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ON SOME INEQUALITIES FOR CONVEX SEQUENCES

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1. A sequence $a = (a_1, a_2, \dots)$ is said to be convex if $\Delta^2 a_n \geq 0, n = 1, 2, \dots,$ where

$$\Delta^2 a_n = \Delta(\Delta a_n) = a_{n+2} - 2a_{n+1} + a_n, \quad \Delta a_n = a_{n+1} - a_n.$$

If a and p are real sequences, then the well-known Abel identity holds:

$$\sum_{i=1}^{n} p_i a_i = a_1 P_1 + \sum_{k=2}^{n} P_k \Delta a_{k-1}, \quad \left(P_k = \sum_{j=k}^{n} p_i\right). \tag{1}$$

The following generalization of (1) is given in [**2**]: I

If x_{ij}, a_i, b_j $(1 \leq i \leq n, 1 \leq j \leq m)$ are real numbers, then

$$\begin{aligned} \sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij} a_i b_j = a_1 b_1 X_{1,1} + b_1 \sum_{r=2}^{n} X_{r,1} \Delta a_{r-1} + a_1 \sum_{s=2}^{m} X_{1,s} \Delta b_{s-1} + \\ + \sum_{r=2}^{n}\sum_{s=2}^{m} X_{r,s} \Delta a_{r-1} \Delta b_{s-1} \end{aligned} \tag{2}$$

where

$$X_{r,s} = \sum_{i=r}^{n}\sum_{j=s}^{m} x_{ij}. \tag{3}$$

Using (1) and (2), we can get the following identity:

$$\begin{aligned} \sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij} a_i b_j = a_1 b_1 X_{1,1} + b_1 \Delta a_1 X^1_{2,1} + a_1 \Delta b_1 X^2_{1,2} + \Delta a_1 b_1 X^3_{2,2} + \\ + b_1 \sum_{r=3}^{n} X^1_{r,1} \Delta^2 a_{r-2} + a_1 \sum_{s=3}^{m} X^2_{1,s} \Delta^2 b_{s-2} + \Delta a_1 \sum_{s=3}^{m} X^3_{2,s} \Delta^2 b_{r-2} + \\ + \Delta b_1 \sum_{r=3}^{n} X^3_{r,2} \Delta^2 a_{r-2} + \sum_{r=3}^{n}\sum_{s=3}^{m} X^3_{r,s} \Delta^2 a)r - 2\Delta^2 b_{s-2}, \end{aligned} \tag{4}$$

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where

$$
(5)\qquad \begin{gathered}
X^1_{r,1} = \sum_{i=r}^{n}\sum_{j=1}^{m}(i-r+1)x_{ij},\ \ X^2_{1,s} = \sum_{i=1}^{n}\sum_{j=s}^{m}(j-s+1)x_{ij},\\
X_{r,s} = \sum_{i=r}^{n}\sum_{j=s}^{m}(i-r+1)(j-s+1)x_{ij}.
\end{gathered}
$$

Indeed, we have

$$
\begin{aligned}
\sum_{r=2}^{n} X_{r,1}\Delta a_{r-1} &= \Delta a_1 \sum_{r=2}^{n} X_{r,1} + \sum_{r=3}^{n}\left(\sum_{k=r}^{n} X_{k,1}\right)\Delta^2 a_{r-2} = \\
&= \Delta a_1 X^1_{2,1} + \sum_{r=3}^{n} X^1_{r,1}\Delta^2 a_{r-2};\\
\sum_{s=2}^{m} X_{1,s}\Delta b_{s-1} &= \Delta b_1 \sum_{s=2}^{m} X_{1,s} + \sum_{s=3}^{m}\left(\sum_{j=s}^{m} X_{1,j}\right)\Delta^2 b_{s-2} = \\
&= \Delta b_1 X^1_{1,2} + \sum_{s=3}^{m} X^2_{1,s}\Delta^2 b_{s-2};
\end{aligned}
$$

$$
\begin{gathered}
\sum_{r=2}^{n}\sum_{s=2}^{m} X_{r,s}\Delta a_{k-1}\Delta b_{s-1} = \Delta a_1\Delta b_1 \sum_{r=2}^{n}\sum_{s=2}^{m} X_{r,s} + \Delta b_1 \sum_{r=3}^{n}\left(\sum_{k=r}^{n}\sum_{j=2}^{m} X_{k,j}\right)\Delta^2 a_{r-1} + \\
+\Delta a_1 \sum_{s=3}^{m}\left(\sum_{k=2}^{n}\sum_{j=s}^{m} X_{k,j}\right)\Delta^2 b_{s-2} + \sum_{r=3}^{n}\sum_{s=3}^{m}\left(\sum_{k=r}^{n}\sum_{j=s}^{m} X_{k,j}\right)\Delta^2 a_{r-2}\Delta^2 b_{s-2} = \\
= \Delta a_1\Delta b_1 X^3_{2,2} + \Delta b_1 \sum_{r=3}^{n} X^3_{r,2}\Delta^2 a_{r-2} + \Delta a_1 \sum_{s=3}^{m} X_{2,s}\Delta^2 b_{r-2} + \\
+\sum_{r=3}^{n}\sum_{s=3}^{m} X^3_{r,s}\Delta^2 a_{r-2} b_{s-2},
\end{gathered}
$$

so, from (2), we obtain (4).

Using (4), we can easily obtain the following theorem:

THEOREM 1. *Let x_{ij} $(1 \le i \le n,\ \ 1 \le j \le m)$ be real numbers. Inequality*

$$
(6)\qquad \sum_{i=1}^{n}\sum_{j=s}^{m} x_{ij}a_i b_j \ge 0
$$

holds for all convex sequences a and b if and only if

$$
(7)\qquad \begin{aligned}
&X_{1,1} = 0,\ \ X^1_{r,1} = 0\ (r = 2,\dots,n),\ \ X^2_{1,s} = 0\ (s = 2,\dots,m),\\
&X^3_{r,2}\ (r = 2,\dots,n),\ \ X^3_{2,s} = 0\ (s = 3,\dots,m),\\
&X^3_{r,s} \ge 0\,, (r = 3,\dots,n:\ s = 3,\dots,m)
\end{aligned}
$$

where $X_{1,1}$, $X^1_{r,1}$, $X^2_{1,s}$ *and* $X_{r,s}$ *are given by* (3) *and* (5).

Remark 1. An analogous result for convex functions can be obtained from Vasić-Lacković's result for bilinear operators ([**3**]).

2. If a is a convex sequence, then

$$[k,l,m,a] \equiv \frac{a_k}{(k-1)(k-m)} + \frac{a_e}{(l-m)(l-k)} + \\ + \frac{a_m}{(m-k)(m-l)} \geq 0 \quad (k,l,m \in N). \tag{8}$$

Indeed, if a is a convex sequence, then the sequence $((a_n - a_1/(n-1))_{n=2,3,\dots}$ is nondecreasing, i.e. the sequence $((a_n - a_k)/(u-k))_{n=k+1,k+2,\dots}$ is also nondecreasing. If $k < l < m$, we have

$$((a_l - a_k)/(l-k)) \leq ((a_m - a_k)/(m-k))$$

wherefore we obtain (8). Analogously, we can get (8) in other cases ($k < m < l$, etc.).

Using (8), analogously to the proof which is given in [**2**], we can get the following theorem:

THEOREM 2. *Let* a *and* b *be convex sequences,* $e = (1,2,\dots,n), p_k \geq 0$ $(k = 1,\dots,n; P_1 > 0)$*; then*

$$F(ab) - F(a)F(b) \geq (F(ea) - F(e)F(a))(F(eb) - F(e)F(b)) / \left(F(e^2) - F(e)^2\right), \tag{9}$$

where $ab = (a_1b_1,\dots,a_nb_n)$ *and* $F(a) = \frac{1}{P_1}\sum_{i=1}^n p_i a_i$. *If* a *or* b *is an arithmetic sequence, then the equality in* (9) *holds.*

Using Theorem 2, we can prove the following result:

THEOREM 3. *If* a *and* b *are convex sequences, then*

$$\sum_{i=1}^n a_i b_i - \frac{1}{n}\sum_{i=1}^n a_i \sum_{i=1}^n b_i \geq \frac{12}{n(n^2-1)} \sum_{k=1}^k (k-(n+1)/2)a_k \sum_{j=1}^n (j-(n+1)/2)b_j \tag{10}$$

with equality when at least one of the sequences a, b *is an arithmetic sequence.*

Proof. We select $F(a) = \frac{1}{n}\sum_{i=1}^n a_i$. Then $F(e) = (n+1)/2$, $F(e^2) = (n+1)(2n+1)/6$ and from (9), we obtain (10).

COROLLARY 1. *Let* a *and* b *be convex sequences, and assume that*

$$\sum_{k=1}^n \left(k - \frac{n+1}{2}\right) b_k = 0 \tag{11}$$

Then Čebyšev's inequality

$$\sum_{i=1}^n a_i b_i - \frac{1}{n}\sum_{i=1}^n a_i \sum_{i=1}^n b_i \geq 0$$

holds.

Remarks: 2° If $b_k = b_{n-k+1}$ $(k = 1, \dots, n)$, then (11) holds.

3° Theorems 2 and 3 are discrete analogues of Lupas' inequalities [**2**]. Corollary 1 is a discrete analogue of the Atkinson inequality [**4**].

3. We can easily show that (10) can be rewritten in the form of inequality (6) for $n = m$. Using this fact, from Theorem 1, we can obtain the following result:

LEMMA 1. *Let p and w be real sequences. Inequality*

$$\sum_{i=1}^{n} a_i b_i - \frac{1}{n} \sum_{k=1}^{n} a_k \sum_{i=1}^{n} b_i \geq K \sum_{k=1}^{n} p_k a_k \sum_{i=1}^{n} w_i b_i \tag{12}$$

is walid for every pair a and b of convex sequences if and only if

$$K \sum_{i=1}^{n} p_i \sum_{j=1}^{n} w_j = 0 \tag{13}$$

$$K \sum_{k=r}^{n} p_k (k - r + 1) \sum_{j=1}^{n} w_j = 0 \quad (r = 2, \dots, n) \tag{14}$$

$$K \sum_{j=s}^{n} w_j (j - s + 1) \sum_{k=1}^{n} p_k = 0 \quad (s = 2, \dots, n) \tag{15}$$

$$K \sum_{k=r}^{n} p_k (k - r + 1) \sum_{j=2}^{n} w_j (j - 1) = \sum_{i=r}^{n} (i - r + 1)(i - (n+1)/2) \quad (r = 2, \dots, n) \tag{16}$$

$$K \sum_{k=2}^{n} p_k (k - 1) \sum_{j=s}^{n} w_j (j - s + 1) = \sum_{i=s}^{n} (i - s + 1)(i - (n+1)/2) \quad (s = 3, \dots, n) \tag{17}$$

$$K \sum_{k=r}^{n} p_k (k - r + 1) \sum_{j=s}^{n} w_j (j - s + 1) \leq \sum_{i=\max(r,s)}^{n} (i - r + 1)(i - s + 1) - \frac{1}{n} \sum_{i=r}^{n} (i - r + 1) \sum_{i=s}^{n} (i - s + 1). \tag{18}$$

Now we will introduce the following notation:

$$u = \sum_{k=1}^{n} p_k, \quad v = \sum_{i=1}^{n} w_i, \quad P = \sum_{k=2}^{n} (k-1) p_k, \quad Q = \sum_{i=2}^{n} (i-1) w_i, \quad U(r) = \sum_{i=r}^{n} (i - r + 1)(i - (n+1)/2). \tag{19}$$

From condition (16) for $r = 2$, i.e. from

$$KPQ = U(2) = n(n^2 - 1)/12 \tag{20}$$

it follows that we have

$$K \neq 0, \quad P \neq 0, \quad Q \neq 0, \tag{21}$$

and

$$K = n(n^2 - 1)/12PQ. \tag{22}$$

Using (21) again on the basis of (14) and (15) we find that

$$u = v = 0. \tag{23}$$

In such a way we find that if condition (23) is valid, then the conditions (13), (14) and (15) are satisfied.

Conditions (21) and (16), i.e. (17), imply that

$$\sum_{k=r}^{n} p_k(k - r + 1) = 12PU(r)/(n(n^2 - 1)) \ \ (r = 2, \ldots, n) \tag{24}$$

$$\sum_{k=r}^{n} w_k(k - r + 1) = 12QU(r)/(n(n^2 - 1)) \ \ (r = 2, \ldots, n). \tag{25}$$

By substitution of (24) and (25) in (18), we have

$$\begin{aligned} &\sum_{i=\max(r,s)}^{n} (i - r + 1)(i - s + 1) - \frac{1}{n} \sum_{i=r1}^{n} (i - r + 1) \sum_{i=s}^{n} (i - s + 1) \geq \\ &\geq \frac{12}{n(n^2 - 1)} U(r)U(s) = \frac{12}{n(n^2 - 1)} \sum_{i=r}^{n} (i - r + 1)(i - \\ &- (n + 1)/2) \sum_{j=s}^{n} (j - s + 1)(j - (n + 1)/2). \end{aligned} \tag{26}$$

Since the sequences a and b defined by

$$a_k = 0 \ \ (k = 1, \ldots, r - 1) \ \text{ and } \ a_k = k - r + 1 \ \ (k = r, \ldots, n),$$

and

$$b_k = 0 \ \ (k = 1, \ldots, s - 1) \ \text{ and } \ b_k = k - s + 1 \ \ (k = s, \ldots, n)$$

are convex, from (10) we have (26), i.e. (26) is true.

From what we have said above infer the following lemma:

LEMMA 2. *Conditions* (13) – (18) *are valid for a pair of seguences p and w if and only if conditions* (21), (22), (23), (24) *and* (25) *are satisfied, where u, v, P, Q, and U are defined by* (19).

Now, we shall show that $p_k = k_1(k - (n+1)/2)$ where k_1 is a real constant.

From (24), for $r = k$ and $k + 1$, we have

$$p_k = k_1(U(k) - 2U(k+1) + U(k+2)) = k_1(k - (n+1)/2)$$

where $k_1 = 12P/(n(n^2-1))$ and $k = 1, \dots, n-2$.

For $r = n$, from (24) we have

$$p_n = k_1 U(n) = k_1(n - (n+1)/2),$$

and for $r = n - 1$, we have $p_{n-1} + 2p_n = k_1 U(n-1)$, i.e.

$$p_{n-1} = k_1(U(n-1) - 2U(n)) = k_1(n - 1 - (n+1)/2).$$

Analogously, we can get that $w_k = k_2(k - (n+1)/2)$ where k is a real constant. 2
So, in virtue of Lemma 2 we find that the following lemma is valid:

LEMMA 3. *Let us suppose that p and w are real sequences. Then the sequences p and w satisfy conditions* (13)–(18) *if and only if these sequences are of the form*

$$p_k = k_1(k - (n+1)/2), \quad w_k = k_2(k - (n+1)/2), \quad (k = 1, \dots, n), \tag{27}$$

where the real constants are arbitrarily chosen such that $k_i \neq 0$ $(i = 1, 2)$, and where we have

$$K = 12/(n(n^2-1)k_1k_2). \tag{28}$$

On the basis of the results above it can be directly concluded that the following theorem is valid:

THEOREM 4. *Suppose that p and w are real sequences. The inequality of the form* (12) *holds for every pair of convex sequences a and b if and only if these sequences p and w are of the form* (27) *where the real constants $k_1 \neq 0$ and $k_2 \neq 0$ are arbitrary and where the constant K is given by* (28). *In other words, for every pair of convex sequences a and b inequality* (12) *holds true if and only if that inequality is of the form* (10).

Remark 4. Theorem 4 is a discrete analogue of Vasić – Lacković's result from [**5**].

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