alpha - m^\alpha)^2 \le (DX)^\alpha$. If $\alpha \ge 1$, then $1/\alpha \le 1$, and $DX = D(X^\alpha)^{1/\alpha} \le (DX^\alpha)^{1/\alpha}$, or $DX^\alpha \ge (DX)^\alpha$.

LEMMA 3. *Let* $0 < \alpha \le \beta \le 1$. *Then*

(i) $DX^\alpha \le (DX)^\beta / m^{2(\beta-\alpha)}$,

(ii) *Following* (i), *the best upper bound for* DX^α *is*

$$DX^\alpha \le \begin{cases} (DX)^\alpha, & DX \ge m^2 \\ (DX)/m^{2(1-\alpha)}, & DX \le m^2 \end{cases}$$

Proof. (i) The statement follows from Lemma 1 and 2

(ii) $(DX)^\beta / m^{2(\beta-\alpha)} = m^{2\alpha}(DX/m^2)^\beta$. If $(DX)/m^2 \ge 1$, we take $\beta = \alpha$, and if $(DX)/m^2 \le 1$, we take $\beta = 1$.

LEMMA 4. *Let* $0 < \alpha \le 1$. *Then*

$$0 \le m^\alpha - EX^\alpha \le (1-\alpha)(DX)/m^{1-\alpha}.$$

Proof. From Jensen's ineq. and ineq. (d) it follows that

$$\begin{aligned} 0 \le m^\alpha - EX^\alpha = m^\alpha E(1 - (X/m)^\alpha) \le m^\alpha[-\alpha E(X/m - 1) + \\ + (1-\alpha)E(X/m - 1)^2) = (1-\alpha)(DX)/m^{2-\alpha}. \end{aligned}$$

Some other estimations for $m^\alpha - EX^\alpha$ we can deduce from Lemma 3 for example $m^\alpha - EX^\alpha \le \sqrt{DX}/m^{1-\alpha}$.

In particular, for $\alpha = 1/2$, we have

LEMMA 5. $D\sqrt{X} \le (DX)/m$, $D\sqrt{X} \le \sqrt{D}X$,

$\sqrt{m} - E\sqrt{X} \le (DX)/(2m^{3/2})$, $E\sqrt{X} \ge (m - (DX)/m)^{1/2}$ if $DX \le m^2$.

4. Applications. Let $X_1, X_2, \dots$ be a sequence of i.i.d.r.v. and

$$EX_i = m, \ \ DX_i = \sigma^2 < \infty, \ \ S_n = X_1 + \cdots + X_n,$$

$\overline{X}_n = S_n/n$. By Lemma 5 we have "unexpected" result

$$D\sqrt{S_n} \le (DS_n)/(ES_n) = \sigma^2/m.$$

It is of interest to consider the behavior of DS_n^α or $D\overline{X}_n^\alpha$, $0 < \alpha \le 1$. The direct consequence of Lemma 4, is

LEMMA 6. $0 \le (ES_n)^\alpha - ES^\alpha \le (1-\alpha)\sigma^2/n^{1-\alpha}m^{2-\alpha}$,

$0 \le m^\alpha - E\overline{X}_n^\alpha \le (1-\alpha)\sigma^2/nm^{2-\alpha}$.

LEMMA 7. *Let* $Z_1, Z_2, \dots$ *be a sequence of i.i.d.r.v. such that* $Z_i \ge -1$, $EZ_i = 0$, $DZ_i = d^2$, *and* $\overline{Z}_n = (Z_1 + \cdots + Z_n)/n$. *Then*

(i) $nE\overline{Z}_n^2 = d^2$, $nE(\overline{Z}_n^2; |\overline{Z}_n| > \varepsilon) \to 0$, $n \to \infty$,

$nE(\overline{Z}_n^2 : |\overline{Z}_n| \leq \varepsilon) \to d^2,\ n \to \infty.$

(ii) $nE|\overline{Z}|^{2\alpha} \to \infty,\ n \to \infty,\ 0 < \alpha < 1,$

$nE(|\overline{Z}|^{2\alpha}; |\overline{Z}_n| > \varepsilon) \to 0,\ n \to \infty,\ 0 \leq \alpha \leq 1.$

Proof. (i) $nE\overline{Z}_n^2 = nD\overline{Z}_n = d^2$. Let $F(y)$ be d.f. of $N(0;1)$ distribution, and a be sufficiently large such that $\int_{-ad}^{ad} y^2 dF(y) \geq 1 - \delta$, δ is a given positive number, and $\sqrt{n} \geq a/\varepsilon$, i.e. $\varepsilon > a/\sqrt{n}$. It follows, in accordance with the Central limit theorem, that

$$d^2 \geq nE(\overline{Z}_n^2; |\overline{Z}_n| \leq \varepsilon) \geq E[(\sqrt{n}\overline{Z}_n)^2;\ |\sqrt{n}\overline{Z}_n| \leq a] \to$$
$$\to d^2 \int_{-ad}^{ad} y^2 dF(y) \geq d^2(1-\delta),\ \text{ or }\ nE(\overline{Z}_n^2; |\overline{Z}_n| \leq \varepsilon) \to d^2.$$

(ii) $nE|\overline{Z}_|^{2\alpha} = n^{1-\alpha}E|\sqrt{n}\overline{Z}_n|^{2\alpha} \geq n^{1-\alpha}E(|\sqrt{n}\overline{Z}_n|^{2\alpha}; |\sqrt{n}\overline{Z}_n| \leq a) \sim$

$$\sim n^{1-\alpha}d^{2\alpha} \int_{-ad}^{ad} y^{2\alpha} dF(y) \to \infty, \quad n \to \infty.$$

$$nE(|\overline{Z}_n|^{2\alpha}; |\overline{Z}_n| > \varepsilon) = n\varepsilon^{2\alpha}E(|\overline{Z}_n/\varepsilon|^{2\alpha}; |\overline{Z}_n/\varepsilon| > 1) \leq$$
$$\leq \varepsilon^{2\alpha} nE(|\overline{Z}_n/\varepsilon|^2; |\overline{Z}_n > \varepsilon) = \varepsilon^{2(\alpha-1)} nE(\overline{Z}_n^2; |\overline{Z}_n| > \varepsilon) \to 0,\ n \to \infty.$$

LEMMA 8. *Let $Z_1, Z_2, \ldots$ be as in Lemma 7. Let $0 < \alpha \leq 1$. Then*

(i) $nE[|(1+\overline{Z}_n)^\alpha - 1|^2; |\overline{Z}_n| > \varepsilon] \to 0,\ n \to \infty,$

(ii) $nE[|(1+\overline{Z}_n)^\alpha - 1|^2; |\overline{Z}_n| \leq \varepsilon] \to \alpha^2 d^2,\ n \to \infty,$

Proof. (i) From the ineq. (b) and Lemma 7. (ii) it follows that

$$nE[|(1+\overline{Z}_n)^\alpha - 1|^2; |\overline{Z}_n| > \varepsilon] \leq nE(|\overline{Z}_n|^{2\alpha}; |\overline{Z}_n| > \varepsilon) \to 0,\ n \to \infty.$$

(ii) By the ineq. (c), for $0 < \delta < 1$, $(\alpha^2/(1+\delta)^2)E(\overline{Z}_n^2; |\overline{Z}_n| \leq \delta) \leq$

$$\leq \alpha^2 E[\overline{Z}_n^2/(1+\overline{Z}_n)^2; |\overline{Z}_n| \leq \delta] \leq E[|(1+\overline{Z}_n)^\alpha - 1|^2; |\overline{Z}_n| \leq \delta] \leq$$
$$\leq \alpha^2 E[\overline{Z}_n^2/(1+\overline{Z}_n)^2; |\overline{Z}_n| \leq \delta] \leq (\alpha^2/(1-\delta)^2)E(\overline{Z}_n^2; |\overline{Z}_n| \leq \delta).$$

Following Lemma 7 (i), we have $E[|(1+\overline{Z}_n)^\alpha - 1|^2; |\overline{Z}_n| \leq \delta] \to \alpha^2 d^2$, $n \to \infty$, $\delta \to 0$, and following (i), we have (ii).

THEOREM 1. *Let $0 < \alpha \leq 1$. Then $nD\overline{X}_n^a lpha \to \alpha^2\sigma^2/m^{2(1-\alpha)}$, or*

$$DS_n^\alpha \sim n^{2\alpha-1}\alpha^2\sigma^2/m^{2(1-\alpha)}.$$

Proof. We consider $nE(\overline{X}_n^\alpha - m^\alpha)^2$ instead $nD\overline{X}_n^\alpha$, because

$$E(\overline{X}_n^\alpha - m^\alpha)^2 = D\overline{X}_n^\alpha + (E\overline{X}_n^\alpha - m^\alpha)^2,$$

and by Lemma 6. $n(m^\alpha - E\overline{X}_n^\alpha)^2 \to 0$, $n \to \infty$. Let $Z_i = (X_i/m) - 1$, $EZ_i = 0$ $DZ_i = \sigma^2/m^2 = d^2$. Then $nE(\overline{X}_n^\alpha - m^\alpha)^2 = m^{2\alpha} nE|(1+\overline{Z}_n)^\alpha - 1|^2 =$

$$= m^{2\alpha} n\{E[\ ; |\overline{Z}_n| \le \varepsilon] + E[\ ; \overline{Z}_n| > \varepsilon]\} \to m^{2\alpha}\alpha^2\sigma^2/m^2 = \alpha^2\sigma^2/m^{2(1-\alpha)}.$$

COROLLARY. $DS_n^\alpha \to \begin{cases} 0, & 0 < \alpha < 1/2 \\ \sigma^2/(4m), & \alpha = 1/2 \\ \infty, & \alpha > 1/2 \end{cases}$.

Example. Let $\overline{S}_n^2 = \sum_1^n (X_i - \overline{X}_n)^2/(n-1)$, $\overline{S}_n = (\overline{S}_n^2)^{1/2}$.

Then $E\overline{S}_n^2 = \sigma^2$, $D\overline{S}_n^2 = (\mu_4 - (n-3)\sigma^4/(n-1))/n$, $\mu_4 = E(X-m)^4$ (see **[3]**), and by Lemma 5

$$D\overline{S}_n \le D\overline{S}_n^2/E\overline{S}_n^2 = (\mu_4 - (n-3)\sigma^4/(n-1))/(n\sigma^2),$$
$$\sigma - (\sigma/2n)(\mu_4/\sigma^4 - (n-3)/(n-1)) \le E\overline{S}_n \le \sigma.$$

Table 1

n=10 p	$\sqrt{p}$	$E(\overline{X}_n)^{1/2}$	$AE(\overline{X}_n)^{1/2}$	$E(\overline{X}'_n)^{1/2}$
0,02500	0,15811	0,07413	0,02500	0,20412
05000	22361	13942	16202	24654
10000	31623	24880	27839	32230
20000	44721	40960	42426	44671
30000	54772	52562	53151	54701
40000	63246	61808	62048	63?07
50000	70711	69707	69821	70693
60000	77460	76751	76811	77452
70000	83666	83182	83217	83663
80000	89443	89144	89163	89442
90000	94868	94729	94736	94868
1,00000	1,0	1,0	1,0	1,0
n=20				
0,01250	0,11180	0,05225	0,01250	0,14417
02500	15811	09802	11319	17465
05000	22361	17438	19526	22819
10000	31623	28663	29791	31589
20000	44721	43344	43589	44677
30000	54772	53890	53968	54755
40000	63246	62612	62650	63238
50000	70711	70246	70267	70707

60000	7?460	77123	77136	77458
70000	83666	83434	83442	83665
80000	89443	89298	89303	89442
90000	94868	94801	94802	94868
1,00000	1,0	1,0	1,0	1,0
n=30				
0,00830	0,09110	0,04245	0,00599	0,11800
01250	11180	06185	06535	13065
02500	15811	09802	11319	17465
05000	22361	19185	20514	22435
10000	31623	29965	30414	31566
20000	44721	43895	43970	44703
30000	54772	54208	54237	54766
40000	63246	62833	62849	63243
50000	70711	70406	70415	70709
60000	77460	77239	77244	77459
70000	83666	83513	83516	83666
80000	89443	89348	89350	89443
90000	94868	94824	94824	94868
1,00000	1,0	1,0	1,0	1,0

Specially if $X : N(m;\sigma^2)$, $\mu_4 = 3\sigma^4$, and

$$\sigma - \frac{\sigma}{n}\left(1 + \frac{1}{n-1}\right) \le E\overline{S}_n \le \sigma.$$

5. Estimation of $\sqrt{m}$. Theorem 1 implies that

$$E(\overline{X}_n) \sim (m - \sigma^2/(4nm))^{1/2} \text{ or } E(\overline{X}_n + \sigma^2/(4nm))^{1/2} \sim (m + O(1/n^2))^{1/2}.$$

If we estimate σ^2/m by $\overline{S}_n^2/\overline{X}_n$ we have estimation for $\sqrt{m}$ in the form

$$(\overline{X}_n + \overline{S}_n/(4n\overline{X}_n))^{1/2}.$$

Let $X_i : B(1;p)$, where $EX_i = p$, $DX_i = p(1-p)$. Then, the estimation for $\sigma^2/m = 1 - p = q$ is $1 - \overline{X}_n$, and the estimation for $\sqrt{p}$ is

$$(\overline{X}_n + (1 - \overline{X}_n)/(4n))^{1/2} = (\overline{X}'_n)^{1/2}.$$

In Table 1 are given the values of p, $\sqrt{p}$, $E(\overline{X}_n)^{1/2}$, $AE(\overline{X}_n)^{1/2}$, $E(\overline{X}'_n)^{1/2}$, where $AE(\overline{X}_n)^{1/2} = (p - q/4n)^{1/2}$, for $n = 10, 20, 30$ and $p \ge 1/4n$.

We see that the estimation $(\overline{X}'_n)^{1/2}$ is very good if p is not so small, and for n not necessarily large (considering unbiasnes). Haldane (see [**4**]) gives another (but similar) improvement for the estimation of $\sqrt{p}$ in the form $(\overline{X}''_n)^{1/2} = ((k + 1/4)(n+1/4))^{1/2}$, where $k/n = \overline{X}_n$. It is easily seen that

$$(4k+1)/(4n+1) \leq k/n + (1-k/n)/(4n).$$

Therefore $E(\overline{X}''_n)^{1/2} \leq E(\overline{X}'_n)^{1/2}$, and following the numerical results in Table 1 we can say that our estimation is better if p is not small (when $\sqrt{p} \geq E(\overline{X}'_n)^{1/2}$), and Haldane's estimation is better for small p.

Note that, following Quenouille method for obtaining unbias estimations (see [**5**] or [**1**, vol. 2.]), we have the estimation T_n for $\sqrt{p}$ in the form

$$T_n = n\sqrt{k/n} - (n-1)[(k/n)\sqrt{(k-1)/(n-1)} + (1-k/n)\sqrt{k/(n-1)}].$$

It is easily shown that $T_n \sim (\overline{X}'_n)^{1/2}$, if k and n are not small.

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