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ON A CLASS OF PROCESSES WITH MULTIPLICITY N=1

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Abstract. Let $x(t) = \int_a^t g(t,u)dz(u)$, $t \in T$, $T = (a,b)$ be the Cramer representation of the stochastic process $x(t)$. We extend a well-known theorem of Cramer concerning sufficient conditions for the process $x(t)$ to have multiplicity $N = 1$, for the case when $x(t)$ satisfies the condition: $g(t,t) = 0$ for all $t \in T$.

' The technique used in this paper is the same as in [**1**] or [**2**]. Let

$$x(t) = \sum_{n=1}^{N} \int_a^t g_n(t,u)dz(u), \quad t \in T = (a,b), \quad u \leq t, \tag{1}$$

be the Cramer representation of the real-valued process $x(t)$.

Let us introduce the following conditions for $g_n(t,u)$ and $z_n(u)$, $n = 1, N$:

(R_1) The functions $g_n(t,u)$ and $\partial g_n(t,u)/\partial t$ are bounded and continuous for $u \leq t$, $u,t \in T$.

(R_2) $g_n(t,t) = 0$ for all $t \in T$.

(R_3) The function $F_n(u) = E(Z_n(u))^2$ is absolutely continuous and not identically constant and $f_n(u) = F_n'(u)$ has at most a finite number of discontinuity points in any finite subinterval of T.

THEOREM. *The process $x(t)$ given by the expression* (1) *which satisfies the conditions R_1, R_2, R_3 has multiplicity $N > 1$.*

Proof. Let us suppose that $N > 1$. For example, let N be two, $n = 1, 2$. From the condition R_3 there exists a finite subinterval $T_1 \subset T$, $T_1 = (a_1, b_1)$ such that the derivatives $f_1(u)$ and $f_2(u)$ are continuous and not equal to zero. Let t

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be any point in T_1 and let $h(u)$ be a function in $L^2(dF_1 \times dF_2)$ such that the next relation is valid:

$$\sum_{n=1}^{2} \int_{a_1}^{s} h_n(u) g_n(s,u) f_n(u) du = 0, \quad s \leq t, t \in T_1.$$

By the condition R_1 the relation may be differentiated with respect to s, and we obtain:

$$\sum_{n=1}^{2} h_n(s) g_n(s,s) f_n(s) + \sum_{n=1}^{2} \int_{a_1}^{s} \frac{\partial g_n(s,u)}{\partial s} h_n(u) f_n(u) du = 0$$

for $s \in (a_1, t]$. By the condition R_2 we obtain:

$$\sum_{n=1}^{2} \int_{a_1}^{s} \frac{\partial g_n(s,u)}{\partial s} h_n(u) f_n(u) du = 0$$

for $s \in (a_1, t]$. This equation is satisfied if for example:

$$(2) \qquad \int_{a_1}^{s} \frac{\partial g_1(s,u)}{\partial s} h_1(u) f_1(u) du = 1, \qquad \int_{a_1}^{s} \frac{\partial g_2(s,u)}{\partial s} h_2(u) f_2(u) du = -1$$

These are the nonhomogeneous integral equations of Volterra of the first type whose (uniquely) determined solutions $f_1(s)h_1(s)$, $f_2(s)h_2(s)$ are bounded and continuous for $s \in (a_1, t]$, and not almost everywhere equal to zero. By our hypothesis about $f_1(u)$ and $f_2(u)$ on T_1, it follows that:

$$(3) \qquad \int_{a_1}^{s} h_1^2(u) dF_1(u) + \int_{a_1}^{s} h_2^2(u) dF_2(u) > 0, \quad t \in T_1.$$

That means that $g(t,u) = (g_1(t,u), g_2(t,u))$ is not complete in $L^2(dF_1 \times dF_2)$, that is $N \neq 2$.

When we suppose that N is equal to any natural number $N > 2$, the conclusion is the same as before. The proof is completed.

The analogous statement is proved in [**1**] for the process $x(t)$ represented by (1) and satisfying the conditions R_1, R_3 and:

$$(R_2') \;\; g_n(t,t) = 1 \;\text{ for all }\; t \in T, n = 1, N,$$

instead of the condition R_2.

Let $g_n(t,t) > 0$ for all $t \in T$, $n = 1$, N, be the condition R_2''.

The conditions R_2' and R_2'' are equivalent. It is easy to see that implys R_2''. Since R_2'' holds then we may introdixce the transformations:

$$\overline{g}_n" = g_n(t,u)/g_n(u,u), \quad \overline{d}z_n(u) = g_n(u,u) \cdot dz_n(u),$$

$u \leq t$, $u, t \in T$, $n = 1, N$, such that R_2' holds for the process $\overline{x}_n(t)$:

$$\overline{x}_n(t) = \int_a^t \overline{g}_n(t,u)\overline{d}z_n(u).$$

Hence we obtain:

THEOREM. *The process $x(t)$ represented by* (1) *and satisfying the conditions R_1, R_3 and*

$$g_n(t,t) > 0 \ \textit{for} \ t \in T, n = 1, N,$$

has multiplicity $N = 1$.

Proof. Let us suppose as before that $N = 2$. By the condition R_1 the function $g_n(t,t)$, $t \in T$ may be equal to zero in a finite number of isolated points or on a finite number of subintervals whose union is the set $A_n \subset T$ of positive measures $(n = 1, 2)$. We choose the subinterval T_1 as before, but we suppose moreover that T_1 does not contain the isolated zeros of the function $g_n(t,t)$, $t \in T$, $n = 1, 2$, and that T_1 is contained in one of the following sets: $A_1 \cap A_2$, $A_1 \cap (T \setminus A_2)$, $A_2 \cap (T \setminus A_1)$, $(T \setminus A_1) \cap (T \setminus A_2)$. For instance, let us take $T_1 subset A_1 \cap (T \setminus A_2)$. Then (2) becomes:

$$\int_{a_1}^{s} \frac{\partial g_1(s,u)}{\partial s} h_1(u) f_1(u) du = 1, \quad h_2(s) f_2(s) + \int_{a_1}^{s} \frac{\partial g_2(s,u)}{\partial s} h_2(u) f_2(u) du = -1,$$

$s \in (a_1, t]$, $t \in T_1$, where the first integral equation of Volterra is the first type and the second one is the second type and its solutions are such that (3) is valid. We can show in a similar way that (3) holds when T_1 is contained in one of the remaining three sets. This ends the proof.

Remark. It is not necessary that the process $x(t)$ be represented by (1). Namely $x(t)$ may be given by

$$x(t) = \int_a^t g(t,u) dz(u), \tag{4}$$

$t \in T = (a, b)$, $u \leq t$, where the nonrandom function $g(t,u) \in L^2(dF)$ and $z(u)$ is the mutually orthognal process of orthogonal increments with $Edz(u) = 0$ and $E(dz(u))^2 = dF(u)$. By the analogous statement in [**1**], such a process $x(t)$ satisfying the conditions R_1, R_2, R_3 has multiplicity $N = 1$ and (4) is its Cramer representation.

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