PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 34 (48), 1983, pp. 121–124

ON Σ_1^0-EXTENSIONS OF $\boldsymbol{\omega}$

Žarko Mijajlović

We shall consider Σ_1^0-extensions of the standard model of arithmetic by models of Peano arithmetic. Even if the notation we shall use is standard in the study of models of Peano arithmetic, we recall some of it.

Peano arithmetic is denoted by P, and its language by L_p. By $\mathbf{M}$, $\mathbf{N}$, etc. we denote models of L_p, or simple expansions of models of this language, and $M, N, \dots$ stand for their domains respectively. The standard model of natural numbers is denoted by $\boldsymbol{\omega}$, thus $\boldsymbol{\omega} = (\omega, +, \cdot, ', 0)$. The sequence $a_0, \dots, a_n \in M$ will be abbreviated by $\vec{a} \in M$. If $a \in M$, then $\mathbf{a}$ denotes the name of a.

The symbols $\subseteq_e, \prec_e, \subseteq_c, \prec_c$ have the usual model-theoretic meaning, namely, they denote respectively the relations of: end-extension, elementary end-extension, cofinal extension, and elementary cofinal extension. If Φ is some of the usual arithmetical hierarchies, e. g. Σ_n^0, Π_n^0, then the symbol $\prec_\Phi$ is used to denote the restriction of the relation $\prec$ to the formulas of Φ.

By Sat $_{\Sigma_k^0}(x, y_1, \dots, y_n)$ we denote a formula of L_p which represents the satisfaction relation for Σ_k^o formulas of P with free variables $y_1, \dots, y_n$. If $\mathbf{M}$ is a model of P, then SSy($\mathbf{M}$) denotes the standard system of $\mathbf{M}$, i.e. the collection of all sets of the form $\{n \in \omega : \mathbf{M} \models \varphi \mathbf{n} \vec{b}\}$ for some $\vec{b} \in M$ and a formula φ of L_p. Observe that SSy($\boldsymbol{\omega}$) is exactly the set of all definable subsets of $\boldsymbol{\omega}$. By Th($\mathbf{M}$) we denote the set of all sentences of L_p true in $\mathbf{M}$.

We are now able to state precisely the question we are interested in. Assume $\boldsymbol{\omega} \prec_{\Sigma_1^0} \mathbf{M}$, where $\mathbf{M}$ is a model of P. By a classical compactness argument there is a model $\mathbf{N}$ such that $\omega \subseteq \mathbf{M} \subseteq \mathbf{N}$, and $\boldsymbol{\omega} \prec \mathbf{N}$. Using Gaifman's splitting theorem, cf. [**4 b** Theorem 2.5]; we can find $\mathbf{K}$ such that $\mathbf{M} \prec_c \mathbf{K} \subseteq_e \mathbf{N}$. Thus, we proved the following sandwich-type fact:

Let $\omega \prec_{\Sigma_1^0} \mathbf{M}, \mathbf{M} \models P$. Then there are $\mathbf{K}$, $\mathbf{N}$ so that $\boldsymbol{\omega} \subseteq \mathbf{M} \prec_c \mathbf{K} \subseteq_e \mathbf{N}$, and Th($\mathbf{N}$)=Th($\boldsymbol{\omega}$).

AMS Subject Classification (1980): Primary 03 H 15, 03 H 20.

Now we turn to the natural question raised by this proposition: which Σ_1^0 exstansions of $\boldsymbol{\omega}$ have end-extension elementary equivalent to $\boldsymbol{\omega}$? We shall answer this question completely for countable models.

THEOREM 1. *Let* $\mathbf{N}$ *be a countable model of* P *such that* $\omega \prec_{\Sigma_1^0} \mathbf{N}$. *Then the following are equivalent:*

1° *There is a model* $\mathbf{M}$ *of* L_p *such that* $\mathbf{N} \subseteq_e \mathbf{M}$, *and* $\boldsymbol{\omega} \prec \Sigma_1^0\mathbf{M}$

2° $SSy(\boldsymbol{\omega}) \subseteq SSy(\mathbf{N})$.

3° *For each* $n \in \omega$ *the code set of* $Th(\boldsymbol{\omega}) \cap \Sigma_n^0$ *belongs to* $SSy(\mathbf{N})$.

Proof. (1° $\to$ 2°) If φx defines $A \subseteq \omega$, then $A = \{n \in \omega : \boldsymbol{\omega} \models \varphi\mathbf{n}\} = \{n \in \omega : \mathbf{M} \models \varphi\mathbf{n}\}$, hence $A \in SSy(\mathbf{M})$. As $\mathbf{N} \subseteq_e \mathbf{M}$, it follows that $SSy(\mathbf{N}) = SSy(\mathbf{M})$, i.e. $A \in SSy(\mathbf{N})$.

(2° $\to$ 3°) Assume $SSy(\boldsymbol{\omega}) \subseteq SSy(\mathbf{N})$. The code set A of $Th(\boldsymbol{\omega}) \cap \Sigma_n^0$ is definable in $\boldsymbol{\omega}$ for each n as $A = \{\ulcorner\varphi\urcorner : \varphi \in \Sigma_n^0,\ \boldsymbol{\omega} \models \mathrm{Sat}_{\Sigma_n^0}(\ulcorner\varphi\urcorner)\}$, where $\ulcorner\varphi\urcorner$ denotes the code number of the formula φ. Hence $A \in SSy(\mathbf{N})$.

(3° $\to$ 1°) This part follows immediately by the following theorem of D. Jensen and A. Ehrenfeucht (cf. [**2**, Theorem 3]):

If $\mathbf{M}$ is a countable nonstandard model of P and T is a complete extension of P, then there is an end-extension $\mathbf{N}$ of $\mathbf{M}$ with Th($\mathbf{N}$)=T iff both of the following holds:

(i) $\mathrm{Th}(\mathbf{M}) \cap \Sigma_1^0 \subseteq T$.

(ii) For each $n \in \omega$ the code set of $T \cap \Sigma_n^0$ belongs to SSy($\mathbf{M}$).

Originally, it was assumed in (ii) that $T \cap B_n$ belongs to SSy($\mathbf{M}$), where B_n denotes the set of all Boolean combinations of Σ_n^0 formulas. However, this is not a stronger assumption as SSy($\mathbf{M}$) is a field of sets.

In [**5**] a criterion is given for a model $\mathbf{M}$ having an end-extension regardless of the cardinality of $\mathbf{M}$. A variant of Theorem 1 in [**5**] is:

THEOREM 2. *Let* $\mathbf{M}$ *be a nonstandard model of* P *and* T *a complete extension of* P. *Then there is a model* $\mathbf{M}_1$ *of* T *satisfying* $\mathbf{M} \subseteq_e \mathbf{M}_1$ *iff both of the following conditions hold:*

1° $\mathrm{Th}(\mathbf{M}) \cap \Sigma_1^0 \subseteq T$.

2° *There is a model* $\mathbf{M}_2$ *of* T *such that* $\mathrm{SSy}(\mathbf{M}_2) = \mathrm{SSy}(\mathbf{M})$.

Thus, taking $T = \mathrm{Th}(\boldsymbol{\omega})$ we have the following

COROLLARY 2.1. *Let* $\mathbf{N}$ *be a model of* P *such that* $\boldsymbol{\omega} \prec_{\Sigma_1^0} \mathbf{N}$. *Then the following propositions are equivalent:*

1° *There is a model* $\mathbf{M}$ *such that* $\mathbf{N} \subseteq_e \mathbf{M}$ *and* $\boldsymbol{\omega} \prec \mathbf{M}$.

2° *There is a model* $\mathbf{M}$ *of* $\mathrm{Th}(\boldsymbol{\omega})$ *such that* $\mathrm{SSy}(\mathbf{M}) = \mathrm{SSy}(\mathbf{N})$.

Some further improvements can be made. For example, using the Arithmetizated Completeness Theorem we obtain at once for any model $\mathbf{M}$ of P such that

$\boldsymbol{\omega} \prec_{\Sigma_1^0} \mathbf{M}$ and $\mathrm{Th}(\boldsymbol{\omega}) \in \mathrm{SSy}(\mathbf{M})$, that there is a (recursively saturated) model $\mathbf{N}$ which satisfies $\mathbf{M} \subseteq_e \mathbf{N}$ and $\mathbf{N} \models \mathrm{Th}(\boldsymbol{\omega})$. For related details one may consult [**4a**, p. 860], and also Lemma 2 in [**5**].

Now, we shall exhibit a countable Σ_1^0-elementary extension of $\boldsymbol{\omega}$ which has no end-extension elementary equivalent to $\boldsymbol{\omega}$. This result shows that Theorem 1 is not vacuous.

THEOREM 3. *There is a model* $\mathbf{M}$ *of* P *such that* $\boldsymbol{\omega} \prec_{\Sigma_1^0} \mathbf{M}$ *and* $\mathrm{SSy}(\boldsymbol{\omega}) - \mathrm{SSy}(\mathbf{M}) \neq \varnothing$.

Proof. Let $T = P + \mathrm{Th}(\boldsymbol{\omega}) \cap \Sigma_2^0$. Then the code set of T is Σ_2^0 as

$$\mathrm{Th}(\boldsymbol{\omega}) \cap \Sigma_2^0 = \{\varphi : \varphi \in \Sigma_2^0, \quad \boldsymbol{\omega} \models \mathrm{Sat}_{\Sigma_2^0}(\ulcorner \varphi \urcorner).$$

Now, define $A = \{n \in \omega \mid \boldsymbol{\omega} \models \mathrm{Sat}_{\Sigma_2^0}(\mathbf{m}, \mathbf{m})\}$. Thus, A belongs to Σ_2^0, i.e. $A \in \mathrm{SSy}(\boldsymbol{\omega})$. If A would be Π_2^0, then A^c would be Σ_2^0, so for some $e \in \omega$ $A^c = \{m \in \omega : \boldsymbol{\omega} \models \mathrm{Sat}_{\Sigma_2^0}(\mathbf{e}, \mathbf{m})\}$. Hence, $e \in A$ iff $\boldsymbol{\omega} \models \mathrm{Sat}_{\Sigma_2^0}(\mathbf{e}, \mathbf{e})$ iff $e \in A^c$, a contradiction. Therefore A is $\Sigma_2^0 - \Pi_2^0$. As $\mathrm{Th}(\boldsymbol{\omega} \cap \Sigma_2^0 \subseteq T$, we see that A is recursively enumerable in T. However, A is not recursive relative to T as A is not representable in T: if there were ψ of L_p such that $n \in A$ implies $T \vdash \psi\mathbf{n}$, and $n \notin A$ implies $T \vdash \rceil\psi\mathbf{n}$ then A and A^c would be recursively enumerable in T, and so, as T is Σ_2^0, it would follow that A^c is Σ_2^0, a contradiction.

Let $\varphi_0 x, \varphi_1 x, \ldots$ be an enumeration of formulas of L_p with x as the only free variable. Define a sequence T_n, $n \in \omega$, of theories as follows:

Let $T_0 = T$. Assume T_n has been defined. We distinguish two cases:

1° If $\{m \in \omega : T_n \vdash \varphi_n\mathbf{m}\}^c = \{m \in \omega : T_n \vdash \rceil\varphi_n\mathbf{m}\}$, then $T_{n+1} = T_n$.

2° Assume for some $m \in \omega$ not $T_n \vdash \varphi_n\mathbf{m}$ and not $T_n \vdash \rceil\varphi_n\mathbf{m}$.

If $m \in A$, then let $T_{n+1} = T_n \cup \{\rceil\varphi_n\mathbf{m}\}$

If $m \notin A$, then let $T_{n+1} = T_n \cup \{\varphi_n\mathbf{m}\}$.

Let $T' = \cup_{n\in\omega} T_n$ and let $\mathbf{M}$ be any prime model of T'. Then every $X \in \mathrm{SSy}(\mathbf{M})$ is definable in $\mathbf{M}$ without parameters. So if $A \in \mathrm{SSy}(\mathbf{M})$, then for some n, $A = \{m \in \omega : \mathbf{M} \models \varphi_n\mathbf{m}\}$. If φ_n satisfied case 1°, then A would be represented in T, a contradiction. Thus φ_n satisfies case 2°; so for some $m \in \boldsymbol{\omega}$ not $T_n \vdash \varphi_n\mathbf{m}$, and not $T_n \vdash \rceil\varphi_n\mathbf{m}$. But then $A \in \mathrm{SSy}(\mathbf{M})$ contradicts the choice of T_{n+1}. Therefore $A \notin \mathrm{SSy}(\mathbf{M})$

COROLLARY 3.1. *There is a model* $\mathbf{M}$ *of* P *such that* $\boldsymbol{\omega} \prec_{\Sigma_1^0} \mathbf{M}$ *and this model has no end-extension which is elementary equivalent to* $\boldsymbol{\omega}$.

REFERENCES

[1] H. Gaifman, *Models and types of Peano's arithmetic*, Ann. Math. Logic **9** (1976), 223–306.

[2] D. Jensen, A. Ehrenfeucht, *Some problem in elementary arithmetic*, Fund. Math. **92** (1976), 223–245.

[3] D. Scott, "Algebra of sets binumerable in complete extensions of arithmetic", *Recursive function theory* (J. C. E. Dekker, Editor), AMS, Providence, R. I. 1962.

[4] C. Smorinski, a) "The incompleteness theorems", J. Barwise (ed.), *Handbook of Mathematical logic*, North Hollandd, Amsterdam 1977.

b) *Recursively saturated nonstandard models of arithmetic*, J. Symbolic Logic **46** (1981) 259–286.

[5] A. Wilkie, "On the theories of end-extension of models of arithmetic", *Set Theory and Hierarchy Theory V* (A. Lachlan, M. Srebrny and A. Zarach, Editors), Springer-Verlag, Heidelberg 1977.

Institut za matematiku
Prirodno-matematički fakultet
11000 Beograd
Jugoslavija

(Received 14 12 1982)