PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 34 (48), 1983, pp. 125–135

DERIVATIONAL FORMULAS OF A SUBSPACE OF A GENERALIZED RIEMANNIAN SPACE

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Summary. L. P. Einsenhart has defined in [**1, 2**] a generalized Riemannian space with nonsymmetric basical tensor, R. S. Maishra and M. Prvanović in [**3,4**] have studied derivational formulas and Gauss-Codazzi equations in a subspace of a generalized Riemannian space. In [**5,6**] we used 4 kinds of covariant derivation of a tenzor in a sbspaces of this space.

In the present work we obtain 4 kinds of derivational fdormulas, using the above mentioned kinds of derivation.

0. Introduction. Let V_N be a generalized Riemanian space with cordinates $y^\alpha(\alpha=1,\dots,N)$ and a basical tensor $a_{\alpha\beta}(a_{\alpha\beta}\neq a_{\alpha\beta})$

The equations

(1) $$y^\alpha = y^\alpha(x^1,\dots,x^M)$$

define a subspace V_M of the space V_N. The coorinates in the V_M are $x^i(i=1,\dots,M)$, and the basical tensor is $g_{ij}\,(g_{ij}\neq g_{ji})$.

The next relations (see [**1–3**]) are valid;

(2) $$a_{\alpha\beta}y^\alpha_{,i}y^\beta_{,j} = a_{\alpha\beta}t^\alpha_i t^\beta_j = g_{ij},$$

where the comma (,) signifies the usual partial derivation, and we have

(3) $$\partial y^\alpha/\partial x^i = y^\alpha_{,i} = t^\alpha_i.$$

If we denote by $\underline{\alpha\beta}$ and $\underset{\vee}{\alpha\beta}$ the symmetrisation and antisymmetrisation over indices α, $beta$ and analogically in other cases, we have

(4a, b) $$g_{\underline{ij}} = a_{\underline{\alpha\beta}}t^\alpha_i t^\beta_j,\ \ g_{\underset{\vee}{ij}} = a_{\underset{\vee}{\alpha\beta}}t^\alpha_i t^\beta_j,$$

(5a, b) $$g_{\underline{ij}}g^{\underline{jk}} = \delta^k_j,\ a_{\underline{\alpha\beta}}a^{\underline{\beta\gamma}} = \delta^\gamma_\alpha.$$

The Christoffel symbols for V_N are

(6) $$\Gamma_{\alpha,\beta\gamma} = (a_{\beta\alpha,\gamma} - a_{\beta\gamma,\alpha} + a_{\alpha\gamma,\beta})/2$$

(7) $$\Gamma^\alpha_{\beta\gamma} = a^{\underline{\pi\alpha}}\Gamma_{\pi,\beta\gamma}\ (\Gamma^\alpha_{\beta\gamma}\neq\Gamma^\alpha_{\gamma\beta})$$

and analogousely for V_M. Further, for example, it is

(7') $$a_{\underline{\alpha\sigma}}\Gamma^{\sigma}_{\beta\gamma} = a_{\underline{\alpha\sigma}}a^{\overline{\pi\sigma}}\Gamma_{\pi,\beta,\gamma} = \delta^{\pi}_{\alpha}\Gamma_{\pi,\beta\gamma} = \Gamma_{\alpha,\beta\gamma}.$$

For unit, mutally orthogonal vectors $N^{\alpha}_{(\varrho)}$, which are also orthogonal on the V_M, we have

(8a) $$a_{\underline{\alpha\beta}}N^{\beta}_{(\sigma)} = e_{(\varrho)}\delta_{\varrho\sigma} \quad (e_{(\varrho)} = \pm 1),$$

(8b) $$a_{\underline{\alpha\beta}}N^{\alpha}_{(\sigma)}t^{\beta}_{j} = 0$$

where the Greek indices in the brackets take values from $M+1$ to N and have not a tensor nature.

Because of non-symmetry of the connexion coefficients, we can define 4 kinds of covariant derivation in the generalized Riemannian space (see [**5, 6**]). For example, for a tensor $a^{\alpha i}_{\beta j}$, whose Greek indices are related to the space V_N, and Latin to the subspace V_M, for a covariant derivation on x^m, we have

(9a,b)
$$a^{\alpha i}_{\beta j \underset{\substack{1\\2}}{|} m} = a^{\alpha i}_{\beta,j,m} + (\Gamma^{\alpha}_{\underset{\mu\pi}{\pi\mu}}a^{\pi i}_{\beta j} - \Gamma^{\pi}_{\underset{\mu\beta}{\beta\mu}}a^{\alpha i}_{\pi j})t^{\mu}_{m} + \Gamma^{i}_{\underset{mp}{pm}}a^{\alpha p}_{\beta j} - \Gamma^{p}_{\underset{mj}{jm}}a^{\alpha i}_{\beta p},$$

(9c, d)
$$a^{\alpha i}_{\beta j \underset{\substack{3\\4}}{|} m} = a^{\alpha i}_{\beta,j,m} + (\Gamma^{\alpha}_{\underset{\mu\pi}{\pi\mu}}a^{\pi i}_{\beta j} - \Gamma^{\pi}_{\underset{\mu\beta}{\beta\mu}}a^{\alpha i}_{\pi j})t^{\mu}_{m} + \Gamma^{i}_{\underset{mp}{pm}}a^{\alpha p}_{\beta j} - \Gamma^{p}_{\underset{mj}{jm}}a^{\alpha i}_{\beta p},$$

If all indices of a tensor are related to the space, we can find its derivation by y^{μ} or by x^m. For example

(10a,b) $$a^{\alpha\beta}_{\gamma \underset{\substack{1\\2}}{|} \mu} = a^{\alpha\beta}_{\gamma,\mu} + \Gamma^{\alpha}_{\underset{\mu\pi}{\pi\mu}}a^{\pi\beta}_{\gamma} + \Gamma^{\beta}_{\underset{\mu\pi}{\pi\mu}}a^{\alpha\pi}_{\gamma} - \Gamma^{\pi}_{\underset{\mu\gamma}{\gamma\mu}}a^{\alpha\beta}_{\pi},$$

(10c,d) $$a^{\alpha\beta}_{\gamma \underset{\substack{3\\4}}{|} \mu} = a^{\alpha\beta}_{\gamma,\mu} + \Gamma^{\alpha}_{\underset{\mu\pi}{\pi\mu}}a^{\pi\beta}_{\gamma} + \Gamma^{\beta}_{\underset{\mu\pi}{\pi\mu}}a^{\alpha\pi}_{\gamma} - \Gamma^{\pi}_{\underset{\mu\gamma}{\gamma\mu}}a^{\alpha\beta}_{\pi},$$

(11) $$a^{\alpha\beta}_{\gamma \underset{\theta}{|} m} = a^{\alpha\beta}_{\gamma \underset{\theta}{|} \mu}t^{\mu}_{m}.$$

Further, using (6) we have

(12a,b) $$\Gamma_{\alpha,\beta\gamma} + \Gamma_{\beta,\alpha\gamma} = a_{\underline{\alpha\beta},\gamma}, \ \Gamma_{\alpha,\beta\gamma} + \Gamma_{\gamma,\beta\alpha} = a_{\underline{\alpha\gamma},\beta}$$

(12c) $$\Gamma_{\alpha,\beta\gamma} + \Gamma_{\alpha,\gamma\beta} = a_{\underline{\alpha\beta},\gamma} - a_{\underline{\beta\gamma},\alpha} + a_{\underline{\gamma\alpha},\beta}.$$

From (10a) and (7')

$$a_{\underline{\alpha\beta}\underset{1}{|}\gamma} = a_{\underline{\alpha\beta},\gamma} - \Gamma^{\pi}_{\alpha\gamma}a_{\underline{\pi\beta}} - \Gamma^{\pi}_{\beta\gamma}a_{\underline{\alpha\pi}} = a_{\underline{\alpha\beta},\gamma} - \Gamma_{\beta,\alpha\gamma} - \Gamma_{\alpha,\beta\gamma},$$

whence, using (12a) we obtain

(13) $$a_{\underline{\alpha\beta}\underset{1}{|}\gamma} = 0.$$

It the same way, we prove that;

(13')
$$g_{\underline{ij}\underset{1}{|}m}=0,$$
$$a_{\underline{\alpha\beta}\underset{2}{|}\gamma}=a_{\underline{\alpha\beta}\underset{3}{|}\gamma}=0,\ \ a_{\underline{\alpha\beta}\underset{4}{|}\gamma}=a_{\underline{\alpha\beta}\underset{2}{|}\gamma}=0,$$
$$g_{\underline{ij}\underset{2}{|}m}=g_{\underline{ij}\underset{3}{|}m}=0,\ \ g_{\underline{ij}\underset{4}{|}m}=g_{\underline{ij}\underset{1}{|}m}=0.$$

Therefore, the following is valid:

(14a,b)
$$a_{\underline{\alpha\beta}\underset{\theta}{|}\gamma}=0,\ \ g_{\underline{ij}\underset{\theta}{|}m}=0,\quad (\theta=1,\dots 4)$$

and, based on (11), it is

(14c)
$$a_{\underline{\alpha\beta}\underset{\theta}{|}\gamma}=0\quad (\theta=1,\dots,4).$$

1. First derivational formulas. By derivational formulas one expresses covariant derivatives of the tensors t_i^α, $N_{(\varrho)}^\alpha$ as linear combinations of these tensors.

From (9a-d) we have

(15a)
$$t^\alpha_{i\underset{1}{|}m}=t^\alpha_{i,m}+\Gamma^\alpha_{\pi\mu}t^\mu_m t^\pi_i-\Gamma^p_{im}t^\alpha_p,$$

(15b)
$$t^\alpha_{i\underset{2}{|}m}=t^\alpha_{i,m}+\Gamma^\alpha_{\mu\pi}t^\mu_m t^\pi_i-\Gamma^p_{mi}t^\alpha_p,$$

(15c)
$$t^\alpha_{i\underset{3}{|}m}=t^\alpha_{i,m}+\Gamma^\alpha_{\pi\mu}t^\mu_m t^\pi_i-\Gamma^p_{mi}t^\alpha_p,$$

(15d)
$$t^\alpha_{i\underset{4}{|}m}=t^\alpha_{i,m}+\Gamma^\alpha_{\mu\pi}t^\mu_m t^\pi_i-\Gamma^p_{im}t^\alpha_p,$$

and we can interpret the tensors $t^\alpha_{i\underset{\theta}{|}m}(\theta=1,\dots,4)$ in the following manner:

(16)
$$t^\alpha_{i\underset{\theta}{|}m}=\underset{\theta}{\Phi}{}^p_{im}t^\alpha_p+\sum_{\varrho=M+1}^{N}\underset{\theta}{\Omega}_{(\varrho)im}N^\alpha_{(\varrho)}$$

For $\theta=1,\dots,4$ we obtain from (16) 4 kinds of **I** derivational formula of the subspace V_M of the generalized Riemannian space V_N.

Multiplying the last equation by $a_{\alpha\beta t^\beta_h}$ and using (4a) and (8b) be obtain

(17)
$$a_{\underline{\alpha\beta}}t^\alpha_{i\underset{\theta}{|}m}t^\beta_h=\underset{\theta}{\Phi}{}^p_{im}a_{\underline{\alpha\beta}}t^\alpha_p t^\beta_h=\underset{\theta}{\Phi}{}^p_{im}g_{\underline{ph}}=\underset{\theta}{\Phi}_{him},\ \text{i.e.}$$

(17')
$$\underset{\theta}{\Phi}{}^h_{im}=\underset{\theta}{\Phi}_{q\,im}g^{\underline{hq}}=a_{\underline{\alpha\beta}}t^\alpha_{i\underset{\theta}{|}m}t^\beta_q g^{\underline{hq}}.$$

Multiplying the equation (16) by $a_{\alpha\beta}N^\beta_{(\sigma)}$ based on (8a, b) it follows

$$a_{\underline{\alpha\beta}}t^\alpha_{i\underset{\theta}{|}m}N^\beta_{(\sigma)}=e_{(\sigma)}\underset{\theta}{\Omega}_{(\sigma)im}\quad (e_{(\sigma)}=\pm 1),$$

whence

$$\underset{\theta}{\Omega}_{(\sigma)im} = e_{(\sigma)} a_{\underline{\alpha\beta}} t^{\alpha}_{i\underset{\theta}{|}m} N^{\beta}_{(\sigma)}. \tag{18}$$

We shall prove now that the tensors $\underset{\theta}{\Phi}^{h}_{im}$ are antisymmetric on lower indices, and $\underset{\theta}{\Phi}_{him}$ on all indices. Since

$$t^{\alpha}_{m,l} = t^{\alpha}_{i,m} = y^{\alpha}_{,im} = \partial^2 y^{\alpha} \backslash \partial x^i \, \partial x^m, \tag{19}$$

we conclude from (15a-d) that

$$\underline{t^{\alpha}_{i\underset{\theta}{|}m}} = (t^{\alpha}_{i,\underset{\theta}{|}m} + t^{\alpha}_{m\underset{\theta}{|}i})/2 = t^{\alpha}_{i,m} + \Gamma^{\alpha}_{\pi\mu} t^{\mu}_{m} t^{\pi}_{i} - \Gamma^{p}_{\underline{im}} t^{\alpha}_{p}. \tag{20}$$

If, on the same manifold on which generalized Riemannian space V_N and its subspace V_M are defined, we define a usual Riemannian space $\overline{V}_N$ and its subspace $\overline{V}_M$ using as basical tensors $a_{\underline{\alpha\beta}}$ and $g_{\underline{ij}}$ the equation (2) becomes

$$t^{\alpha}_{i\underset{\theta}{|}m} = t^{\alpha}_{i;m} \tag{20'}$$

where by the semicolon (;) we denote covariant derivation by using symmetric connexion coefficients $\Gamma^{\alpha}_{\underline{\beta\gamma}}$ respectively $\Gamma^{i}_{\underline{jk}}$ obtained by $a_{\underline{\alpha\beta}}(g_{\underline{ij}})$.

The derivational formulas of the subspace V_M (see [**7**, § 47]) are

$$t^{\alpha}_{i;m} = \sum_{\varrho} \Omega_{(\varrho)im} N^{\alpha}_{(\varrho)} \tag{21a}$$

$$N^{\alpha}_{(\sigma);m} = -e_{(\sigma)} g^{\underline{ps}} \Omega_{(\sigma)sm} t^{\alpha}_{p} + \sum_{\varrho} \Psi_{(\varrho\sigma)m} N^{\alpha}_{(\varrho)} \quad (\Psi_{\varrho\varrho)m} = 0). \tag{21b}$$

Using (18), (20)'

$$\underset{\theta}{\Omega}_{(\sigma)\underline{im}} = e_{(\sigma)} a_{\underline{\alpha\beta}} \underline{t^{\alpha}_{i\underset{\theta}{|}m}} N^{\beta}_{(\sigma)} = e_{(\sigma)} a_{\underline{\alpha\beta}} t^{\alpha}_{i;m} N^{\beta}_{(\sigma)},$$

and from (21a):

$$\underset{\theta}{\Omega}_{(\sigma)\underline{im}} = \Omega_{(\sigma)im}. \tag{22}$$

According to (21a), it is

$$a_{\underline{\alpha\beta}} t^{\alpha}_{i;m} t^{\beta}_{h} = a_{\underline{\alpha\beta}} \underline{t^{\alpha}_{i\underset{\theta}{|}m}} t^{\beta}_{h} = 0 \tag{23}$$

and, based on (17'), we have

$$\underset{\theta}{\Phi}^{h}_{\underline{im}} = a_{\underline{\alpha\beta}} g^{\underline{hq}} \underline{t^{\alpha}_{i\underset{\theta}{|}m}} t^{\beta}_{q}.$$

From this, according to (23), one obtains

(24) $$\underset{\theta}{\Phi}{}^h_{im} = 0 \;\text{ i.e.}$$

(24') $$\underset{\theta}{\Phi}{}^h_{im} = -\underset{\theta}{\Phi}{}^h_{im} = \underset{\theta}{\Phi}{}^h_{\vee im},$$

which means that the tensors $\underset{\theta}{\Phi}{}^h_{im}$ are antisymmetric on indices.

Taking into consideration (14 b, c) from (4a) there follows

(25a) $$a_{\underline{\alpha\beta}} t^\alpha_{i\underset{\theta}{|}m} t^\beta_j + a_{\underline{\alpha\beta}} t^\alpha_i t^\beta_{j\underset{\theta}{|}m} = 0$$

By cyclical interchange of the indices we obtain

(25b) $$a_{\underline{\alpha\beta}} t^\alpha_{j\underset{\theta}{|}i} t^\beta_m + a_{\underline{\alpha\beta}} t^\alpha_j t^\beta_{m\underset{\theta}{|}i} = 0,$$

(25c) $$a_{\underline{\alpha\beta}} t^\alpha_{m\underset{\theta}{|}j} t^\beta_i + a_{\underline{\alpha\beta}} t^\alpha_m t^\beta_{i\underset{\theta}{|}j} = 0.$$

Adding (25b, c) and subrtacting (25a), we get

$$a_{\underline{\alpha\beta}}(\underline{t^\alpha_m t^\beta_{i\underset{\theta}{|}j}} + t^\alpha_j t^\beta_{m\underset{\theta\vee}{|}i} + t^\alpha_i t^\beta_{m\underset{\theta\vee}{|}j}) = 0.$$

From this and (23)

(26) $$a_{\underline{\alpha\beta}} t^\alpha_i t^\beta_{m\underset{\theta\vee}{|}j} + a_{\underline{\alpha\beta}} t^\alpha_j t^\beta_{m\underset{\theta\vee}{|}i} = 0.$$

On the basc ofd (16) and (24') we have

$$T^\alpha_{i\underset{\theta\vee}{|}m} = \underset{\theta}{\Phi}{}^p_{i\underset{\vee}{m}} t^\alpha_p + \sum_\varrho \underset{\theta}{\Omega}_{(\varrho)i\underset{\vee}{m}} N^\alpha_{(\varrho)} = \underset{\theta}{\Phi}{}^p_{im} t^\alpha_p + \sum_\varrho \underset{\theta}{\Omega}_{(\varrho)i\underset{\vee}{m}} N^\alpha_{(\varrho)},$$

which put into (26), gives

$$a_{\underline{\alpha\beta}} t^\alpha_i \underset{\theta}{\Phi}{}^p_{mj} t^\beta_p + a_{\underline{\alpha\beta}} t^\alpha_j \underset{\theta}{\Phi}{}^p_{mi} t^\beta_p = 0, \quad \text{i.e..}$$

$$\underline{g}_{ip} \underset{\theta}{\Phi}{}^p_{mj} + \underline{g_j}_p \underset{\theta}{\Phi}{}^p_{mi} = 0 \Rightarrow \underset{\theta}{\Phi}_{imj} = -\underset{\theta}{\Phi}_{jmi}.$$

From here and (24'), we have

(27) $$\underset{\theta}{\Phi}_{imj} = -\underset{\theta}{\Phi}_{jmi} = -\underset{\theta}{\Phi}_{ijm} = -\underset{\theta}{\Phi}_{mij}.$$

We see that tensors $\underset{\theta}{\Phi}_{ijm}$ are antisymmetric on all pairs of indices.

2. Second derivational formulas. From (9a-d), for covariant derivation of normals we have

(28a) $$N^\alpha_{(\sigma)\underset{1}{|}m} = N^\alpha_{(\sigma)\underset{3}{|}m} = N^\alpha_{(\sigma)m} + \Gamma^\alpha_{\pi\mu} t^\mu_m n^\pi_{(\sigma)},$$

(28b) $$N^\alpha_{(\sigma)\underset{2}{|}m} = N^\alpha_{(\sigma)\underset{4}{|}m} = N^\alpha_{(\sigma)m} + \Gamma^\alpha_{\mu\pi} t^\mu_m N^\pi_{(\sigma)}.$$

From (8a) we get

$$a_{\underline{\alpha\beta}} N^\alpha_{(\varrho)} N^\beta_{(\varrho)} = e_{(\varrho)} \quad (e_{(\varrho)} = \pm 1),$$

and, based on (14c):

$$a_{\underline{\alpha\beta}} N^\alpha_{(\varrho)\underset{\theta}{|}m} N^\beta_{(\varrho)} + a_{\underline{\alpha\beta}} N^\alpha_{(\varrho)} N^\beta_{(\varrho)\underset{\theta}{|}m} = 0 \Rightarrow a_{\underline{\alpha\beta}} N^\alpha_{(\varrho)\underset{\theta}{|}m} N^\beta_{(\varrho)} = 0, \text{ i.e.}$$

(29) $$N^\alpha_{(\varrho)\underset{\theta}{|}m} \perp N^\alpha_{(\varrho)}.$$

We can express the tensors $N^\alpha_{(\varrho)\underset{\theta}{|}m}$ in the following way:

(30) $$N^\alpha(\sigma \underset{\theta}{|} m = \underset{\theta}{B}{}^p(\sigma)m + \sum_\varrho \Psi_{\varrho\sigma)m} N^\alpha_{(\varrho)}, \quad \underset{\theta}{\Psi}_{(\sigma\sigma)m} = 0,$$

where the condition $\underset{\theta}{\Psi}_{(\sigma\sigma)m} = 0$ follows from (29). From (28a, b) and (30)

(31a, b) $$\underset{1}{B}{}^h_{(\sigma)m} = \underset{3}{B}{}^h_{(\sigma)m}, \ \underset{3}{B}{}^h_{(\sigma)m} = \underset{4}{B}{}^h_{(\sigma)m},$$

(32a, b) $$\underset{1}{\Psi}_{(\varrho\sigma)m} = \underset{3}{\Psi}_{(\varrho\sigma)m}, \underset{2}{\Psi}_{(\varrho\sigma)m} = \underset{4}{\Psi}_{(\varrho\sigma)m}.$$

n order to determine the tensors $= \underset{\theta}{B}{}^h_{(\sigma)m}$, we shall didfferentiate covariantly the relation (8b). Taking into acount (14c), we obtain

$$a_{\underline{\alpha\beta}} N^\alpha_{(\sigma)\underset{\theta}{|}m} t^\beta_j + a_{\underline{\alpha\beta}} N^\alpha_{(\sigma)} t^\alpha_{j\underset{\theta}{|}m} = 0.$$

From here, by a change based on (30), (16) we obtain

$$a_{\underline{\alpha\beta}} t^\beta_j \Big(\underset{\theta}{B}{}^p_{(\sigma)m} t^\alpha_p + \sum_\varrho \underset{\theta}{\Psi}_{(\varrho\sigma)m} N^\alpha_{(\varrho)} \Big) + a_{\underline{\alpha\beta}} N^\alpha_{(\sigma)} \Big(\underset{\theta}{\Phi}{}^p_{jm} t^\beta_p + \sum_\varrho \underset{\theta}{\Omega}_{(\varrho)jm} B^\beta_{(\varrho)} \Big) = 0, \text{ i.e.}$$

(33) $$g_{\underline{jp}} \underset{\theta}{B}{}^p_{(\sigma)m} + e_{(\sigma)} \underset{\theta}{\Omega}_{(\sigma)jm} = 0.$$

Multiplying the previous relation by $g^{\underline{ij}}$, we get

$$\delta^i_p \underset{\theta}{B}{}^p_{(\sigma)m} + e_{(\sigma)} g^{\underline{ij}} \underset{\theta}{\Omega}_{(\sigma)jm} = 0,$$

respectively

(34) $$\underset{\theta}{B}{}^i_{(\sigma)m} = -e_{(\sigma)} g^{\underline{is}} \underset{\theta}{\Omega}_{(\sigma)sm} = -e_{(\sigma)} \underset{\theta}{\Omega}{}^i_{(\sigma)m}.$$

With respect to (31, 34) we conclude

(35a,b) $$\underset{1}{\Omega}{}^i_{(\sigma)m} = \underset{3}{\Omega}{}^i_{(\sigma)m}, \ \underset{2}{\Omega}{}^i_{(\sigma)m} = \underset{4}{\Omega}{}^i_{(\sigma)m}.$$

From (33)

(36) $$\underset{\theta}{\Omega}_{(\sigma)im} = -e_{(\sigma)} g_{\underline{ip}} \underset{\theta}{B}{}^p_{(\sigma)m},$$

and from this and (31a, b) we have

(35'a, b) $$\underset{1}{\Omega}_{(\sigma)im} = \underset{3}{\Omega}^{i}_{(\sigma)im}, \ \underset{2}{\Omega}_{(\sigma)im} = \underset{4}{\Omega}^{i}_{(\sigma)im}.$$

Changing $\underset{\theta}{B}^{p}_{(\sigma)m}$ based on (34) into (30), we obtain

(37) $$N^{\alpha}_{(\sigma)\underset{\theta}{|}m} = -e_{(\sigma)}\underset{\theta}{\Omega}^{p}_{(\sigma)m}t^{\alpha}_{p} + \sum_{p}\underset{\theta}{\Psi}_{(\varrho\sigma)m}N^{\alpha}_{(\varrho)}, \ \underset{\theta}{\Psi}_{(\sigma\sigma)m} = 0, \quad \text{i.e.}$$

(37') $$N^{\alpha}_{(\sigma)\underset{\theta}{|}m} = -e_{(\sigma)}g^{\underline{ps}}\underset{\theta}{\Omega}_{(\sigma)sm}t^{\alpha}_{p} + \sum_{p}\underset{\theta}{\Psi}_{(\varrho\sigma)m}N^{\alpha}_{(\varrho)}, \ \underset{\theta}{\Psi}_{(\sigma\sigma)m} = 0. \ \text{i.e.}$$

From this and (28a, b) we obtain 2 kinds of the second derivational formula of the subspace V_M (for $\theta = 1, 2$).

We shall now investigate some properites of the tensors

$$\underset{\theta}{\Psi}^{h}_{im}, \ \underset{\theta}{\Psi}_{(\sigma)im}, \ \underset{\theta}{\Psi}_{(\varrho\sigma)m}.$$

From (37') and (8a, b) we have

$$a_{\underline{\alpha\beta}}N^{\alpha}_{(\sigma)\underset{\theta}{|}m}N^{\beta}_{(\pi)} = \underset{\theta}{\Psi}_{(\pi\sigma)m}a_{\underline{\alpha\beta}}N^{\alpha}_{(\pi)}N^{\beta}_{(\pi)} = e_{(\pi)}\underset{\theta}{\Psi}_{(\pi\sigma)m},$$

from where it is

(38) $$\underset{\theta}{\Psi}_{(\pi\sigma)m} = e_{(\pi)}a_{\underline{\alpha\beta}}N^{\alpha}_{(pi)}N^{\beta}_{(\sigma)\underset{\theta}{|}m} \quad (e_{(\pi)} = \pm 1).$$

On the other hand, by covariant differentiation of the relation

$$a_{\underline{\alpha\beta}}N^{\alpha}_{(pi)}N^{\beta}_{(\sigma)} = e_{(\pi)}\delta_{\pi\sigma}, \quad \text{we get}$$
$$a_{\underline{\alpha\beta}}\Big(N^{\alpha}_{(\pi)\underset{\theta}{|}m}N^{\beta}_{(\sigma)} + N^{\alpha}_{(\pi)}B^{\beta}_{(\sigma)\underset{\theta}{|}m}\Big) = 0,$$

whence, from (28), we have

(39) $$e_{(\pi)}\underset{\theta}{\Psi}_{(\pi\sigma)m} + e_{(\sigma)}\underset{\theta}{\Psi}_{(\sigma\pi)m} = 0, \quad \text{or}$$

(39') $$\underset{\theta}{\Psi}_{(\pi\sigma)m} = -e_{(\pi)}e_{(\sigma)}\underset{\theta}{\Psi}_{(\sigma\pi)m} = \pm\underset{\theta}{\Psi}_{(\sigma\pi)m}.$$

With regard to (8a), one obtains the sing – in the case when both normals $N^{\alpha}_{(\pi)}$ and $N^{\alpha}_{(\sigma)}$ are either real or imaginary, and the sign + in the case when one of these normals is real and the other is imaginary.

From the definitions (15a-d) it is

(40a, b) $$t^{\alpha}_{i\underset{2}{|}m} = t^{\alpha}_{m\underset{1}{|}i}, \ t^{\alpha}_{i\underset{4}{|}m} = t^{\alpha}_{m\underset{3}{|}i},$$

and from this and (16), 24'), we have

(41a, b) $$\underset{2}{\Phi}^{h}_{im} = \underset{1}{\Phi}^{h}_{mi} = -\underset{1}{\Phi}^{h}_{im}, \ \underset{4}{\Phi}^{h}_{im} = \underset{3}{\Phi}^{h}_{mi} = -\underset{3}{\Phi}^{h}_{im}.$$

From (40a, b), (16) and (35'a, b)

$$\underset{2}{\Omega}_{(\sigma)im}=\underset{1}{\Omega}_{(\sigma)mi}=\underset{3}{\Omega}_{(\sigma)mi},\ \underset{4}{\Omega}_{(\sigma)im}=\underset{3}{\Omega}_{(\sigma)mi}=\underset{1}{\Omega}_{(\sigma)mi},\ \text{i.e.}$$

(42) $$\underset{1}{\Omega}_{(\sigma)im}=\underset{2}{\Omega}_{(\sigma)mi}=\underset{3}{\Omega}_{(\sigma)im}=\underset{4}{\Omega}_{(\sigma)mi}.$$

Further, using (15a-d)

(43a) $$t^\alpha_{i\underset{1}{|}m}=t^\alpha_{i;m}+\Gamma^\alpha_{\underset{\vee}{\pi\mu}}t^\mu_m t^\pi_i-\Gamma^p_{\underset{\vee}{im}}t^\alpha_p,$$

(43b) $$t^\alpha_{i\underset{2}{|}m}=t^\alpha_{i;m}+\Gamma^\alpha_{\underset{\vee}{\mu\pi}}t^\mu_m t^\pi_i-\Gamma^p_{\underset{\vee}{mi}}t^\alpha_p,$$

(43c) $$t^\alpha_{i\underset{3}{|}m}=t^\alpha_{i;m}+\Gamma^\alpha_{\underset{\vee}{\pi\mu}}t^\mu_m t^\pi_i-\Gamma^p_{\underset{\vee}{mi}}t^\alpha_p,$$

(43d) $$t^\alpha_{i\underset{4}{|}m}=t^\alpha_{i;m}+\Gamma^\alpha_{\underset{\vee}{\mu\pi}}t^\mu_m t^\pi_i-\Gamma^p_{\underset{\vee}{im}}t^\alpha_p,$$

From the previous equations we see that

(44) $$\Big(t^\alpha_{i\underset{1}{|}m}+t^\alpha_{i\underset{2}{|}m}\Big)/2=\Big(t^\alpha_{i\underset{3}{|}m}+t^\alpha_{i\underset{4}{|}m}\Big)/2=t^\alpha_{i;m},$$

and, taking into account (16) and (21a), we obtain

$$\underset{1}{\Phi}{}^h_{im}+\underset{2}{\Phi}{}^h_{im}=\underset{3}{\Phi}{}^h_{im}+\underset{4}{\Phi}{}^h_{im}=0,$$

which affirm the equations (41a, b). We have also

(45) $$(\underset{1}{\Omega}_{(\sigma)im}+\underset{2}{\Omega}_{(\sigma)im})/2=(\underset{3}{\Omega}_{(\sigma)im}+\underset{4}{\Omega}_{(\sigma)im})/2=\Omega_{(\sigma)im}.$$

From (22), (45) one affirms (42).

If we make a change in (43a) using (16), (21a), (22), we get

(46) $$\underset{1}{\Phi}{}^p_{im}t^\alpha_p+\sum_p[\underset{1}{\Omega}_{(\varrho)im}-\underset{1}{\Omega}_{(\varrho)im}]N^\alpha_{(\varrho)}=\Gamma^\alpha_{\underset{\vee}{\pi\mu}}t^\pi_i t^\mu_m-\Gamma^p_{\underset{\vee}{im}}t^\alpha_p.$$

Multiplying this equation by $a_{\underline{\alpha\beta}}t^\beta_h$, we have

(47a) $$\underset{1}{\Phi}_{him}=a_{\underline{\alpha\beta}}\Gamma^\alpha_{\underset{\vee}{\pi\mu}}t^\beta_h t^\pi_i t^\mu_m-\Gamma_{h.\underset{\vee}{im}}.$$

In the same way, using (43b - d), we obtain

(47b) $$\underset{2}{\Phi}_{him}=a_{\underline{\alpha\beta}}\Gamma^\alpha_{\underset{\vee}{\mu\pi}}t^\beta_h t^\pi_i t^\mu_m-\Gamma_{h.\underset{\vee}{mi}},.$$

(47c) $$\underset{3}{\Phi}_{him}=a_{\underline{\alpha\beta}}\Gamma^\alpha_{\underset{\vee}{\pi\mu}}t^\beta_h t^\pi_i t^\mu_m-\Gamma_{h.\underset{\vee}{mi}}.$$

(47d) $$\underset{4}{\Phi}_{him}=a_{\underline{\alpha\beta}}\Gamma^\alpha_{\underset{\vee}{\mu\pi}}t^\beta_h t^\pi_i t^\mu_m-\Gamma_{h.\underset{\vee}{im}}.$$

Considering (41a, b) and (47a-d), we conclude that

(48a, b) $$\underset{2}{\Phi}_{him} = -\underset{1}{\Phi}_{him}, \ \underset{3}{\Phi}_{him} = \underset{1}{\Phi}_{him} + 2\Gamma_{h.\underset{\vee}{im}},$$

(48c) $$\underset{4}{\Phi}_{him} = -\underset{1}{\Phi}_{him} - 2\Gamma_{h.\underset{\vee}{im}},$$

respectively

(48'a, b) $$\underset{2}{\Phi}^{h}_{im} = -\underset{1}{\Phi}^{h}_{im}, \ \underset{3}{\Phi}^{h}_{im} = \underset{1}{\Phi}^{h}_{im} + 2\Gamma^{h}_{\underset{\vee}{im}},$$

(48'c.) $$\underset{4}{\Phi}^{h}_{im} = -\underset{1}{\Phi}^{h}_{im} - 2\Gamma^{h}_{\underset{\vee}{im}}.$$

If me multiply equation (46) by $a_{\underline{\alpha\beta}}N^{\beta}_{(\sigma)}$, we obtain

$$e_{(\sigma)}[\underset{1}{\Omega}_{(\sigma}im - \Omega_{(\sigma)im}] = a_{\underline{\alpha\beta}}\Gamma^{\alpha}_{\underset{\vee}{\pi\mu}}N^{\beta}_{(\sigma)}t^{\mu}_{m}t^{\pi}_{i}, \quad \text{i.e.}$$

(49) $$\underset{1}{\Omega}_{(\sigma)\underset{\vee}{im}} = e_{(\sigma)}a_{\underline{\alpha\beta}}\Gamma^{\alpha}_{\underset{\vee}{\pi\mu}}N^{\beta}_{(\sigma)}t^{\mu}_{m}t^{\pi}_{i}.$$

By a direct ckeck based on (43b-d), or using (49), (42), we get

(49') $$\underset{1}{\Omega}_{(\sigma)\underset{\vee}{im}} = \underset{2}{\Omega}_{(\sigma)\underset{\vee}{mi}} = \underset{3}{\Omega}_{(\sigma)\underset{\vee}{im}} = \underset{4}{\Omega}_{(\sigma)\underset{\vee}{mi}} = e_{(\sigma)}a_{\underline{\alpha\beta}}\Gamma^{\alpha}_{\underset{\vee}{\pi\mu}}N^{\beta}_{(\sigma)}t^{\pi}_{i}t^{\mu}_{m}.$$

From this

$$\underset{1}{\Omega}_{(\sigma)im} = \underset{3}{\Omega}_{(\sigma)im} = \underset{1}{\Omega}_{(\sigma)\underline{im}} + \underset{1}{\Omega}_{(\sigma)\underset{\vee}{im}} = \Omega_{(\sigma)im} + \underset{1}{\Omega}_{(\sigma)\underset{\vee}{im}},$$

$$\underset{2}{\Omega}_{(\sigma)im} = \underset{4}{\Omega}_{(\sigma)im} = \underset{1}{\Omega}_{(\sigma)\underline{im}} + \underset{1}{\Omega}_{(\sigma)\underset{\vee}{im}} = \Omega_{(\sigma)im} - \underset{1}{\Omega}_{(\sigma)\underset{\vee}{im}}, \quad \text{i.e.}$$

(50) $$\underset{\theta}{\Omega}_{(\sigma)im} = \Omega_{(\sigma)im} \pm \underset{1}{\Omega}_{(\sigma)\underset{\vee}{im}},$$

where for $\theta = 1, 3$ we take the sign $+$, and for $\theta = 2, 4$ the sign $-$, and $\Omega_{(\sigma)\underset{\vee}{im}}$ is given by (49').

Further, from (28a, b)

(51a) $$N^{\alpha}_{(\sigma)\underset{1}{|}m} = N^{\alpha}_{(\sigma)\underset{3}{|}m} = N^{\alpha}_{(\sigma);m} + \Gamma^{\alpha}_{\underset{\vee}{\pi\mu}}N^{\pi}_{(\sigma)}t^{\mu}_{m},$$

(51b) $$N^{\alpha}_{(\sigma)\underset{2}{|}m} = N^{\alpha}_{(\sigma)\underset{4}{|}m} = N^{\alpha}_{(\sigma);m} + \Gamma^{\alpha}_{\underset{\vee}{\mu\pi}}N^{\pi}_{(\sigma)}t^{\mu}_{m},$$

and this

(52) $$(N^{\alpha}_{(\sigma)\underset{1}{|}m} + N^{\alpha}_{(\sigma)\underset{2}{|}m})/2 = (N^{\alpha}_{(\sigma)\underset{3}{|}m} + N^{\alpha}_{(\sigma)\underset{4}{|}m})/2 = N^{\alpha}_{(\sigma);m}.$$

From here, considering (37') and (21b) we obtain (45) and also

(53) $$(\underset{1}{\Psi}_{(\varrho\sigma)m} + \underset{2}{\Psi}_{(\varrho\sigma)m})/2 = (\underset{3}{\Psi}_{(\varrho\sigma)m} + \underset{4}{\Psi}_{(\varrho\sigma)m})/2 = \Psi_{\varrho\sigma)m}.$$

If we put the covariant derivation into (51a) based on (37') and (21b), we get

$$-e_{(\sigma)}g^{\underline{ps}}\underset{1}{\Omega}_{(\sigma)s\underset{\vee}{m}}t^{\alpha}_{p}+\sum_{p}\Big(\underset{1}{\Psi}_{(\varrho\sigma)m}-\Psi_{(\varrho\sigma)m}\Big)N^{\alpha}_{(\varrho)}=\Gamma^{\alpha}_{\underset{\vee}{\pi\mu}}N^{\pi}_{(\sigma)}t^{\mu}_{m}. \tag{54}$$

If we multiply this equation by $a_{\underline{\alpha\beta}}t^{\beta}_{i}$, we get

$$-e_{(\sigma)}g^{\underline{ps}}g_{\underline{pi}}\underset{1}{\Omega}_{(\sigma)s\underset{\vee}{m}}=a_{\underline{\alpha\beta}}\Gamma^{\alpha}_{\underset{\vee}{\pi\mu}}t^{\beta}_{i}t^{\mu}_{m}N^{\pi}_{(\sigma)},\quad\text{i.e.}$$

$$\underset{1}{\Omega}_{(\sigma)i\underset{\vee}{m}}=e_{(\sigma)}a_{\underline{\alpha\beta}}\Gamma^{\alpha}_{\underset{\vee}{\pi\mu}}t^{\beta}_{i}t^{\mu}_{m}N^{\pi}_{(\sigma)}, \tag{55}$$

which is another form for (49). From here, in accordance with (42), we have

$$\underset{1}{\Omega}_{(\sigma)i\underset{\vee}{m}}=\underset{2}{\Omega}_{(\sigma)\underset{\vee}{m}i}=\underset{3}{\Omega}_{(\sigma)i\underset{\vee}{m}}=\underset{4}{\Omega}_{(\sigma)\underset{\vee}{m}i}=e_{(\sigma)}a_{\underline{\alpha\beta}}\Gamma^{\alpha}_{\underset{\vee}{\mu\pi}}t^{\beta}_{i}\,t\,t^{\mu}_{m}N^{\pi}_{(\varrho)}, \tag{55'}$$

which is another form of (49').

Multiplying the equation (54) by $a_{\underline{\alpha\beta}}N^{\beta}_{(\tau)}$, we get

$$e_{(\tau)}(\underset{1}{\Psi}_{(\tau\sigma)m}-\Psi_{(\tau\sigma)m})=a_{\underline{\alpha\beta}}\Gamma^{\alpha}_{\underset{\vee}{\pi\mu}}N^{\beta}_{(\tau)}N^{\pi}_{(\sigma)}t^{\mu}_{m}, \tag{56}$$

respectively

$$\underset{1}{\Psi}_{(\tau\sigma)m}=\underset{3}{\Psi}_{(\tau\sigma)m}=\Psi_{(\tau\sigma)m}+e_{(\tau)}a_{\underline{\alpha\beta}}\Gamma^{\alpha}_{\underset{\vee}{\pi\mu}}N^{\beta}_{(\tau)}N^{\pi}_{(\sigma)}t^{\mu}_{m}. \tag{56'}$$

Analogously to (56), we get

$$e_{(\tau)}\Big(\underset{2}{\Psi}_{(\tau\sigma)m}-\Psi_{(\tau\sigma)m}\Big)=a_{\underline{\alpha\beta}}\Gamma^{\alpha}_{\underset{\vee}{\pi\mu}}N^{\pi}_{(\sigma)}t^{\mu}_{m},\ \text{ i.e.} \tag{57}$$

$$\underset{2}{\Psi}_{(\tau\sigma)m}=\underset{4}{\Psi}_{(\tau\sigma)m}=\Psi_{(\tau\sigma)m}-e_{(\tau)}a_{\underline{\alpha\beta}}\Gamma^{\alpha}_{\underset{\vee}{\pi\mu}}N^{\beta}_{(\tau)}N^{\pi}_{(\varrho)}t^{\mu}_{m}. \tag{57'}$$

From (56', 57') one gets (53). If we introduce a tensor

$$h_{(\tau\sigma)m}=e_{(\tau)}a_{\underline{\alpha\beta}}\Gamma^{\alpha}_{\underset{\vee}{\pi\mu}}N^{\beta}_{(\tau)}N^{\pi}_{(\sigma)}t^{\mu}_{m}, \tag{58}$$

from (56', 57')

$$\underset{\theta}{\Psi}_{(\tau\sigma)m}=\Psi_{(\tau\sigma)m}\pm h_{(\tau\sigma)m}, \tag{59}$$

where for $\theta=1,3$ we taje the sing $+$ and for $\theta=2,4$ tje sing $-$.

REFERENCES

[1] L. P. Eisenhart, *Generalized Riemannian spaces, I*, Preoc. Nat. Acad. Sci. U. S.A. **37** (1951), 311–315.

[2] L. P. Eisenhart, *Generalized Riemannian spaces, II*, Preoc. Nat. Acad. Sci. U. S.A. **38** (1952), 505–508.

[3] R. S. Mishra, *Subspace of generalized Riemannian space*, Bull. Acad. Roy. Belg. Bull. Cl. Sci. (1954), 1058–1071.

[4] M. Prvanović, *Équations de Gauss d'un sous-espace plongé dans l'espace Riemannian généralizé*, Acad. Roy. Belg. Bull. Cl. Sci. (5) **41** (1955), 615–621.

[5] S. M. Minčić, *Ricci type identities in a subspace of a space of non-symmetric affine connexion*, Publ. Inst. Math. (Beograd) (n.DSD.) **18** (**32**) (1975), 137–148.

[6] S. M. Minčić, *Novye tožderstva tippa Ričči v podprostranstve nesimetričnoj affinoj svjaznosti*, Izv. Vysš. Učebn. Zaved. Matematika, **5** (203) (1979), 17–27.

[7] L. P. Ejzenhart, Rimanova geometrija, Moskva, 1948.

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(Received 23 05 1980)