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CONDITIONS FOR THE INTEGRABILITY OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATION, II

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Abstract. Conditions for the integrability of second order nonlinear differential equation (0.1) are derived. The obtained result contains, as a special cases a number of known results.

0. In this paper we continue the investigation of integrability of non- linear second order differential equations. We consider the following equation:

$$y'' + P(y)y'^2 + Q(x,y)y' + R(x,y) = 0, \tag{0.1}$$

where P, Q, R are given functions.

Equations of this type were considered in a number of papers (see [**1-16**]). Only in Kamke's collection [**1**] 103 equations of the form (0.1) are noted. Also, Painlevé [2] considered many equations of the same form.

We purpose a method for solving equation (0.1) by reducing it to an equation of the form:

$$Y'' + A(Y,x)Y' + B(Y,x) = 0, \tag{0.2}$$

which is considered in [**5**]. In [**5**] was proved that the equation (0.2) can be reduced to the autonomous equation

$$d^2z/dt^2 + f(z)dz/dt + g(z) = 0 \tag{0.3}$$

by means of transformations

$$y = q(x)z(t) + r(x), \quad dt = p(x)dx, \tag{0.4}$$

if the following conditions are fulfilled:

$$A(Y,x) = pf((Y-r)/q) - 2q'/q - p'/p, \tag{0.5}$$
$$B(Y,x) = p^2qg((Y-r)/q) - (q''/q + A(Y,x)q'/q)(Y-r) - (r'' + A(Y,x)r')$$

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(r, g are some functions depending of one variable, p is a differentiable function of $x; q, r$ are twice differentiable functions of x).

In Section 1 we shall derive the conditions for the integrability of (0.1) by using the above result from [**5**]. Also, some remarks and examples are given.

A certain special class of equations of the form (0.1) is treated in Section 2. Some particular cases are given and compared with some known results from [**7-9**], [**13-17**].

1. Substituting, in (0.1)

$$Y = F(y), \tag{1.1}$$

where

$$F(y) = K(\int_{y_0}^{y} \exp(\int_{y_0}^{y} P(y)dy)dy + L) \tag{1.2}$$

(K, L are constants) we obtain (0.2), where the functions A, B are given by:

$$A(K(\int_{y_0}^{y} \exp(\int_{y_0}^{y} P(y)dy)dy + L), x) = Q(x, y),$$

$$B(K\int_{y_0}^{y} \exp(\int_{y_0}^{y} P(y)dy)dy + L), x) = K\exp(\int_{y_0}^{y} (P(y)dyR(x, y) \tag{1.3}$$

This means that the equations (0.1) and (0.2) are equivalent.

Furthermore, using (0.3) — (0.5) and (1.1) — (1.3), we conclude that (0.1) can be reduced to the autonomous form if the functions Q, R are given by:

$$\begin{aligned} Q(x,y) &= pf((F(y)-r)/q) - 2q'/q - p'/p, \\ R(x,y) &= p^2qg((F(y)-r)/q)/F'(y) - (q''/q + Q(x,y)q'/q)(F(y)-r) - \\ &\quad -(r'' + Q(x,y)r'). \end{aligned} \tag{1.4}$$

In this case under the transformation

$$F(y) = q(x)z(t) + r(x), \quad dt = p(x)dx, \tag{1.5}$$

where F is given by (1.2), the equation (0.1) reduces to the autonomous equation (0.3).

Then we obtain that the general solution of (0.1) is given by:

$$\int (U(F(y)-r)/q,\ C)^{-1} d((F(y)-r)/q) = \int p(x)dx + D, \tag{1.6}$$

(C, D) are arbitrary constants), $u = U(z, C)$ is the general solution of the first order equation

$$u(z)du/dz + f(z)u(z) + g(z) = 0 \tag{1.7}$$

Remarks and *examples*. Here are some remarks and examples related to the above result.

1^0 Equation

$$y'' + P(y)y'^2 + a(x)y' + b(x)T(y) = 0, \tag{1.8}$$

under the transformation (1.1) — (1.2) reduces to the

$$Y'' + a(x)Y' + b(x)S(Y) = 0$$

where $S(F(y)) = F'(y)T(y)$. If $b(x) = k \exp(-2\int a(x)dx)(k =\text{const})$ then the above equation is integrable (see e.g. [**5, 10, 11**]), so in this case, (1.8) is also integrable. This result is also obtained by J. D. Kečkić [**10**] and by L. M. Berkovič and N. N. Rozov [**11**].

2^0 In Kamke's collection [**1**] 103 equations of the type 0.1 are noted. These are equations (6.45)—(6.53), (6.86), (6.107),(6.109)—(6.131), (6.137)—(6.153), (6.155)—(6.176), (6.179), (6.180), (6.183)—(6.186), (6.189)—(6.192), (6.196)—(6.204), (6.206), (6.207), (6.210), (6.212)— (6.215), (6.222)—(6.224). Many of these equations can be solved by applying the above result.

3^0 Let $r(z) = 0$ and $Q(x, y) = a(x)$. Then the following equation

$$y'' + F''(y)F'(y^{-1}y'^2 + a(x)y' + c\exp(-2\int a\,dx)q^{-3}F'(y)^{-1}g(F(y)/q)-$$
$$-(q''/q + a(x)q'/q)F(y)F'(y)^{-1} = 0, \quad (c = \text{const})$$

has the general solution

$$\int(-2\int g(F/q)d(F/q) + C)^{-1/2}d(F/q) = \int q^{-2}\exp(-\int a\,dx)dx + D \tag{1.9}$$

$(C, D$ are arbitrary constants, $F = F(y))$.

In particular, the following equations

$$y'' + (k-1)y^{-1}y'^2 + a(x)y' + c\exp(-2\int a\,dx)y^{1-k}q^{-3}g(y^k/q)-$$
$$-(q''/q + a(x)q'/q)y/k = 0,$$

$$y'' + ky'^2 + a(x)y + c\exp(-2\int adx)q^{-3}e^{-ky}g(e^{ky}/q) - (q''/q + a(x)q'/q)/k = 0$$

$$y'' - y'^2/y + a(x)y' + c\exp(-2\int adx)yq^{-3}g(\log y/q) - (q''/q + a(x)q'/q)\log y/y = 0$$

have the general solutions given by (1.9), where $F(y) = y^k$, $F(y) = e^{ky}$, $F(y) = \log y$, respectively. The above equations (first and third) appear in [**12**].

4^0 Equations

$$y'' + F''(y)F'(y)^{-1}y^{2'} + a(x)y' + \sum_{i=1}^{m} b_i(x)F(y)^{n_i}F'(y)^{-1} = 0,$$

$$y'' + ky^{2'} + a(x)y' + \sum_{i=1}^{m} b_i(x)\exp((n_i - 1)ky)/k = 0,$$

$$y'' + (k-1)y^{-1}y'^2 + a(x)y' + \sum_{i=1}^{m} b_i(x)y^{(n_i-1)k+1}/k = 0,$$

where

(1.10)
$$b_i(x) = c_i \exp(-2\int a\,dx)(C_1 + C_2\int \exp(-\int a\,dx)dx + C_3(\int \exp(-\int a\,px)dx)^2)$$

$(C_1, C_2, C_3, c_i, n_i)$ are constants, $i = 1, \dots, m$), are integrable by quadratures.

The above equations are equivalent to

$$Y'' + a(x)Y' + \sum_{i=1}^{m} b_i(X)Y^{n_i} = 0,$$

which is also integrable if b_i have the form (1.10) (see [**5**]).

5° Let $q(x) = 1$, and $Q(x, y) = a(x)$. Then the following equation

$$y'' + F''(y)F'(y)^{-1}y'^2 + a(x)y + c\exp(-2\int a\,dx)F'(y)^{-1}g(F(y) - r) -$$
$$-(r'' + a(x)r')F'(y)^{-1} = 0, \quad (c = \text{const})$$

has the general solution

(1.11) $$\int(-2\int g(F - r)d(F - r) + C)^{-1/2}d(F - r) = \int \exp(-\int a\,dx)dx + D$$

$(C, D$ are arbitrary constants).

6° Equations

$$y'' + F''(y)F'(y)^{-1}y'^2 + a(x)y' + \sum_{i=1}^{m} b_i(x)\exp(n_iF(y))F'(y)^{-1} = 0,$$

$$y'' - y'^2/y + a(x)y' + \sum_{i=1}^{m} b_i(x)y^{n_i+1} = 0,$$

where $n_i (i = 1, \dots, m)$ are constants and b_i are given by

(1.12) $$b_i(x) = c_i \exp(-2\int a\,dx - n_i r(x)) \quad (c_i, n_i \text{ are constants } i = 1, \dots, m)$$

$$r(x) = C_1 + C_2\int \exp(-\int a\,dx)dx + C_3(\int \exp(-\int a\,dx)dx)^2$$

(C_1, C_2, C_3 are constants), are integrable. This follows from the equation

$$Y'' + a(x)Y' + \sum_{i=1}^{m} b_i(x)\exp(n_i Y) = 0,$$

which is integrable if b_i iz given by (1.12) (see [**5**]).

2. Let he equation (0.2) be of the form:

$$Y'' + a(x)Y' + b(x)Y + c(x)Y'' = 0 \quad (n = \text{const}), \tag{2.1}$$

(a, b, c are given functions). Equation of the type (0.1) corresponding to (2.1) is:

$$y'' + P(y)y'^2 + a(x)y' + b(x)Q_1(y) + c(x)Q_2(y) = 0, \tag{2.2}$$

where P, Q_1, Q_2, satisfy the following system

$$Q_1'(y) = 1 - P(y)Q_1(y), \; Q_1(y)Q_2'(y) = (n - P(y)Q_1(y))Q_2(y). \tag{2.3}$$

Equation (2.2) — (2.3) reduces to the (2.1) by the transformation

$$Y = F(y) = (Q_2(y)/Q_1(y))^{1/(n-1)}. \tag{2.4}$$

For the equation (2.1) the following result is known (see e.g. [**7**]):

Equation (2.1) has the solution $Y = u(x)Z(v(x)/u(x))$, where u and v are linearly independent solutions of the linear equation:

$$u'' + a(x)/u' + b(x)u = 0. \tag{2.5}$$

with the Wronskian $W = v'u - u'v$; Z is a solution of the following nonlinear equation

$$d^2Z/dt^2 + h(t)Z^n = 0, \tag{2.6}$$

where h is determined by

$$h(v(x)/u(x)) = u(x)^{n+3}c(x)W^{-2}. \tag{2.7}$$

Using the above we can formulate the following result for the equation (2.2)—(2.3):

Equation (2.2)—(2.3) has the solution

$$F(y) = u(x)Z(v(x)/u(x)), \tag{2.8}$$

where u and v are linearly independent solutions of (2.5); F is given by (2.4); Z satisfies the equation (2.6)—(2.7).

Remarks and *examples.* 1° Using (2.4) we obtain that the equation (2.2)—(2.3) can be represented in the following form:

$$y'' + F''(y)F'(y)^{-1}y'^2 + a(x)y' + b(x)F(y)F'(y)^{-1} + c(x)F(y)^nF'(y)^{-1} = 0.$$

2° Let $h(t) = k(C_1 + C_2 t + C_3 t^2)^{-(n+3)/2} (k, C_1, C_2, C_3, = \text{const})$. Then the equation (2.6) is integrable by quadratures (see e.g. [**7**]) and from (2.7) we find the function c:

$$c(x) = kW^2(C_1u^2 + C_2uv + C_3v^2)^{-(n+3)/2}. \tag{2.9}$$

In this case the equation (2.2) — (2.3) — (2.11) has the solution

$$F(y) = (C_1u^2 + C_2uv + C_3v^2)^{1/2} U(\int (C_1 + C_2v/u + C_3v^2/u^2)^{-1} d(v/u)), \tag{2.10}$$

where $U(s)$ is given by

$$\int (C - (C_1C_3 - C_2^2/4)U^2 - 2kU^{n+1}/(n+1))^{-1/2} dU = s + D \ \ (n \neq -1),$$
$$\int (C - (C_1C_3 - C_2^2/4)Y^2 - 2k \log U)^{-1/2} dU = s + D \ \ (n = -1). \tag{2.11}$$

(C, D are arbitrary constants, F is given by (2.4)).

3° Furthermore, let

$$c(x) = W^2 u^{-n-3} (\sum_{i=1}^{m} (a_i (v/u)^{b_i})^q \sum_{i,j=1}^{m} A_{ij} (v/u)^{B_{ij}}), \tag{2.12}$$

where $q = k(1-n) - 2$, $A_{ij} = ka_ia_jb_i(1 - b_i + (1-k)b_j)$, $B_{ij} = b_i + b_j - 2, k, a_i, b_i$ are constants $(i, j = 1, \dots, m)$.

In this case the equation (2.2)—(2.3)—(2.12) has the solution:

$$F(y) = u(\sum_{i=1}^{m} a_i (v/u)^{b_i})^k \tag{2.13}$$

where F is determined by (2.4).

4° In particular if $C_1 = 1$, $C_2 = C_3 = 0$, the equation (2.2) — (2.3) — (2.9) becomes

$$y'' + P(y)y'^2 + a(x)y' + b(x)Q_1(y) + k(W^2u^{-n-3}Q_2(y) = 0 \tag{2.14}$$

(P, Q_1, Q_2 satisfy (2.3)) and has the general solution given by (2.10) — (2.11). with $C_1 = 1$, $C_2 = C_3 = 0$.

In the special case $n = -3$ the equation (2.14) reduces to the well known Herbst's equation (see e.g. [**3**, pp. 62–64], [**4**, pp. 187–188] [**6, 8, 9**]). Also, from the above we obtain the general solution of the Herbst's equation.

If $a(x) = 0$, $b(x) = b$=const, $c(x) = c$ =const and $n = 3$ equation (2.14) — (2.3) reduces to the equation (2.5) from [**15**].

5° Let $P(y) = (p-1)/y \, (p \neq 0)$. Then the equation (2.2) becomes

$$y'' + (p-1)y^{-1}y'^2 + a(x)y' + b(x)p^{-1}y = c(x)p^{-1}y^{(n-1)p+1} = 0. \tag{2.15}$$

In this case $F(y) = y^p$. Some special cases of the equation (2.15) with $c(x)$ given by (2.15) and $k = 2/(n-1)$, i.e. $q = 0$ are treated in [**13, 14, 15**]. These results can be obtained as a particular cases of the above. Also, equation (2.15) is treated in [**17**].

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