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EXTRAPOLATION OF MOVING AVERAGE AND AUTOREGRESSIVE PROCESSES WHEN THE ENTIRE PAST OF THE PROCESSES IS KNOWN

Pavle Mladenović

1. Introduction. Let $X(t) = (X_1(t), \ldots, X_n(t))$, $t \in R$, be a multidimensional and wide sense stationary random process with the mean value zero, the correlation matrix $\|B^X_{jk}(\tau)\|$ and the spectral process $Z^X(\lambda) = (Z^X_1(\lambda), \ldots, Z^X_n(\lambda))$ $\lambda \in R$. Suppose that all elements $B_{jk}(\tau)$ of the correlation matrix $\|B_{jk}(\tau)\|$ fall off sufficiently rapidly at infinity so that they can be represented in the form:

$$B^X_{jk}(\lambda) = \int\limits_{-\infty}^{+\infty} e^{i\tau\lambda} f^X_{jk}(\lambda) d\lambda, \quad i, k = 1, 2, \ldots, n \tag{1.1}$$

(i.e. there exist the spectral densities $f^X_{jk}(\lambda)$).

Let H denote the Hilbert space of all second-order random functions, where a scalar product is defined by $(\xi, \eta) = E\xi\overline{\eta}$, and let $H(X)$ be the Hilbert space generated by $\{X_k(t) : t \in R, k = 1, 2, \ldots, n\}$. Let $L^2(F)$ be the space of functions $\Phi(\lambda) = (\Phi_1(\lambda), \ldots, \Phi_n(\lambda))$ satisfying the following condition:

$$\sum_{k=1}^{n}\sum_{j=1}^{n}\int\limits_{-\infty}^{+\infty} \Phi_k(\lambda)\overline{\Phi_j(\lambda)} f^X_{j,k}(\lambda) d\lambda < \infty \tag{1.2}$$

The space $L^2(F)$ becomes a Hilbert space if we define the scalar product $(\Phi(\lambda), \Psi(\lambda))$ by:

$$(\Phi(\lambda), \Psi(\lambda)) = \int\limits_{-\infty}^{+\infty} \sum_{k=1}^{n}\sum_{j=1}^{n} \Phi_k(\lambda)\overline{\Psi_j(\lambda)} f^X_{j,k}(\lambda) d\lambda < \infty \tag{1.3}$$

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A random function ξ belongs to $H(X)$ if and only if there exists $\Phi(\lambda) \in L^2(F)$ so that

$$\xi = \sum_{k=1}^{n} \int_{-\infty}^{+\infty} \Phi_k(\lambda) dZ_k(\lambda) \tag{1.4}$$

The mapping of $H(X)$ to $L^2(F)$ given by $\xi = \sum_{k=1}^{n} \int \Phi_k(\lambda) dZ_k(\lambda) \to (\Phi_1(\lambda), \ldots, \Phi_n(\lambda))$ is an isometry.

A stationary random process $X(t)$, $t \in R$, is said to be a process with a rational spectrum if all the functions $f^X_{jk}(\lambda)$ are rational functions, i.e. $f_{jk}(\lambda) = Q_{jk}(\lambda)/P_{jk}(\lambda)$, where $P_{jk}(\lambda)$ and $Q_{jk}(\lambda)$ are polynomials in λ. The spectral densities $f_{jj}(\lambda)$ can be represented in the form:

$$B_0|\lambda^M + B_1\lambda^{M-1} + \cdots + B_N|^2 |\lambda^P + A_1\lambda^{P-1} + \cdots + A_P|^{-2} \tag{1.5}$$

where $B_0 > 0$ and $M < P$. Denote the determinant of the spectral matrix $\|f_{jk}(\lambda)\|$ by $D(\lambda)$ $(= Q(\lambda)/P(\lambda))$. Let $2N_{jj}$, $2(N_{jj} - m_j)$, $2K$, $2L$ be the degrees of $P_{jj}(\lambda), Q_{jj}(\lambda), P(\lambda), Q(\lambda)$ respectively. Then $2K - 2L \geq 2\sum_{k=1}^{n} m_k$.

A stationary random process with a rational spectrum is nonsingular if the determinant $D(\lambda)$ has no real zeros and $2K - 2L \geq 2\sum_{k=1}^{n} m_k$. The polynomial $Q(\lambda)$ can be represented in the form $B(\lambda - \theta_1)\ldots(\lambda - \theta_L)(\lambda - \overline{\theta}_1)\ldots(\lambda - \overline{\theta}_L)$ where $\Im\theta_i > 0$, $i = 1, 2, \ldots, L$.

Now, suppose we know the values $X_k(s)$, $s \leq t$, $k = 1, 2, \ldots, n$. The simplest problem of linear extrapolation of a stationary random process $X(t)$ is the following: Find a random variable

$$\tilde{X}_1(t, T) = \sum_{k=1}^{n} \int_{-\infty}^{+\infty} \Phi_k(\lambda) dZ_k(\lambda) \in H(X)$$

which is the best approximation of $X_1(t + \tau)$, depending linearly on $X_k(s)$, $s \leq t$, $k = 1, 2, \ldots, n$. As an index of approximation quality we shall use the mean square error. We shall call $\tilde{X}_1(t, \tau)$ the linear least-squares estimator of $X_1(t + \tau)$. We see that $X_1(t, \tau)$ is the projection of $X_1(t + \tau)$ into the smallest Hilbert space $H(X, t)$ spaned by $\{X_k(s), s \leq t, k = 1, 2, \ldots, n\}$.

In the paper [**7**] the following theorem was proved:

THEOREM 1. (Yaglom): *If $X(t) = (X_1(t), \ldots, X_n(t))$ is a nonsingular stationary random process with a rational spectrum, then the linear least-squares estimator of $X_1(t + \tau)$ is given by:*

$$\tilde{X}_1(t, \tau) = \sum_{k=1}^{n} \left\{ \sum_{j=0}^{m_k - 1} wk^{(j)}(t) + \int_0^{\infty} w_k(s, \tau) X_k(t - s) ds \right\} \tag{1.6}$$

where the coefficients $w_k^{(j)}(\tau)$ *and the functions* $w_k(s,\tau)$ *can be found (in a unigue way) from the condition: the functions*

$$\Phi_k(\lambda) = \sum_{j=0}^{m_k-1} w_k^{(j)}(\tau)(i\lambda)^j + \int_0^\infty e^{-is\lambda} w_k(s,\tau)ds \tag{1.7}$$

have the form

$$\Phi_k(\lambda) = \omega_k(\lambda)\{(\lambda-\theta_1)\dots(\lambda-\theta_l)\}^{-1}, \quad k=1,2,\dots,n. \tag{1.8}$$

where $w_k(\lambda)$ *is a polynomial of degree* $L+m_k-1$ *(but not of a greater degree) such that the functions*

$$\Psi_k(\lambda) = (e^{i\tau\lambda} - \Phi_1(\lambda))f_{1k}(\lambda) - \sum_{j=2}^{n} \Phi_j(\lambda) f_{jk}(\lambda) \tag{1.9}$$

are analytic in the upper half plane.

The functions $\Phi_k(\lambda)$, $k=1,2,\dots,n$, will be called the spectral characteristic for extrapolation of the random process $X(t)$ at the point $t+\tau$.

2. Moving average and autoregressive processes

Let $X(t) = (X_1(t),\dots,X_n(t))$ and $Y(t) = (Y_1(t)),\dots,Y_n(t))$ be two multidimensional stationary random processes satisfying the following equation

$$Y(t) = \sum_{\nu=0}^{N} a_\nu X(t-\nu\theta), \quad a_0 = 1, \quad \theta > 0 \tag{2.1}$$

The process $Y(t)$ is called the moving average process of order N associated with the process $X(t)$. We shall denote it by $MA_X(N,\theta)$. The process $X(t)$ is called the autoregressive process of order N associated with the process $Y(t)$. We shall denote it by $AR_Y(N,\theta)$.

It the roots of

$$t^N + a_1 t^N + \dots + a_N = 0 \tag{2.2}$$

are less than one in absolute value, then the following equations also hold:

$$X_k(t) = \sum_{\nu=0}^{\infty} c_\nu Y_k(t-\nu\theta), \quad k=1,2,\dots,n. \tag{2.3}$$

The series on the right side of (2.3) converges in quadratic mean and the coefficients c_ν satisfy the homogeneous difference equation

$$a_0 c_k + a_1 c_{k-1} + \dots + a_N c_{k-N} = 0, \quad k > N \tag{2.4}$$

and the initial conditions

$$\begin{aligned} &a_0 = 1 \\ &a_0 c_k + a_1 c_{k-1} + \dots + a_k c_0 = 0, \quad O < k < N. \end{aligned} \tag{2.5}$$

THEOREM 2. *Let $X(t)$ and $Y(t)$ be a multidimensional stationary random processes with spectral processes $(Z_1^X(\lambda),\dots,Z_n^X(\lambda))$ and $(Z_1^Y(\lambda),\dots,Z_n^Y(\lambda))$ and let $X(t)$ and $Y(t)$ satisfy (2.1). Then:*

$$\text{a)}\quad dZ_k^Y(\lambda)=\left\{\sum_{\nu=0}^{N}a_\nu e^{-i\lambda\nu\theta}\right\}dZ_k^X(\lambda),\quad k=1,2,\dots,n \tag{2.6}$$

$$\text{b)}\quad f_{jk}(\lambda)=\left|\sum_{\nu=0}^{N}a_\nu e^{-i\nu\theta\lambda}\right|^2 f_{jk}(\lambda),\quad j,k=1,2,\dots,n \tag{2.7}$$

c) *If the roots of* (2.2) *are less than one in absolute value then*

$$H(X,t)=H(Y,t). \tag{2.8}$$

Proof: The equation (2.6) follows from

$$\int_{-\infty}^{+\infty}e^{it\lambda}dZ_k^Y(\lambda)=Y(t)=\sum_{\nu=0}^{N}a_\nu\int_{-\infty}^{+\infty}e^{i(t-\nu\theta)\lambda}dZ_k^X(\lambda)=\int_{-\infty}^{+\infty}e^{it\lambda}\sum_{\nu=0}^{N}a_\nu e^{-i\nu\theta\lambda}dZ_k^X(\lambda).$$

The equation (2.7) follows from

$$\int_{-\infty}^{+\infty}e^{i\tau\lambda}f_{jk}^Y(\lambda)d\lambda=B_{jk}^Y(\tau)=EY_j(t+\tau)\overline{Y_k(t)}=$$

$$=E\sum_{\nu=0}^{N}a_\nu X_j(t+\tau-\nu\theta)\sum_{\mu=0}^{N}\overline{a}_\mu\overline{X_k(t-\mu\theta)}=\sum_{\nu=0}^{N}\sum_{\mu=0}^{N}a_\nu\overline{a}_\mu B_{jk}^X(\tau-(\nu-\mu)\theta)$$

$$=\sum_{\nu=0}^{N}\sum_{\mu=0}^{N}a_\nu\overline{a}_\mu\int_{-\infty}^{+\infty}e^{i\tau\lambda}e^{i(\nu-\mu)\theta\lambda}f_{jk}^X(\lambda)=\int_{-\infty}^{+\infty}e^{i\tau\lambda}\left|\sum_{nu=0}^{N}a_\nu e^{-i\nu\theta\lambda}\right|^2 f_{jk}^X(\lambda)d\lambda.$$

The result (2.8) follows from (2.1) and (2.3).

In this paper, we shall solve the simplest extrapolation problem for the moving average and autoregressive processes associated with a nonsingular stationary process with a rational spectrum.

3. Linear extrapolation of moving average processes

THEOREM 3. *Let $X(s)$ and $Y(s)$ be two multidimensional stationary processes satisfying* (2.1) *and let the roots of the equation* (2.2) *be less than one in absolute value. If $\Phi_{k,\tau}^X(\lambda)$ and $\Phi_{k,\tau}^Y(\lambda)$ are the spectral characteristics for extrapolation of the processes $X(s)$ and $Y(s)$ at the point $t+\tau$ $(\tau>0)$, then:*

a) *If τ/θ is not an integer and $[\tau/\theta] = S < N$, then:*

$$\Phi_{1,\tau}^{Y}(\lambda) = \left\{ \sum_{\nu=0}^{S} a_\nu \Phi_{1,\tau-\nu\theta}^{X}(\lambda) + \sum_{\nu=S+1}^{N} a_\nu e^{i(\tau-\nu\theta)\lambda} \right\} \left\{ \sum_{\nu=0}^{N} a_\nu e^{-i\nu\theta\lambda} \right\}^{-1} \tag{3.1}$$

$$\Phi_{k,\tau}^{Y}(\lambda) = \sum_{\nu=0}^{S} a_\nu \Phi_{1,\tau-\nu\theta}^{X}(\lambda) \left\{ \sum_{\nu=0}^{N} a_\nu e^{-i\nu\theta\lambda} \right\}^{-1}, k = 2, \ldots, n. \tag{3.2}$$

b) *If $\tau = S\theta$ and $S \leq N$, it is only necessary to replace S by $S-1$ in the equations* (3.1) *and* (3.2).

c) *If τ/θ is not an integer and $S \geq N$ or if $\tau = S\theta$ and $S > N$, then:*

$$\Phi_{k,\tau}^{Y}(\lambda) = \sum_{\nu=0}^{N} a_\nu \Phi_{1,\tau-\nu\theta}^{X}(\lambda) \left\{ \sum_{\nu=0}^{N} a_\nu e^{-i\nu\theta\lambda} \right\}^{-1}, k = 2, \ldots, n. \tag{3.3}$$

Proof: a) If τ/θ is not an integer and $S < N$, we have $\tau - \nu\theta > 0$ for $\nu = 0, 1, 2, \ldots, S$ and $\tau - \nu\theta \leq 0$ for $\nu = S+1, \ldots, N$.

It follows from (2.8) that

$$\tilde{Y}_1(t,\tau) = \mathrm{Proj}_{H(Y,t)} Y_1(t+\tau) = \sum_{\nu=0}^{N} \mathrm{Proj}_{H(X,t)} X_1(t+\tau-\nu\theta) =$$
$$= \sum_{\nu=0}^{S} a_\nu \tilde{X}_1(t, \tau-\nu\theta) + \sum_{\nu=S+1}^{N} a_\nu X_1(t+\tau-\nu\theta).$$

If we apply (1.6) and (2.6), we obtain

$$\tilde{Y}_1(t,\tau) = \int_{-\infty}^{+\infty} e^{it\lambda} \left\{ \sum_{\nu=0}^{S} a_\nu \Phi_{k,\tau-\nu\theta}^{X}(\lambda) + \sum_{\nu=S+1}^{N} a_\nu e^{i(\tau-\nu\theta)\lambda} \right\} \left\{ \sum_{\nu=0}^{N} a_\nu e^{-i\nu\theta\lambda} \right\}^{-1} dZ_1^X(\lambda) +$$
$$+ \sum_{k=2}^{n} \int_{-\infty}^{+\infty} e^{it\lambda} \sum_{\nu=0}^{S} a_\nu \Phi_{k,\tau-\nu\theta}(\lambda) \left\{ \sum_{\nu=0}^{N} a_\nu e^{-i\nu\theta\lambda} \right\}^{-1} dZ_k^X(\lambda)$$

and the desired result follows.

b) and c) The proof is similar.

COROLLARY: *If $X(s), s \in R$, is a nonsingular stationary random process with rational spectrum, then the spectral characteristics $\Phi_{k,\tau}^{Y}(\lambda)$, $k = 1, 2, \ldots, n$, can be represented in the form:*

a) *If τ/θ is not an integer and $S < N$ then:*

$$\Phi_{1,\tau}^{Y}(\lambda) R_1(\lambda) \sum_{\nu=0}^{\infty} c_\nu^{(1)} e^{-i\nu\theta\lambda} + \sum_{\nu=S+1}^{\infty} c_\nu^{(2)} e^{i(\tau-\nu\theta)\lambda} \tag{3.4}$$

$$\Phi_{k,\tau}^{Y}(\lambda) R_k(\lambda) \sum_{\nu=0}^{\infty} c_\nu e^{-i\nu\theta\lambda}, \quad k = 2, \ldots, n. \tag{3.5}$$

b) *If* $\tau = s\theta$ *and* $S \leq N$, *it is necessary to replace* S *by* $S-1$ *in the formulae* (3.4) *and* (3.5).

c) *If* τ/θ *is not an integer and* $S \geq N$ *or if* $\tau = S\theta$ *and* $S > N$ *then:*

$$\Phi^{Y}_{k,\tau}(\lambda) = R_k(\lambda)\sum_{\nu=0}^{\infty} c_\nu e^{-i\nu\theta\lambda}, \quad k = 1, 2, \ldots, n. \tag{3.6}$$

The functions $R_k(\lambda) = \sum_{\nu} a_\nu \Phi_{k,\tau-\nu\theta}(\lambda)$ are rational functions.

Proof: If α_ν, $\nu = 1, 2, \ldots, n$, are the roots of the equation (2.2), we have

$$\left\{\sum_{\nu=0}^{N} a_\nu e^{-i\nu\theta\lambda}\right\}^{-1} = \prod_{nu=0}^{N}(1-\alpha e^{-i\nu\theta\lambda})^{-1} = \sum_{\nu=0}^{\infty} c_\nu e^{-i\nu\theta\lambda}$$

and the proof follows.

THEOREM 4. *Let* $X(s)$ *and* $Y(s)$ *satisfy the assumptions of Theorem* 3, *and let* $X(s)$ *be a nonsingular stationary process with rational spectrum. Suppose we know the values* $Y_k(s)$, $s \leq t$, $k = 1, 2, \ldots, n$. *If* τ/θ *is not an integer and* $S < N$, *then the linear least-squares estimator of* $Y_1(t+\tau)$ *is given by:*

$$\begin{aligned}\tilde{Y}_1(t,\tau) = \sum_{k=1}^{n}\left\{\sum_{j=-}^{m_k-1}\sum_{\alpha=0}^{\infty} A^{(j)}_{k,\alpha} Y_k(j)(t-\alpha\theta) + \int_0^\infty B_k(u)Y_k(t-u)\,du\right\}+ \\ + \sum_{\mu=0}^{\infty} D_\mu Y_1(t+\tau-(S+1)\theta-\mu\theta),\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}A^{(j)}_{k,\alpha} &= c_\alpha \sum_{\nu=0}^{S} a_\nu w^{(j)}_k(\tau-\nu\theta), \\ B_k(u) &= \sum_{\alpha=0}^{\infty} c_\alpha D_k(u-\alpha\theta,\tau),\end{aligned} \tag{3.8}$$

$$D_k(u-\alpha\theta,\tau) = \begin{cases} 0 & 0 \leq u < \alpha\theta \\ \sum_{\nu=0}^{S} a_\nu w_k(u-\alpha\theta,\tau-\nu\theta) & u \geq \alpha\theta \end{cases}, \quad \alpha = 0, 1, \ldots, \tag{3.10}$$

$$D_\mu = \sum_{\nu=S+1}^{N} a_\mu F(\mu,\nu) \tag{3.11}$$

$$F(\mu,\nu) = \begin{cases} 0 & 0 \leq \mu < \nu - S - 1 \\ c_{\mu-\nu+S+1} & \mu \geq \nu - S - 1 \end{cases}, \quad \nu = S+1, \ldots, N \tag{3.12}$$

where the coefficients c_ν *were given by* (2.4) *and* (2.5) *and* $w^{(j)}_k$, $w_k(s,\tau)$ *were determined by Theorem* 1.

Proof: Using the formulae (1.6) and (2.3) we obtain

$$\tilde{Y}_1(t,\tau)=\sum_{\nu=0}^{N}a_\nu \tilde{X}_1(t,\tau-\nu\theta)+\sum_{\nu=S+1}^{N}a_\nu X(t+\tau-\nu\theta)=$$

$$=\sum_{\nu=0}^{N}a_\nu\left\{\sum_{k=1}^{n}\left[\sum_{j=0}^{m_k-1}w_k^{(j)}(\tau-\nu\theta)X_k^{(j)}(t)+\int_0^\infty w_k(s,\tau-\nu\theta)X_k(t-s)ds\right]\right\}+$$

$$+\sum_{\nu=S+1}^{N}a_\nu X_1(t+\tau-\nu\theta)=\sum_{k=1}^{n}\sum_{\alpha=0}^{\infty}c_\alpha\left\{\sum_{j=0}^{m_k-1}\sum_{\nu=0}^{S}a_\nu w_k^{(j)}(\tau-\nu\theta)Y_k^{(j)}(t-\alpha\theta)+\right.$$

$$\left.+\int_0^\infty\sum_{\nu=0}^{S}a_\nu w_k(s,\tau-\nu\theta)Y_k(t-s-\alpha\theta)ds\right\}+$$

$$+\sum_{\alpha=0}^{\infty}\left\{c_\alpha\sum_{\nu=S+1}^{N}a_\nu a_\nu\right\}Y_1(t+\tau-\nu\theta-\alpha\theta),$$

and (3.7) follows from (3.8) – (3.12) and

$$\sum_{\alpha=0}^{\infty}c_\alpha\int_0^\infty\sum_{\nu=0}^{S}a_\nu w_k(s,\tau-\nu\theta)Y_k(t-s-\tau-\alpha\theta)ds=$$

$$=\sum_{\alpha=0}^{\infty}c_\nu\int_{\alpha\theta}^\infty\sum_{\nu=0}^{S}a_\nu w_k(u-\alpha\theta,\tau-\nu\theta)\cdot Y_k(t-u)du=$$

$$=\sum_{\alpha=0}^{\infty}c_\nu\int_0^\infty D_k(u-\alpha\theta,\tau)Y_k(t-u)du=\int_0^\infty B_k(u)Y_k(t-u)du,$$

$$\sum_{\nu=S+1}^{N}a_\nu\sum_{\alpha=0}^{\infty}c_\alpha Y_1(t+\tau-\nu\theta-\alpha\theta)Y_k(t-u)du=$$

$$=\sum_{\nu=S+1}^{N}a_\nu\sum_{\mu=\nu-S-1}^{\infty}c_{\mu-\nu+S+1}Y_1(t+\tau-(S+1)\theta-\mu\theta)=$$

$$=\sum_{\nu=S+1}^{N}a_\nu F(\mu,\nu)Y_1(t+\tau-(S+1)\theta-\mu\theta)=\sum_{\mu=0}^{\infty}D_\mu Y_1(t+\tau-(S+1)\theta-\mu\theta).$$

THEOREM 5. *If $T=S\theta$ and $S\le N$, we can obtain $\tilde{Y}_1(t,\tau)$ if we replace S by $S-1$ in the formulae (3.7) – (3.12).*

If τ/θ is not an integer and $S\ge N$, or if $\tau=S\theta$ and $S>N$, then $\tilde{Y}_1(t,\tau)$ is given by:

$$\tilde{Y}(t,\tau)=\sum_{k=1}^{n}\left\{\sum_{j=0}^{m_k-1}\sum_{\alpha=0}^{\infty}A_{k,\alpha}^{(j)}Y_k^{(j)}(t-\alpha\theta)+\int_0^\infty B_k(u)Y_k(t-u)du\right\}\qquad(3.13)$$

where

$$A_{k,\alpha}^{(j)} = c_\alpha \sum_{\nu=0}^{N} a_\nu w_k^{(j)}(\tau - \nu\theta), \tag{3.14}$$

$B_k(u)$ *is given by* (3.9) *and*

$$D_k(u-\alpha\theta,\tau) = \begin{cases} 0 & 0 \le u < \alpha\theta \\ \sum_{\nu=0}^{S} a_\nu w_k(u-\alpha\theta, \tau-\nu\theta) & u \ge \alpha\theta \end{cases}, \tag{3.15}$$

$\alpha = 0, 1, \dots$

4. Linear extrapolation of autoregressive processes

THEOREM 6. *Let* $X(s)$ *and* $Y(s)$ *satisfy the assumptions of Theorem* 3. *Then:*
a) *If* $\tau\theta$ *is not an integer and* $[\tau/\theta] = S$, *then*

$$\Phi_{1,\tau}^X(\lambda) = \sum_{\nu=0}^{S} c_\nu \Phi_{1,\tau-\nu\theta}^Y(\lambda) \sum_{\nu=0}^{N} a_\nu e^{-i\nu\theta\lambda} + \sum_{\mu=S+1}^{S+N} D_\mu e^{i(\tau-\mu\theta)\lambda} \tag{4.1}$$

$$D_\mu = \sum_{\nu=0}^{\mu-S-1} a_\nu c_{\mu-\nu}, \quad \mu = S+1, \dots, S+N. \tag{4.2}$$

$$\Phi_{k,\tau}^X(\lambda) = \sum_{\nu=0}^{S} c_\nu \Phi_{k,\tau-\nu\theta}^Y(\lambda) \sum_{\nu=0}^{N} a_\nu e^{-i\nu\theta\lambda}, \quad k = 2, \dots, n. \tag{4.3}$$

b) *If* $\tau = S\theta$, *then it is necessary to replace* S *by* $S-1$ *in the equations* (4.1) – (4.3).

Proof. a) It follows from (2.3), (2.6) and (2.8) that

$$\begin{aligned}
\tilde{X}_1(t,\tau) &= \text{Proj}_{H(X,t)} X_1(t+\tau) = \text{Proj}_{H(Y,t)} \sum_{\nu=0}^{\infty} c_\nu Y_1(t+\tau-\nu\theta) \\
&= \sum_{\nu=0}^{S} c_\nu \tilde{Y}_1(t,\tau-\nu\theta) + \sum_{\nu=S+1}^{\infty} c_\nu Y_1(t+\tau-\nu\theta) \\
&= \sum_{\nu=0}^{S} c_\nu \sum_{k=1}^{n} \int_{-\infty}^{+\infty} e^{it\lambda} \Phi_{k,\tau-\nu\theta}(\lambda) dZ_k^Y(\lambda) + \sum_{\nu=S+1}^{\infty} c_\nu \int_{-\infty}^{+\infty} e^{i(t+\tau-\nu\theta)\lambda} dZ_1^Y(\lambda) \\
&= \int_{-\infty}^{+\infty} e^{it\lambda} \left\{ \sum_{\nu=0}^{S} c_\nu \Phi_{1,\tau-\nu\theta}(\lambda) + \sum_{\nu=S+1}^{\infty} c_\nu e^{i(\tau-\nu\theta)\lambda} \right\} \{ \sum_{\nu=0}^{N} a_\nu e^{-i\nu\theta\lambda} dZ_1^X(\lambda) \\
&+ \sum_{k=2}^{n} \int_{-\infty}^{+\infty} e^{it\lambda} \sum_{\nu=0}^{S} c_\nu \Phi_{k,\tau-\nu\theta}(\lambda) \sum_{\nu=0}^{N} a_\nu e^{-i\nu\theta\lambda} dZ_k^X(\lambda).
\end{aligned}$$

The term $\sum_{\nu=0}^{\infty} c_\nu e^{i(\tau-\nu\theta)\lambda} \sum_{\nu=0}^{N} a_\nu e^{-i\nu\theta\lambda}$ can be represented in the form

$$\sum_{\mu=S+1}^{S+N} \sum_{\nu=0}^{\mu-S-1} a_\nu c_{\mu-\nu} e^{i(\tau-\mu\theta)\lambda} + \sum_{\mu=S+N+1}^{\infty} \sum_{\nu=0}^{N} a_\nu c_{\mu-\nu} e^{i(\tau-\mu\theta)\lambda}$$

and the desired result follows if we use (2.4) and (2.5).

COROLLARY: *If $Y(s)$, $s \in R$, is a nonsingular stationary process with a rational spectrum and $X = AR_Y(N,\theta)$, then the spectral characteristics $\Phi_{k,\tau}(\lambda)$ can be represented in the form:*

$$\Phi_{1,\tau}^X(\lambda = R_1(\lambda) \sum_{\nu=0}^{N} a_\nu e^{-i\nu\theta\lambda} + \sum_{\mu=S'}^{S'+N-1} D_\mu e^{i(\tau-\mu\theta)\lambda}$$

$$\Phi_{k,\tau}^X(\lambda = R_k(\lambda) \sum_{\nu=0}^{N} a_\nu e^{-i\nu\theta\lambda}, \quad k = 2,\dots,n$$

where $S' = S+1$ if τ/θ is not an integer and $S' = S$ if $\tau = S\theta$..

The functions $R_k(\lambda)$, $k = 1,2,\dots,n$, are rational functions.

THEOREM 7. *Let $X(s)$ and $Y(s)$ be two multidimensional stationary processes satisfying (2.1) and let the roots of the equation (2.2) be less then one in absolute value. Let $Y(s)$ be a nonsingular random process with a rational spectrum. Suppose we know the values of the stationary process $X(s) = AR_{Y(s)}(N,\theta)$ for $s \le t$. Then, the linear least-squares estimator of $X_1(t+\tau)$, $\tau > 0$, is given by:*

a) *If τ/θ is not an integer and $[\tau/\theta] = S$, then:*

$$\begin{aligned}\tilde{X}_1(t,\tau) &= \sum_{k=1}^{n} \left\{ \sum_{j=0}^{m_k-1} \sum_{\alpha=0}^{N} B_{k,\alpha}^{(j)} X_k^{(j)}(t-\alpha\theta) + \int_0^\infty B_k(u) X_k(t-u)du \right\} \\ &= \sum_{\mu=0}^{N-1} D_\mu X_1(t+\tau-(S+1)-\mu\theta),\end{aligned} \tag{4.4}$$

where

$$B_{k,\alpha}^{(j)} = a_\alpha \sum_{\nu=0}^{S} c_\nu w_k^{(j)}(\tau-\nu\theta), \tag{4.5}$$

$$B_k(u) = \sum_{\alpha=0}^{N} a_\alpha D_k(u-\alpha\theta,\tau), \tag{4.6}$$

$$D_k(u-\alpha\theta,\tau)=\begin{cases}0 & 0\le u<\alpha\theta\\ \sum_{\nu=0}^{S}c_\nu w_k(u-\alpha\theta,\tau-\nu\theta) & u\ge\alpha\theta\end{cases}\tag{4.7}$$

$$D_\mu=\sum_{\alpha=0}^{N}a_\alpha F(\mu,\alpha),\quad \mu=0,1,\dots,N-1\tag{4.8}$$

$$F(\mu,\nu)=\begin{cases}0 & 0\le\mu<\alpha\\ c_{S+1+\mu-\alpha} & \mu\ge\alpha\end{cases},\quad \alpha=0,1,2\dots,N\tag{4.9}$$

b) *If* $\tau=S\theta$, *it is necessary to replace* S *by* $S-1$ *in the equations* (4.4) – (4.9).

Proof: a) Since $Y(s)$ is a nonsingular stationary process with a rational spectrum, we have

$$\tilde{Y}_1(t,\tau)=\sum_{k=1}^{n}\left\{\sum_{j=0}^{m_k-1}w_k^{(j)}(\tau)Y_k^{(j)}(t)+\int_0^\infty w_k(s,\tau)Y_k(t-s)ds\right\}$$

and

$$\tilde{X}_1(t,\tau)=\sum_{\nu=0}^{S}c_\nu\left\{\sum_{k=1}^{n}\left[\sum_{j=0}^{m_k-1}w_k^{(j)}(\tau-\nu\theta)Y_k^{(j)}(t)+\int_0^\infty w_k(s,\tau-\nu\theta)Y_k(t-s)ds\right]+\right.$$

$$\left.+\sum_{\nu=S1}^{\infty}c_\nu Y_1(t+\tau-\nu\theta)\right\}=$$

$$=\sum_{k=1}^{n}\left\{\sum_{j=0}^{m_k-1}\sum_{\alpha=0}^{N}\left(a_\alpha\sum_{\nu=0}^{S}c_\nu w_k^{(j)}(\tau-\nu\theta)\right)X_k^{(j)}(t-\alpha\theta)+\right.$$

$$\left.+\int_0^\infty\sum_{\nu=0}^{S}c_\nu w_k(s,\tau-\nu\theta)\sum_{\alpha=0}^{N}a_\alpha X_k(t-s-\alpha\theta)ds\right\}+$$

$$+\sum_{\nu=S+1}^{\infty}c_\nu\sum_{\alpha=0}^{N}a_\alpha X_1(t+\tau-\nu\theta-\alpha\theta)$$

and (4.4) follows from (4.5) – (4.9) and

$$\sum_{\alpha=0}^{N}a_\alpha\int_0^\infty\sum_{\nu=0}^{S}c_\nu w_k(s,\tau-\nu\theta)X_k(t-s-\alpha\theta)ds=\sum_{\alpha=0}^{N}a_\alpha\int_0^\infty\sum_{\nu=0}^{S}c_\nu w_k(u-\alpha\theta,\tau-\nu\theta)\cdot$$

$$\cdot X_k(t-u)du=\sum_{\alpha=0}^{N}a_\alpha\int_0^\infty D_k(u-\alpha\theta,\tau)X_k(t-u)du=\int_0^\infty B_k(u)X_k(t-u)du,$$

$$\sum_{\nu=S+1}^{\infty}c_\nu\sum_{\alpha=0}^{N}a_\alpha X_1(t+\tau-\nu\theta-\alpha\theta)=\sum_{\alpha=0}^{N}\sum_{\mu=\alpha}^{\infty}c_{S+1+\mu-\alpha}X_1(t+\tau-(S+1)\theta-\mu\theta)$$

$$=\sum_{\alpha=0}^{N}a_\alpha\sum_{\mu=\alpha}^{\infty}F(\mu,\alpha)X_1(t+\tau-(S+1)\theta-\mu\theta)=\sum_{\mu=0}^{N-1}D_\mu X_1(t+\tau-(S+1)\theta-\mu\theta).$$

The last equation follows from (2.4).

5. Let us now give some examples.

Example 1. Let $x_1(s)$ and $x_2(s)$ be two independent stationary random processes with the rational spectral densities

$$f_1(\lambda) = c_1(\lambda^2+\alpha_1^2)^{-1},\ f_2(\lambda) = c_2(\lambda^2+\alpha_2^2)^{-1},\quad c_1,c_2,\alpha_1,\alpha_2>0,\ \ \alpha_1 \neq, \alpha_2,$$

and let $X_1(s) = x_1(s)+x_2(s)$, $X_2(s) = x_1(s)-x_2(s)$. It is easy to see that the spectral density matrix $\|f_{ij}^k(\lambda)\|_{2\times 2}$ of the stationary random process $X(s) = (X_1(s), X_2(s))$ is given by

$$\begin{aligned} &M^X(\lambda) = \|f_{ij}^{(X)}(\lambda)\| = \\ &= \begin{Vmatrix} c_1(\lambda^2+\alpha_1^2)^{-1}+c_2(\lambda^2+\alpha_2^2)^{-1} & c_1(\lambda^2+\alpha_1^2)^{-1}-c_2(\lambda^2+\alpha_2^2)^{-1} \\ c_1(\lambda^2+\alpha_1^2)^{-1}-c_2(\lambda^2+\alpha_2^2)^{-1} & c_1(\lambda^2+\alpha_1^2)^{-1}+c_2(\lambda^2+\alpha_2^2)^{-1} \end{Vmatrix} \end{aligned} \tag{5.1}$$

In this case $D(\lambda) = 4c_1c_2(\lambda^2+\alpha_1^2)^{-1}(\lambda^2+\alpha_2^2)^{-1}$ has no real zeros, $n=2$, $2L=0$, $2K=4$, $m_1=m_2=1$, $2K-2L=2(m_1+m_2)$, and thus $X(s)$ is a nonsingular stationary random process with a rational spectrum.

Suppose we know the values of the process $X(s)$ for $s\le t$. Using Theorem 1 we have:

$$\begin{aligned} &\Phi_1(\lambda) = \omega_1(\lambda) = K_1,\ \ \Phi_2(\lambda) = \omega_2(\lambda) = K_2, \\ &\Psi_1(\lambda) = (e^{i\tau\lambda}-K_1)f_{11}(\lambda) - K_2 f_{21}(\lambda), \\ &\Psi_2(\lambda) = (e^{i\tau\lambda}-K_1)f_{12}(\lambda) - K_2 f_{22}(\lambda). \end{aligned}$$

From the fact that the functions $\Psi_1(\lambda)$ and $\Psi_2(\lambda)$ are analytic in the upper half-plane, it follows that

$$2K_1 = 2K_1(T) = e^{-\tau\alpha_1}+e^{-\tau\alpha_2},\ \ 2K_2 = K2(\tau) = e^{-\tau\alpha_1}-e^{-\tau\alpha_2} \tag{5.2}$$

Then, we have $w_1^{(0)}(\tau) = K_1(\tau)$, $w_2^{(0)}(\tau) = K_2(\tau)$, $w_1(s,\tau) = w_2(s,\tau) = 0$, and

$$2X_1(t,\tau) = (e^{-\tau\alpha_1}+e^{-\tau\alpha_2})X_1(t) + e^{-\tau\alpha_1} - e^{-\tau\alpha_2}X_2(t). \tag{5.3}$$

Example 2. Let $X(s)$ be the process given in the example 1, $Y(s) = X(s) - \beta X(s-1)$, $|\beta|<1$, and suppose we know the values of the process $X(s)$ for $s\le t$. In this case: $N=1$, $a_0=1$, $a_1=-\beta$, $Y(s) = MA_X(1,\theta)$, $X(s) = \sum_{\alpha=0}^{\infty}\beta^\alpha Y(s-\alpha\theta)$.

a) If $0<\tau<\theta$ $(0=S<N=1)$, using the formulae (3.8) – (3.12) we obtain

$$\begin{aligned} &A_{1,\alpha}^{(0)} = \beta^\alpha a_0 w_1^{(0)}(\tau) = \beta^\alpha K_1(\tau),\ \ A_{2,\alpha}^{(0)} = \beta^\alpha a_0 w_2^{(0)}(\tau) = \beta^\alpha K_2(\tau), \\ &B_1(u) = B_2(u) = 0,\ \ D_\mu = a_1 F(\mu,1) = a_1 c_\mu = a_1\beta^\mu = -\beta^{\mu+1}, \end{aligned}$$

where $K_1(\tau)$ and $K_2(\tau)$ were given in the example 1, and finally

$$\tilde{Y}_1(t,\tau) = \frac{1}{2}(e^{-\tau\alpha_1} + e^{-\tau\alpha_2}) \sum_{\alpha=0}^{\infty} \beta^\alpha Y_1(t-\alpha\theta) + \frac{1}{2}(e^{-\tau\alpha_1} - e^{-\tau\alpha_2}) \sum_{\alpha=0}^{\infty} \beta^\alpha Y_2(t-\alpha\theta) - \sum_{\mu=0}^{\infty} \beta^{\mu+1} Y_1(t+\tau-\theta-\mu\theta). \tag{5.4}$$

b) If $\tau = \theta$, $(S = N = 1)$, then replacing τ by θ, $\tilde{Y} - 1(t,\theta)$ can be obtained from (5.4).

c) If $\tau = \theta$, using the formulae (3.9) and (3.13) – (3.15) we obtain:

$$B_1(u) = B_2(u) = 0,$$

$$2A_{1,\alpha}^{(0)} = 2\beta^\alpha(a_0 K_1(\tau-\theta)) = \beta^\alpha(e^{-\tau\alpha_1} + e^{-\tau\alpha_2} - \beta e^{-(\tau-\theta)\alpha_1} - \beta e^{-(\tau-\theta)\alpha_2}),$$

$$2A_{2,\alpha}^{(0)} = 2\beta^\alpha(a_0 K_2(\tau-\theta)) = \beta^\alpha(e^{-\tau\alpha_1} - e^{-\tau\alpha_2} - \beta e^{-(\tau-\theta)\alpha_1} - \beta e^{-(\tau-\theta)\alpha_2}),$$

and finally

$$2\tilde{Y}_1(t,\tau) = (e^{\tau\alpha_1} + e^{-\tau\alpha_2} - \beta e^{\tau\alpha_1} - \beta e^{-\tau\alpha_2} \sum_{\alpha=0}^{\infty} \beta^\alpha Y_1(t-\alpha\theta) + + (e^{\tau\alpha_1} - e^{-\tau\alpha_2} - \beta e^{-(\tau-\theta)\alpha_1} - \beta e^{-(\tau-\theta)\alpha_2}) \sum_{\alpha=0}^{\infty} \beta^\alpha Y_2(t-\alpha\theta) \tag{5.5}$$

Example 3. Let $Y(s)$ be a process with the spectral density matrix (5.1) and $X(s) - \beta X(s-1) = Y(s)$, $|\beta| < 1$. Then $X(s) = AR_Y(1,\theta)$, $N = 1$, $a_0 = 1$, $a_1 = -\beta$ and using the formulae (4.4) – (4.9) we obtain:

$$2B_{1,\alpha}^{(0)} = 2a_\alpha \sum_{\nu=0}^{S} \beta^\nu K_1(\tau-\nu\theta) = a_\alpha \sum_{\nu=0}^{S} \beta^\nu (e^{-(\tau-\nu\theta)\alpha_1} + e^{-(\tau-\nu\theta)\alpha_2})$$

$$2B_{2,\alpha}^{(0)} = 2a_\alpha \sum_{\nu=0}^{S} \beta^\nu K_2(\tau-\nu\theta) = a_\alpha \sum_{\nu=0}^{S} \beta^\nu (e^{-(\tau-\nu\theta)\alpha_1} + e^{-(\tau-\nu\theta)\alpha_2})$$

$D_\mu = a_0 F(\mu,0) + a_1 F(\mu,1) = a_0 c_{S+1-\mu} + a_1 c_{S+\mu} = \beta^{S+1+\mu} - \beta\beta^{S+\mu} = 0,\ \ \mu \geq 1$

$D_0 = a_0 c_{S+1} = \beta^{S+1}$ and

$$\tilde{X}_1(t,\tau) = \frac{1}{2}\sum_{\nu=0}^{S} \beta^\nu \{e^{-(\tau-\nu\theta)\alpha_1} + e^{-(\tau-\nu\theta)\alpha_2}\}(X_1(t) - \beta X_1(t-\theta)) + + \frac{1}{2}\sum_{\nu=0}^{S} \beta^\nu \{e^{-(\tau-\nu\theta)\alpha_1} + e^{-(\tau-\nu\theta)\alpha_2}\}(X_2(t) - \beta X_2(t-\theta)) + + \beta^{S+1} X_1(t+\tau-(S+1)\theta). \tag{5.6}$$

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Odsek za matematiku
Prirodno-matematički fakultet
11000 Beograd
Jugoslavija

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