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MODEL THEORY FOR $\mathbf{L}_{\mathcal{A}M}$ LOGIC

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Abstract. In [**3**] Keisler introduced several probability logics ($L_{\mathcal{A}M}, L(\int)_{w1w}$, etc.) and developed model theory for them together with Hoover. We introduce $L_{\mathcal{A}M}$ which, instead of probability measure, has a σ-finite one and give a method how to transfer results from $L_{\mathcal{A}P}$ to our logic.

Measure logic $L_{\mathcal{A}M}$ is the logic which is like the infinitary logic $L_{\mathcal{A}}$ except that quantifiers $(M\mathbf{x} \geq r)$, $(M\mathbf{x} > r)$, $(M\mathbf{x} \leq r)$ and $(M\mathbf{x} < r)$ ($\mathbf{x}$ is a finite sequence of variables and $r \in [0, \infty)$) are used instead of ordinary quantifiers $(\forall x)$ and $(\exists x)$.

Let $HC_{\mathfrak{m}}$ be the set of hereditarily countable sets over $\mathfrak{M}$. We asume that $\mathcal{A} \subseteq HC_{\mathfrak{m}}$ is an admissible set with urelements so that $\omega \in \mathcal{A}$.

Thus $L_{\mathcal{A}M} = \mathcal{A} \cap L_{\omega_1 M}$, where $L_{\omega_1 M}$ denotes $L_{HC_{\mathfrak{M}} M}$. Similarly $L_{\omega M}$ denotes $L_{HF\mathfrak{M}M}$ in case that the reals are contained in $HF_{\mathfrak{M}}$.

Let L be, throughout the paper, a countable $\mathcal{A}$-recursive set of finitary relation and constant symbols. We suppose that the equality and a countable sequence $(R_n : n \in \omega)$ of unary relational symbols are logical symbols.

Let $(A, \mathcal{B}, \mu)$ be a σ-finite measure space such that singletons are measurable. Then $(A^n, \mathcal{B}^{(n)}, \mu^{(n)})$ is measure space such that $\mathcal{B}^{(n)}$ is the σ-algebra generated by the measurable rectangles and the diagonal sets $\{\mathbf{x} \in A^n : x_k = x_s\}$ and measure $\mu^{(n)}$ is the resticion of the completion of μ^n to $\mathcal{B}^{(n)}$.

Definition 1. A measure model for L is an order pair $(\mathfrak{A}, \mu)$, where $\mathfrak{A}$ is an ordinary model and μ is a σ-finite measure on A such that each singleton is measurable and each relation $R^{\mathfrak{A}} \subseteq A^m$ is $\mu^{(m)}$-measurable.

The sequence $(R_i : i \in \omega)$ is a σ-cover of the set A.

The satisfaction relation $(\mathfrak{A}, \mu) \models [\mathbf{a}]$ for $\varphi(\mathbf{x}) \in L_{\mathcal{A}M}$, measure model $(\mathfrak{A}, \mu)$ and $\mathbf{a} \in A^n$ is defined as usually for propositional connectives. For the quantifier $(M\mathbf{x} > r)$.

$$(\mathfrak{A}, \mu) \models (M\mathbf{x} > r)\varphi(\mathbf{x}, \mathbf{y})\,[\mathbf{a}] \text{ iff } \mu^{(n)}(\{\mathbf{b} \in A^n : (\mathfrak{A}, \mu) \models \varphi[\mathbf{a}, \mathbf{b}]\}) > r$$

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and similarly for other quantifiers.

It can be shown that for each model $(\mathfrak{A}, \mu)$, formula $\varphi(\mathbf{x}, \mathbf{y}) \in L_{\mathcal{A}M}$ and m-tuple $\mathbf{a}$ in A^m, the set $\{\mathbf{b} \in A^m : (\mathfrak{A}, \mu) \models \varphi[\mathbf{a}, \mathbf{b}]\}$ is $\mu^{(n)}$-measurable. So the satisfaction relation is well defined.

Let $L_{\mathcal{A}M}$ be a logic similar to $L_{\mathcal{A}p}$ except that real numbers occurring in quantifiers are from the interval $[0, m]$. Specially, for $m = 1$, $L_{\mathcal{A}1} = L_{\mathcal{A}P}$.

We say that $(\mathfrak{A}, \mu)$ is Γ-elementary equivalent to $(\mathfrak{B}, \lambda)$ and write $(\mathfrak{A}, \mu) \equiv_\Gamma (\mathfrak{B}, \lambda)$ provided that for each $\varphi \in \Gamma \subseteq L_{\omega_1 M}$

$$(\mathfrak{A}, \mu) \models \varphi \text{ iff } (\mathfrak{B}, \lambda) \models \varphi$$

For $\Gamma = L_{\mathcal{A}m}$ $(\Gamma = L_{\mathcal{A}m})$ we write simply $(\mathfrak{A}, \mu) \equiv (\mathfrak{B}, \lambda)((\mathfrak{A}, \mu) \equiv_m (\mathfrak{B}, \lambda))$

By using the following theorem for $L_{\mathcal{A}p}$ logic we will prove similar result for our logic.

THEOREM 1. **[2]** *Let* $\Gamma = \{(M\mathbf{x} \geq r)\psi : \psi$ *is a conjuction of atomic formulas*$\}$. *Then:*

(i) $(\mathfrak{A}, \mu) \equiv_1 (\mathfrak{B}, \lambda)$ *iff* $(\mathfrak{A}, \mu) \equiv_\Gamma (\mathfrak{B}, \lambda)$

(ii) *For each formula* $\varphi \in L_{\omega_1 M}$ *there is a formula* ψ *which is a propositional (i. e. Boolean) combination of formulas from* Γ *such that* $\models \varphi \leftrightarrow \psi$.

Let $R_i^n(\mathbf{x})$ be an abbreviation for the conjuction $R_i(x_{m_1}) \wedge \cdots \wedge R_i(x_{m_n})$.

The following definition is of great importance.

Definition 2. The set of bounded formulas (that is B-formulas) of $L_{\mathcal{A}M}$ is the least set such that:

i) each atomic formula is a B-formula,

ii) a conjuction and a negation of B-formulas is a B-formula,

iii) if φ is a B-formula then $(M\mathbf{x} \leq r)(\varphi(\mathbf{x}) \wedge R_i^n(\mathbf{x}))$ and $(M\mathbf{x} > r)(\varphi(\mathbf{x}) \wedge R_i^n(\mathbf{x}))$ are B-formulas for $r \in R$ and $i \in \omega$.

LEMMA 1. *For each formula* φ *of* $L_{\mathcal{A}M}$ *there is a* B*-formula* τ *of* $L_{\mathcal{A}M}$ *such that* $\models \varphi \leftrightarrow \tau$.

Sketch of the proof. The proof follows by induction on the complexity of formulas.

Let $\psi(\mathbf{x}, \mathbf{y})$ be a B-formula such that $\models \theta(\mathbf{x}, \mathbf{y}) \leftrightarrow \psi(\mathbf{x}, \mathbf{y})$

In the main quantifier step we use the following equivalences:

$(M\mathbf{x} \leq r)\theta(\mathbf{x}, \mathbf{y}) \leftrightarrow \bigwedge_{i \geq 0}(M\mathbf{x} \leq r)(\psi(\mathbf{x}, \mathbf{y}) \wedge R_i^n(\mathbf{x}))$

$(M\mathbf{x} > r)\theta(\mathbf{x}, \mathbf{y}) \leftrightarrow \bigvee_{i \geq 0}(M\mathbf{x} > r)(\psi(\mathbf{x}, \mathbf{y}) \wedge R_i^n(\mathbf{x}))$

$(M\mathbf{x} \geq r)\theta(\mathbf{x}, \mathbf{y}) \leftrightarrow \bigwedge_{n > 0} \bigvee_{i \geq 0}(M\mathbf{x} > r - 1/n)(\psi(\mathbf{x}, \mathbf{y}) \wedge R_i^n(\mathbf{x}))$

$(M\mathbf{x} < r)\theta(\mathbf{x}, \mathbf{y}) \leftrightarrow \bigvee_{n > 0} \bigwedge_{i \geq 0}(M\mathbf{x} \leq r - 1/n)(\psi(\mathbf{x}, \mathbf{y}) \wedge R_i^n(\mathbf{x}))$

for $r \in R$

$(M\mathbf{x} \leq \infty)\theta(\mathbf{x},\mathbf{y}) \leftrightarrow \bigwedge_{i\geq 0} \bigvee_{n>0} (M\mathbf{x} \leq n)(\psi(\mathbf{x},\mathbf{y}) \wedge R_i^n(\mathbf{x}))$

$(M\mathbf{x} > \infty)\theta(\mathbf{x},\mathbf{y}) \leftrightarrow \bigvee_{i\geq 0} \bigwedge_{n>0} (M\mathbf{x} > n)(\psi(\mathbf{x},\mathbf{y}) \wedge R_i^n(\mathbf{x}))$

$(M\mathbf{x} \geq \infty)\theta(\mathbf{x},\mathbf{y}) \leftrightarrow \bigwedge_{n>0} \bigvee_{i\geq 0} (M\mathbf{x} > n)(\psi(\mathbf{x},\mathbf{y}) \wedge R_i^n(\mathbf{x}))$

$(M\mathbf{x} < \infty)\theta(\mathbf{x},\mathbf{y}) \leftrightarrow \bigvee_{n>0} \bigwedge_{i\geq 0} (M\mathbf{x} \leq n)(\psi(\mathbf{x},\mathbf{y}) \wedge R_i^n(\mathbf{x}))$

We say that $\bigwedge_{m\geq 0} \varphi_m$ is a monotone conjuction if $\models \varphi_{m+1} \to \varphi_m$ for each $m \geq 0$, and $\bigvee_{m\geq 0} \varphi_m$ is a monotone disjunction if $\models \varphi_m \vec{\vee} \varphi_{m+1}$ for each $m \geq 0$.

Definition 3. The set monotone bounded formulas (MB-formulas) of $L_{\mathcal{A}M}$ is the least set such that:

i) each B-formula of $L_{\omega M} \cap \mathcal{A}$ is a MB-formula,

ii) a monotone conjunction of MB-formulas is a MB-formula,

iii) a monotone disjunction of MB-formulas is a MB-formula.

Similarly, as in [**3**], we can prove the following lemma.

LEMMA 2. *Each B-formula φ is equivalent to some MB-formula ψ.*

Now we produce a sequence of finite measure models $(\mathfrak{A}(i), \mu_i)_{i\in\omega}$ which approximate the given mesure model $(\mathfrak{A}, \mu)$.

Definition 4. Let $(\mathfrak{A}, \mu)$ be a measure model and $(R_i : i \in \omega)$ a σ-cover of A. For each $i \in \omega$ we define the finite model $(\mathfrak{A}(i), \mu_i)_{i\in\omega}$ such that:

i) $A_i = A \cap R_i^{\mathfrak{A}}$

ii) for each n-placed relation $S : S^{\mathfrak{A}(i)} = S^{\mathfrak{A}} \cap (R_i^{\mathfrak{A}})^n$

iii) for each constant symbol $c^{\mathfrak{A}}$, if $c^{\mathfrak{A}} \in R_i^{\mathfrak{A}}$ then $c^{\mathfrak{A}(i)} = c^{\mathfrak{A}}$

iv) $\mu_i = \mu \restriction \mathrm{dom}(\mu) \cap R_i^{\mathfrak{A}}$

THEOREM 2. *Let $(\mathfrak{A}, \mu)$ and $(\mathfrak{B}, \lambda)$ be measure models of some similarity type and $\Gamma = \{(M\mathbf{x} \geq r)\psi : \psi$ is conjunction of atomic formulas $\}$. Then:*

i) $(\mathfrak{A}, \mu) \equiv (\mathfrak{B}, \lambda)$ *iff* $(\mathfrak{A}, \mu) \equiv_\Gamma (\mathfrak{B}, \lambda)$

ii) *For each formula φ of $L_{\omega_1 M}$ there is a formula ψ which is a Boolean combination of formulas from Γ such that $\models \varphi \leftrightarrow \psi$.*

Sketch of the proof. A consequence of the theorem 1 and $(\mathfrak{A}, \mu) \equiv_\Gamma (\mathfrak{B}, \lambda)$ is $(\mathfrak{A}(i), \mu_i) \equiv_{m_i} \mathfrak{B}(i), \lambda_i)$ for each $i \in \omega$ (Where is $m_i = \mu(A_i) = \lambda(B_i)$).

The proof follows at once from clauses (i) and (ii) of the lemmas above.

What follows is the list of axioms and rules of inference:

1. All axioms of $L_{\omega_1\omega}$ without quantifiers,

2. Axioms about measure quantifiers

$(M\mathbf{x} \geq r)\varphi(\mathbf{x}) \to (M\mathbf{x} \geq s)\varphi(\mathbf{x}),\ s < r;\ (M\mathbf{x} > r)\varphi(\mathbf{x}) \to (M\mathbf{x} \geq r)\varphi(\mathbf{x})$

$(M\mathbf{x} \geq r)\varphi(\mathbf{x}) \to (M\mathbf{y} \geq r)\varphi(\mathbf{y});\ (M\mathbf{x} > r)\varphi(\mathbf{x}) \to (M\mathbf{y} > r)\varphi(\mathbf{y});\ (Mx \geq 0)\varphi(x)$

3. Axioms of finite additivity

$$((M\mathbf{x} \leq r)\varphi(\mathbf{x}) \wedge (M\mathbf{x} \leq s)\varphi(\mathbf{x})) \rightarrow (M\mathbf{x} \leq r+s)(\varphi(\mathbf{x}) \vee \psi(\mathbf{x}))$$

$$(M\mathbf{x} \geq r)\varphi(\mathbf{x}) \wedge (M\mathbf{x} \geq s)\psi(\mathbf{x}) \wedge (M\mathbf{x} \leq 0)(\varphi(\mathbf{x}) \wedge \psi(\mathbf{x})) \rightarrow (M\mathbf{x} \geq r+s) (\varphi(\mathbf{x}) \vee \psi(\mathbf{x}))$$

4. Axioms of continuity

$$\bigwedge_n \bigvee_m (M\mathbf{y} < 1/n)((M\mathbf{x} \geq r - 1/m)\varphi(\mathbf{x},\mathbf{y}) \wedge (M\mathbf{x} < r)\varphi(\mathbf{x},\mathbf{y}) \wedge R_i^s(\mathbf{y}))\ i \in \omega,\ r \in R^+$$

$$\bigwedge_n \bigvee_m (M\mathbf{y} < 1/n)((M\mathbf{x} > r)\varphi(\mathbf{x},\mathbf{y}) \wedge (M\mathbf{x} \leq r + 1/m)\varphi(\mathbf{x},\mathbf{y}) \wedge R_i^s(\mathbf{y}))$$

$$\bigwedge_n \bigvee_{\Phi 0} (M\mathbf{y} < 1/n)(\wedge\Phi_0(\mathbf{y}) \wedge \neg \wedge \Phi(\mathbf{y}) \wedge R_i^s(\mathbf{y})) \quad \Phi_0 \text{ finite}, \ \ \Phi_0 \subseteq \Phi$$

5. Axioms of σ-finitsnes

$$(Mx < \infty)R_i(x) \wedge (Mx > i)R_i(x),\ i \in \omega;\ (Mx \leq 0)(R_i(x) \wedge \neg R_{i+1}(x))$$

6. Fubini axioms

$$(M\mathbf{x} \geq r)\varphi(\mathbf{x}) \leftrightarrow (Mx^\pi \geq r)\varphi(\mathbf{x}),\ (M\mathbf{x} > r)\varphi(\mathbf{x}) \leftrightarrow (Mx^\pi > r)\varphi(\mathbf{x})$$

Where π is a permutation of $\{1, \dots, n\}$.

$$(\bigwedge_{1 \leq i \leq n-1} (M\mathbf{x} \leq r_i)((M\mathbf{y} \geq s_i)\varphi(\mathbf{x},\mathbf{y}) \wedge (M\mathbf{y} < s_{i+1})\varphi(\mathbf{x},\mathbf{y}) \wedge R_j^s(\mathbf{x},\mathbf{y})) \wedge$$
$$(M\mathbf{x} \leq 0)((M\mathbf{y} \geq s_n)\varphi(\mathbf{x},\mathbf{y}) \wedge (M\mathbf{y} \leq \infty)\varphi(\mathbf{x},\mathbf{y}) \wedge R_j^s(\mathbf{x},\mathbf{y}))) \rightarrow$$
$$(M\mathbf{x}\mathbf{y} \leq \sum r_i s_{i+1})(\varphi(\mathbf{x},\mathbf{y}) \wedge R_j^s(\mathbf{x},\mathbf{y}))\ \ j \in \omega$$

$$\bigwedge_{1 \leq i \leq n-1} (M\mathbf{x} \geq r_i)((M\mathbf{y} \geq s_i)\varphi(\mathbf{x},\mathbf{y}) \wedge (M\mathbf{y} < s_{i+1})\varphi(\mathbf{x},\mathbf{y}) \wedge R_j^s(\mathbf{x},\mathbf{y})) \rightarrow$$
$$(M\mathbf{x}\mathbf{y} \geq \sum r_i s_i)(\varphi(\mathbf{x},\mathbf{y}) \wedge R_j^s(\mathbf{x}.\mathbf{y}))$$

Where $0 = s_1 \leq s_2 \leq \ldots \leq s_n$ and s is the length of the sequence $\mathbf{x}, \mathbf{y}$

7. Axiom of product measurability

$$(M\mathbf{x}\mathbf{y}\mathbf{z} \geq s)R_n^k(\mathbf{x},\mathbf{y},\mathbf{z}) \rightarrow ((M\mathbf{x} \geq r)(M\mathbf{y} > 0)(M\mathbf{z} \geq r)(\varphi(\mathbf{x},\mathbf{y}) \leftrightarrow \varphi(\mathbf{y},\mathbf{z})) \wedge R_n^k(\mathbf{x},\mathbf{y},\mathbf{z}))\ \text{ for }\ n > 0,\ \ s^t > r^k (\text{if}\ x = x_1, \dots, x_t)$$

8. Axiom for constants

$\bigvee_{i \in \omega} R_i(d)$ for each constant d

Ruels of inference

9. $\frac{\varphi,\ \varphi\to\psi}{\psi}$ Modus Ponens

10. $\frac{\varphi\to\psi\ \ \psi\in\Psi}{\varphi\Psi}$ Conjuction

11. $\frac{\psi\to\varphi(\mathbf{x})}{\psi\to(M\mathbf{x}\geq\infty)\varphi(\mathbf{x})}\ \frac{\psi\to\varphi(\mathbf{x})}{\psi\to(M\mathbf{x}\leq 0)\neg\varphi(\mathbf{x})}$ Generalization

12. $\frac{\varphi\to(M\mathbf{x}\leq r)(\psi(\mathbf{x})\wedge R_i^n(\mathbf{x}))\ \ i\in\omega}{\varphi\to(M\mathbf{x}\leq r)\psi(\mathbf{x})}$ σ-finiteness

THEOREM 3. (Soundness) *Any set Φ of sentences of $L_{\mathcal{A}M}$ which has a model is consistent.*

Definition 5. A weak measure model for $L_{\mathcal{A}M}$ is a structure

$$(\mathfrak{A},\mu)=\langle A,R_i^{\mathfrak{A}},c_j^{\mathfrak{A}},\mu_n^{\mathfrak{A}}\rangle_{i\in I,j\in J,n\in N}$$

such that each μ_n is a finitely additive σ-finite measure on A^n with each singleton measurable and the set $\{\mathbf{b}\in A^n:(\mathfrak{A},\mu)\models\varphi[\mathbf{a},\mathbf{b}]\}$ is μ_n-measurable for each $\varphi(\mathbf{x},\mathbf{y})\in L_{\mathcal{A}M}$ and $\mathbf{a}$ in A^m.

Of course each measure model is a weak measure model.

Satisfaction is defined in a natural way.

THEOREM 4. (Weak completeness) *Let $\mathcal{A}$ be countable. If a set Φ of sentences is consistent in $L_{\mathcal{A}M}$, then Φ has a weak measure model.*

Sketch of the proof. Let C be a countable set of new constants, and let $S=L\cup C$. Using the notion of consistence property we can extend Φ to a maximal $S_{\mathcal{A}M}$-consistent set Γ_ω of sentences with the following properties:

i) If $\Phi\subseteq\Gamma_\omega$ and $\wedge\Phi\in S_{\mathcal{A}M}$, then $\wedge\Phi\in\Gamma_\omega$

ii) If $(M\mathbf{x}\geq 0)\varphi(\mathbf{x})\in\Gamma_\omega$, then $\varphi(\mathbf{c})\in\Gamma_\omega$ for some $\mathbf{c}\in C^n$

iii) If $(M\mathbf{x}>0)\varphi(\mathbf{x})\in\Gamma_\omega$ then $(M\mathbf{x}>0)(\varphi(\mathbf{x})\wedge R_i^n(\mathbf{x}))\in\Gamma_\omega$ for some $i\in\omega$

Let C^* be the set of all constants.

In the usual way Γ_ω induces a classical model $\mathfrak{A}=\langle A,R_i,c_j\rangle$ for S, where $A=\{c^{\mathfrak{A}}:c\in C^*\}$.

The axioms insure that a finitely aditive σ-finite measure μ_n can be well defined in the following way:

$$\mu_n\{c^{\mathfrak{A}}:\varphi(\mathbf{c},\mathbf{d})\in\Gamma_\omega\}=\sup\{r:(M\mathbf{x}\geq r)\varphi(\mathbf{x},\mathbf{d})\in\Gamma_\omega\}$$

For the weak model $(\mathfrak{A},\mu)=\langle A,R_i^{\mathfrak{A}},c_j^{\mathfrak{A}},\mu_n\rangle_{i\in I,j\in J,n\in N}$ it is easy to show by induction that $(\mathfrak{A},\mu)\models\Gamma_\omega$ and hence $(\mathfrak{A},\mu)\models\Phi$.

Let $(\mathfrak{A},\mu)$ be a model from the last theorem. Similarly as in Definition 4 we interesect the model $(\mathfrak{A},\mu)$ with $R_i^{\mathfrak{A}}$ to get the finite weak model $(\mathfrak{A}(i),(\mu)_i)$.

In the logic $L_{\mathcal{A}\mu_i}(A_i)$ it is possible to prove the following theorem.

THEOREM 5. *For any weak model $(\mathfrak{A}(i),(\mu)_i)$ there is a finite measure model $(\mathfrak{B}(i),\lambda_i)$ such that $(\mathfrak{A}(i),(\mu)_i)\equiv_{\mu_1(A_i)}(\mathfrak{B}(i),\lambda_i)$*

From Theorem 4 and the next one the completeness result follows.

THEOREM 6. *Let $(\mathfrak{A}, \mu)$ be a weak measure model. Then there is a measure model $(\mathfrak{M}, \lambda)$ such that $(\mathfrak{A}, \mu) = \mathfrak{M}, \lambda)$.*

Proof. Let $(\mathfrak{B}(i), \lambda_i)$ $i \in \omega$ be the sequence as in the Theorem 5. Let us define a sequence $(\mathfrak{N}(i), \eta_i)$ $i \in \omega$ in the following way. Let $(\mathfrak{N}(0), \eta_0) = (\mathfrak{B}(0), \lambda_0)$. If we suppose that $(\mathfrak{N}(n), \eta_n)$ is defined let:

1) $N_{n+1} = N_n \cup (B_{n+1} \backslash B_n)$, $B_n = R_n^{\mathfrak{m}}$

2) If S is k-placed relation symbol then

$$S^{\mathfrak{N}(n+1)} = S^{\mathfrak{N}(n)} \cup (S^{\mathfrak{B}(n+1)} \cap (M_{n+1}^k \backslash M_n^k))$$

3) For each $c \in L$, $[c^{\mathfrak{N}(n+1)}$ is constant iff $c^{\mathfrak{N}(n+1)} \in N_n \cup (B_{n+1} \backslash B_n)]$ and [either $c^{\mathfrak{N}(n+1)} = c^{\mathfrak{N}(n)}$ if $c^{\mathfrak{N}(n+1)} \in N_n$ or $c^{\mathfrak{N}(n+1)} = c^{\mathfrak{B}(n+1)}$otherwise].

4. dom $(\eta_{n+1}^{(k)}) = \{U \cup V : U \in$dom$(\eta_n^{(k)}),\ V \in$dom$(\lambda_{n+1}^{(k)}) \cap (B_{n+1} \backslash B_n)\}$ and $\eta_{n+1}^{(k)}(U \cup V) = \eta_n^{(k)} + \lambda_{n+1}^{(k)}(V)$, where $U \in$dom$(\eta_n^{(k)})$ and $V \in$dom$(\lambda_{n+1}^{(k)})$.

By induction on n is easy to show that for each formula without quantifiers

$$\eta_n^{(k)}(\{\mathbf{x} \in N_n^k : (\mathfrak{N}(n), \eta_n) \models \varphi(\mathbf{x})\}) = \lambda_n^{(k)}(\{\mathbf{x} \in M_n^k : (\mathfrak{B}(n),\ \lambda_n) \models \varphi(\mathbf{x})\})$$

Therefore, by Theorem $2(\mathfrak{N}(n),\ \eta_n) \equiv_{n_1(N_n)} (\mathfrak{B}(n)\lambda_n)$.

Let

1) $M = \bigcup_{n \geq 0} N_n$

2) $S = \bigcup_{n \geq 0} S^{\mathfrak{N}(n)}$ for k-placed relation symbol $S \in L$

3) $c^{\mathfrak{M}} = c^{\mathfrak{N}(n)}$ for some $n \geq 0$

4) $\lambda^{(k)} = \sup_n \eta_n^{(k)}$

It is obvious that $(\mathfrak{M}, \lambda)$ is requested model.

The following completeness theorem is an immediate consequence.

THEOREM 7. *A countable set of sentences Φ of L_{AM} has a measure model iff Φ is consistent L_{AM}.*

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