

HIGHER-LEVEL SEQUENT-SYSTEMS FOR INTUITIONISTIC MODAL LOGIC

Kosta Došen

Abstract. This paper presents higher-level sequent-systems for intuitionistic analogues of $S5$ and $S4$. As in [3] rules for modal constants involve sequents of level 2, i.e. sequents having collections of ordinary sequents of level 1 on the left and right of the turnstile. Starting from a canonical higher-level sequent formulation of $S5$, the restriction of sequents of level 2 to those with the single-conclusion property produces $S4$, without changing anything else. A similar restriction on sequents of level 1 produces Heyting $S5$, and if this restriction is made on sequents of both level 1 and 2, we obtain Heyting $S4$. The paper contains a brief discussion of Kripke-style models for the intuitionistic propositional modal logics in question.

In [3] we have presented sequent formulations of the modal logics $S5$ and $S4$ based on sequents of higher levels: sequents of level 1 are like ordinary sequents, sequents of level 2 have collections of sequents of level 1 on the left and right of the turnstile, etc. The rules we gave for modal constants involved sequents of level 2, whereas rules for other customary logical constants of first-order logic involved only sequents of level 1. In this paper we shall show how starting from the same sequent rules of higher level we can obtain sequent formulations of intuitionistic analogues of $S5$ and $S4$ based on the necessity operator \Box .

We say that a sequent has the single-conclusion property if its right-hand side is either a singleton or empty. It is well-known from [7] that it is possible to obtain a sequent formulation of Heyting's logic out of a sequent formulation of classical logic just by restricting sequents of level 1 to those with the single-conclusion property, without changing anything else. In [3] we have shown that the same restriction applied to sequents of level 2 can produce a sequent formulation of $S4$ out of a sequent formulation of $S5$, without changing anything else. Here we shall show that starting from the same sequent rules with sequents of level 1 restricted to single-conclusion sequents, if we make no restriction upon sequents of level 2, we obtain a sequent formulation of an intuitionistic analogue of $S5$, which we shall call $S5_H$. If we also restrict sequents of level 2 to single-conclusion sequents,

without changing anything else, we obtain a sequent formulation of an intuitionistic analogue of $S4$, which we shall call $S4_H$. This is shown succinctly in the following table:

	$S5$	$S4$	$S5_H$	$S4_H$
level 1			restricted	restricted
level 2		restricted		restricted

To simplify matters we shall concentrate in this paper on propositional logic. Only in the final, fifth section we shall indicate briefly how to obtain sequent-systems for first-order predicate logics corresponding to $S5_H$ and $S4_H$. In the first section we shall present our sequent-systems, and in the second section we shall show to what Hilbert-style systems these sequent-systems correspond. In the third section we shall briefly consider Kripke-style models for our intuitionistic propositional modal logics. In the fourth section we shall consider alternative bases, with strict implication \prec and the possibility operator \diamond , for our intuitionistic modal logics. This paper is a companion to [3], [1] and [4], and we assume the reader is acquainted with these papers. However, to make this paper more self-contained we shall briefly recapitulate some basic notions.

1. Sequent-systems. Let O be the language of modal propositional logic based on the connectives $\rightarrow, \wedge, \vee, \perp, \top$ and \Box . We use A, B, \dots, A_1, \dots as schematic letters for formulae of O . We define $\neg A$ as $A \rightarrow \perp$, and $A \leftrightarrow B$ as $(A \rightarrow B) \wedge (B \rightarrow A)$. Parentheses are omitted with the assumption that \rightarrow and \leftrightarrow bind more strongly than \wedge and \vee .

Starting from O we build the language D as follows. The formulae of O are formulae of level 0. The empty set sign \emptyset is a set term of any level ≥ 1 . Let A_1^n, \dots, A_k^n , where $k \geq 1$, be distinct formulae of level n ; then $\{A_1^n, \dots, A_k^n\}$ is a set term of level $n + 1$. If Γ and Δ are set terms of level n , then $\Gamma \vdash^n \Delta$ is a formula of level n , called a sequent. The set term $\Gamma \cup \Delta$ stands for the union of the sets Γ and Δ , and the set term $\Gamma + \Delta$ for the disjoint union of Γ and Δ (for details see [3, §1]).

A sequent $\Gamma \vdash^n \Delta$ has the *single-conclusion* property iff Δ is either a singleton or \emptyset . A rule is of level n iff the highest level of formulae occurring in it is n . A rule is *level preserving* iff all formulae occurring in it are of the same level. The *horizontalization* of a level-preserving rule

$$\frac{\Psi}{A^n}$$

is the sequent $\Gamma \vdash^{n+1} \{A^n\}$ where Γ is obtained from the set of premises by omitting repetitions of formulae.

For our sequent-systems we shall assume the following *structural* rules:

$$\begin{array}{l}
\textit{Ascending (A)} \quad \frac{A^n}{\emptyset \vdash^{n+1} \{A^n\}} \\
\textit{Descending (D)} \quad \frac{\emptyset \vdash^{n+1} \{A^n\}}{A^n} \\
\textit{Iteration (I)} \quad \frac{A^n}{A^n} \\
\textit{Cut (C)} \quad \frac{\Gamma_1 \vdash^{n+1} \Delta_1 + \{A^n\} \quad \Gamma_2 + \{A^n\} \vdash^{n+1} \Delta_2}{\Gamma_1 \cup \Gamma_2 \vdash^{n+1} \Delta_1 \cup \Delta_2} \\
\textit{Thinning (T)} \quad \frac{\Gamma \vdash^{n+1} \Delta}{\Gamma \cup \Gamma_1 \vdash^{n+1} \Delta \cup \Delta_1}
\end{array}$$

In all these rules we have $n \geq 0$. An instance of a rule R of level n will be denoted by R^n .

Let the *double-line* rule

$$\frac{B_1^n \dots B_k^n}{A^n}$$

be an abbreviation for the following list of rules:

$$\frac{B_1^n \dots B_k^n}{A^n}, \frac{A^n}{B_1^n}, \dots, \frac{A^n}{b_k^n}.$$

If R is the name of this double-line rule, $R \downarrow$ is the name of the first rule in the list, and $R \uparrow$ designates any of the remaining rules in the list. For the connectives of \mathcal{O} we shall give the following double-line rules:

$$\begin{array}{l}
(\rightarrow) \quad \frac{\Gamma + \{A\} \vdash^1 \Delta + \{B\}}{\Gamma \vdash^1 \Delta + \{A \rightarrow B\}} \\
(\wedge) \quad \frac{\Gamma \vdash^1 \Delta + \{A\} \quad \Gamma \vdash^1 \Delta + \{B\}}{\Gamma \vdash^1 \Delta + \{A \wedge B\}} \\
(\vee) \quad \frac{\Gamma + \{A\} \vdash^1 \Delta \quad \Gamma + \{B\} \vdash^1 \Delta}{\Gamma + \{A \vee B\} \vdash^1 \Delta} \\
(\perp) \quad \frac{\Gamma \vdash^1 \emptyset}{\Gamma \vdash^1 \{\perp\}} \\
(\top) \quad \frac{\emptyset \vdash^1 \Delta}{\{\top\} \vdash^1 \Delta} \\
(\Box) \quad \frac{\Pi + \{\emptyset \vdash^1 \{A\}\} \vdash^2 \Sigma + \{\Gamma \vdash^1 \Delta\}}{\Pi \vdash^2 \Sigma + \{\Gamma + \{\Box A\} \vdash^1 \Delta\}}
\end{array}$$

Axioms for our sequent-systems will be systematically generated from the rules by assuming the horizontalizations of all the level-preserving rules, i.e. all the rules mentioned above except **A** and **D**.

The sequent-system $DS5$ is obtained from the rules above, and the corresponding horizontalizations, in the full language D . If we assume these rules and horizontalizations in a restricted version of D where sequents of level 2 must have the single-conclusion property, everything else remaining unchanged, we obtain the sequent-system $DS4$. The sequent-system $DS5_H$ is obtained by an analogous restriction for sequents of level 1 instead of sequents of level 2, and for $DS4_H$ we make this restriction for sequents of both level 1 and level 2. According to Lemma 4 of [3] sequents of all levels higher than 2 are irrelevant for $DS5_H$ and $DS4_H$, as well as for $DS5$ and $DS4$.

2. Hilbert-style systems. Let H be the Heyting propositional calculus axiomatized by the following rule and axiom-schemata:

1. $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$, 3. $A \rightarrow (B \rightarrow A)$,
4. $(C \rightarrow A) \rightarrow ((C \rightarrow B) \rightarrow (C \rightarrow A \wedge B))$, 5. $A \wedge B \rightarrow A$,
6. $A \wedge B \rightarrow B$,
7. $A \rightarrow A \vee B$, 8. $B \rightarrow A \vee B$, 9. $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C))$
10. $\perp \rightarrow A$, 11. \top .

The classical propositional calculus C is obtained from H by adding the axiom-schema:

$$12. A \vee (A \rightarrow B).$$

The system $S4_H$ is obtained from H by adding the following rule and axiom-schemata:

- \square 1. $\frac{A}{\square A}$,
- \square 2. $\square(A \rightarrow B) \rightarrow (\square A \rightarrow \square B)$, \square 3. $\square A \rightarrow A$, \square 4. $\square A \rightarrow \square \square A$.

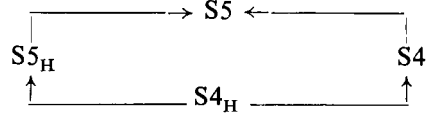
The system $S5_H$ is obtained from $S4_H$ by adding the axiom-schema:

$$\square 5. \square A \vee \square(\square A \rightarrow B).$$

The systems $S4$ and $S5$ are extensions of C obtained from $S4_H$ and $S5_H$ respectively by adding the axiom-schema 12.

The system $S4_H$ is what everybody would take as the natural intuitionistic analogue of $S4$. This system has been considered in a number of works (see [4] and [1] for references). On the other hand, there are various nonequivalent systems which may be considered as intuitionistic analogues of $S5$. The system $S5_H$, axiomatized equivalently with $\square A \vee \square \neg \square A$ rather than our $\square 5$, has been considered in [2], [9], [10], [5], [6] and [4] (in these papers one can find also a number of intuitionistic analogues of $S5$).

From results concerning Kripke-style models of Section 3, it follows easily that the containment relations between our modal systems, as shown in the following chart:



are proper. With these models it is also easy to show that $S4_H$ and $S5_H$ are conservative extensions of the nonmodal system H . The systems $S4$ and $S5$ are, of course, decidable, and the same holds for $S4_H$ and $S5_H$ (see [9] and [10]). It is easy to show that $S4_H$ has the disjunction property, whereas it is clear that $S5_H$ lacks this property (see [4]). So, $S5_H$ may perhaps be considered “intuitionistically spurious”.

It follows from the results of [3] that a formula A of O is provable in $DS4$ (respectively $DS5$) iff A is provable in $S4$ (respectively $S5$). Here we shall demonstrate the following analogous theorem:

THEOREM 1.1. *A formula A of O is provable in $DS4_H$ iff A is provable in $S4_H$.*

1.2. *A formula A of O is provable in $DS5_H$ iff A is provable in $S5_H$.*

This theorem will follow from the following lemmata:

LEMMA 1.1. *The rules of $S4_H$ are derivable and the axioms of $S4_H$ are provable in $DS4_H$.*

1.2. *The axiom-schema $\Box 5$ is provable in $DS5_H$.*

Proof: The proof of 1.1 is exactly as in [3] (see Lemma 7.1). For 1.2 we have

$$(a) \quad \{\emptyset \vdash^1 \{A\}\} \vdash^2 \{\emptyset \vdash^1 \{\Box A\}\}$$

which is proved as in [3, p. 159]. Then we proceed as follows:

$$(\Box) \downarrow \frac{T^2 \frac{\{\emptyset \vdash^1 \{A\}\} \vdash^2 \{\emptyset \vdash^1 \{\Box A\}\}}{\{\emptyset \vdash^1 \{A\}\} \vdash^2 \{\emptyset \vdash^1 \{\Box A\}, \emptyset \vdash^1 \{B\}\}}}{\emptyset \vdash^2 \{\emptyset \vdash^1 \{\Box A\}, \{\Box A\} \vdash^1 \{B\}\}}$$

which by applying the horizontalization of $(\rightarrow) \downarrow$, an instance of (a), and C^2 yields $\emptyset \vdash^2 \{\emptyset \vdash^1 \{\Box A\}, \emptyset \vdash^1 \{\Box(\Box A \rightarrow B)\}\}$. From that with $(\vee) \uparrow$, C^2 , D^2 and D^1 we obtain $\Box A \vee \Box(\Box A \rightarrow B)$. q.e.d.

For the following translation from D into O :

$o(A)$ is A ,

$$\hat{o}(\Gamma) \text{ is } \begin{cases} o(A_1^n) \wedge \cdots \wedge o(A_k^n) & \text{if } \Gamma = \{A_1^n, \dots, A_k^n\}, \quad k \geq 1 \\ \top & \text{if } \Gamma = \emptyset, \end{cases}$$

$$\check{o}(\Gamma) \text{ is } \begin{cases} o(A_1^n) \vee \cdots \vee o(A_k^n) & \text{if } \Gamma = \{A_1^n, \dots, A_k^n\}, \quad k \geq 1 \\ \perp & \text{if } \Gamma = \emptyset, \end{cases}$$

$$o(\Gamma \vdash^{n+1} \Delta) \text{ is } \Box(\hat{o}(\Gamma) \rightarrow \check{o}(\Delta))$$

we have:

LEMMA 2.1. *If A^n is provable in $DS4_H$, then $o(A^n)$ is provable in $S4_H$.*

2.2. *If A^n is provable in $DS5_H$, then $o(A^n)$ is provable in $S5_H$.*

Proof. 2.1. By induction on the length of proof of A^n in $DS4_H$. For **A** and **D** we use:

$$\frac{A}{\frac{}{\Box(\top \rightarrow A)}}.$$

For **I** we use $\Box(A \rightarrow A)$. For **C** we use $\Box(G_1 \rightarrow D_1 \vee A) \wedge \Box(G_2 \vee A \rightarrow D_2) \rightarrow \Box(G_1 \wedge G_2 \rightarrow D_1 \vee D_2)$. For **T** we use $\Box(G \rightarrow D) \rightarrow \Box(G \wedge G_1 \rightarrow D \vee D_1)$. For (\rightarrow) we use $\Box(G \wedge A \rightarrow B) \leftrightarrow \Box(G \rightarrow (A \rightarrow B))$. For (\wedge) we use $\Box(G \rightarrow A) \wedge \Box(G \rightarrow B) \leftrightarrow \Box(G \rightarrow A \wedge B)$. For (\vee) we use $\Box(G \wedge A \rightarrow D) \wedge \Box(G \wedge B \rightarrow D) \leftrightarrow \Box(G \wedge (A \vee B) \rightarrow D)$. For (\perp) and (\top) we use $A \rightarrow A$. Finally, for (\Box) we use:

$$\Box(\Box P \wedge \Box A \rightarrow \Box(G \rightarrow D)) \leftrightarrow \Box(\Box P \rightarrow \Box(G \wedge \Box A \rightarrow D)).$$

To prove this last formula in $S4_H$ from left to right we proceed as follows:

$$\begin{array}{c} \Box(\Box P \wedge \Box A \rightarrow \Box(G \rightarrow D)) \\ \hline \Box(\Box P \wedge \Box A \rightarrow (G \rightarrow D)) \\ \hline \Box(\Box P \rightarrow (G \wedge \Box A \rightarrow D)) \\ \hline \Box\Box(\Box P \rightarrow (G \wedge \Box A \rightarrow D)) \\ \hline \Box(\Box P \rightarrow \Box(G \wedge \Box A \rightarrow D)) \end{array}$$

The converse is obtained analogously.

2.2. We proceed as for 2.1. The only additional case is for (\Box) , where we use:

$$\Box(\Box P \wedge \Box A \rightarrow \Box C \vee (G \rightarrow D)) \leftrightarrow \Box(\Box P \rightarrow \Box C \vee \Box(G \wedge \Box A \rightarrow D)).$$

To prove this formula in $S5_H$ from left to right we show first that $\Box(\Box A \vee B) \rightarrow \Box A \vee \Box B$ is provable in $S5_H$. We have:

$$\begin{array}{l} \Box(\Box A \rightarrow B) \rightarrow (\Box(\Box A \vee B) \rightarrow \Box A \vee \Box B) \text{ and} \\ \Box A \rightarrow (\Box(\Box A \vee B) \rightarrow \Box A \vee \Box B) \end{array}$$

and we apply $\Box 5$. Then we proceed as follows:

$$\begin{array}{c} \frac{\Box(\Box P \wedge \Box A \rightarrow \Box C \vee \Box(G \rightarrow D))}{\Box(\Box A \rightarrow (\Box P \rightarrow \Box C \vee \Box(G \rightarrow D)))} \qquad \frac{\Box(\Box A \vee \Box(\Box A \rightarrow \Box(G \rightarrow D)))}{\Box(\Box A \vee (\Box A \rightarrow \Box(G \rightarrow D)))} \\ \hline \frac{\Box((\Box P \rightarrow \Box C \vee \Box(G \rightarrow D)) \vee (\Box A \rightarrow \Box(G \rightarrow D)))}{\Box(\Box P \rightarrow \Box C \vee (\Box A \rightarrow \Box(G \rightarrow D)))} \\ \hline \frac{\Box(\Box P \rightarrow \Box C \vee (G \wedge \Box A \rightarrow D))}{\Box\Box(\Box P \rightarrow \Box C \vee (G \wedge \Box A \rightarrow D))} \\ \hline \Box(\Box P \rightarrow \Box C \vee \Box(G \wedge \Box A \rightarrow D)) \end{array} .$$

For the last step we use $\Box(\Box C \vee (G \wedge \Box A \rightarrow D)) \rightarrow \Box C \vee \Box(G \wedge \Box A \rightarrow D)$. This proves our formula from left to right. The converse is quite straightforward. q.e.d.

From Lemmata 1.1, 1.2, 2.1 and 2.2 we easily infer Theorem 1.1 and 1.2.

3. Kripke-style models. For $S4_H$ and $S5_H$ we shall now give Kripke-style models based on two accessibility relations, one intuitionistic and the other modal. These models are dealt with extensively in [1] and [4]. We shall first recapitulate briefly the basic notions we need.

An $H\Box$ frame is $\langle X, R_I, R_M \rangle$ where $X \neq \emptyset$, $R_I \subseteq X^2$ is reflexive and transitive, $R_M \subseteq X^2$, and $R_I R_M \subseteq R_M R_I$ ($R_1 R_2$ is short for $R_1 \circ R_2$). The variables $u, v, w, u_1 \dots$ range over X . An $H\Box$ model is $\langle X, R_I, R_M, V \rangle$ where $\langle X, R_I, R_M \rangle$ is an $H\Box$ frame and the valuation V is a mapping from the set of propositional variables of O to the power set of X such that the following Heredity Condition is satisfied for every propositional variable p :

$$\forall u_1, u_2 (u_1 R_I u_2 \Rightarrow (u_1 \in V(p) \Rightarrow u_2 \in V(p))).$$

The relation \models in $u \models A$ is defined as usual, except that for \rightarrow it involves R_I as in intuitionistic Kripke models, whereas for \Box it involves R_M as in modal Kripke models. A formula A holds in an $H\Box$ model iff $(\forall u \in X) u \models A$; and A holds in a frame Fr (i.e. $Fr \models A$) iff A holds in every model with this frame. An $H\Box$ frame is condensed iff $R_I R_M = R_M$, and it is strictly condensed iff $R_I R_M = R_M R_I = R_M$. We use R_\Box as an abbreviation for $R_M R_I$.

In [1] and [4] one can find a proof of the following statements, for every $H\Box$ frame Fr :

all theorems of H plus $\Box 1$ and $\Box 2$ hold in Fr ,

$Fr \models \Box A \rightarrow A$ iff R_\Box is reflexive,

$Fr \models \Box A \rightarrow \Box \Box A$ iff R_\Box is transitive.

Here we shall prove the following lemma:

LEMMA 3. *For every $H\Box$ frame Fr we have $Fr \models \Box A \vee \Box(\Box A \rightarrow B)$ iff $R_\Box^{-1} R_\Box \subseteq R_\Box$ (i.e. R_\Box is Euclidean).*

Proof. (\Rightarrow) Suppose *not* $R_\Box^{-1} R_\Box \subseteq R_\Box$, i.e. for some v, v_1 and v_2 we have $v R_\Box v_1$ and $v R_\Box v_2$ and *not* $v_1 R_\Box v_2$. Next let $\forall u (u \models p \Leftrightarrow v_1 R_\Box u)$ and $\forall u$ *not* $u \models q$. It is easy to show that there is an $H\Box$ model such that this is satisfied. In this model *not* $v_2 \models p$, and hence *not* $v \models \Box p$. Also, $v_1 \models \Box p$ and *not* $v_1 \models q$. Hence, *not* $v \models \Box(\Box p \rightarrow q)$, and so we obtain *not* $v \models \Box p \vee \Box(\Box p \rightarrow q)$.

(\Leftarrow) Suppose *not* $Fr \models \Box A \vee \Box(\Box A \rightarrow B)$. So, for some v we have *not* $v \models \Box A$ and *not* $v \models \Box(\Box A \rightarrow B)$. Hence, there is a v_2 such that $v R_\Box v_2$ and *not* $v_2 \models A$, and there is a v_1 such that $v R_\Box v_1$ and $v_1 \models \Box A$ and *not* $v_1 \models B$. But from the euclideanity of R_\Box it follows that $v_1 R_\Box v_2$, and since $v_1 \models \Box A$ and *not* $v_2 \models A$ we obtain a contradiction. q.e.d.

In [4] one can find a proof of the following statement: A is provable in $S4_H$ iff A holds in every (condensed, strictly condensed) $H\Box$ frame where R_\Box is reflexive and transitive. Analogously, we can establish that A is provable in $S5_H$ iff A holds

in every condensed, strictly condensed) $H\Box$ frame where R_\Box is reflexive, transitive and Euclidean.

Of course, R_\Box is reflexive, transitive and Euclidean iff R_\Box is reflexive, transitive and symmetric. Since $Fr \models A \vee \Box(\Box A \rightarrow B)$ iff R_\Box is symmetric, the axiom-schema $A \vee \Box(\Box A \rightarrow B)$ can replace $\Box 5$ in $S5_H$. As $\Box 5$, the axiom-schema $\Box A \vee \Box\neg\Box A$ corresponds to the euclideanity of R_\Box , and similarly, $A \vee \Box\neg\Box A$ corresponds to the symmetry of R_\Box .

4. Alternative bases. As stated in [3], if the strict implication $A \prec B$ is defined as $\Box(A \rightarrow B)$, in $DS4_H$ and in all stronger sequent-systems we can derive the rules:

$$(\prec) \frac{\Pi + \{\{A\}\} \vdash^1 \{B\} \vdash^2 \Sigma + \{\Gamma \vdash^1 \Delta\}}{\Pi \vdash^1 \Sigma + \{\Gamma + \{A \prec B\} \vdash^1 \Delta\}}$$

and their horizontalizations. Alternatively, we could base O and our modal systems in D on the connective \prec and the rules (\prec) . Then by defining $\Box A$ as $\top \prec A$ we can derive the rules (\Box) and their horizontalizations.

If the possibility operator is defined by $\Diamond A =_{\text{df}} \neg\Box\neg A$, then in $DS5_H$ and $DS4_H$ we can derive the rules:

$$(\Diamond) \frac{\Pi + \{\{A\} \vdash^1 \emptyset\} \vdash^2 \Sigma + \{\Gamma \vdash^1 \Delta\}}{\Pi \vdash^2 \Sigma + \{\Gamma \vdash^1 \Delta + \{\Diamond A\}\}}$$

and their horizontalizations. These rules are now available only with Δ empty, whereas with $DS5$ and $DS4$ it need not be empty. However, \Diamond and the double-line rule (\Diamond) cannot serve as an alternative basis for $DS5_H$ and $DS4_H$. Take the systems exactly like $DS5_H$ and $DS4_H$ save that \Box is replaced by \Diamond , (\Box) is replaced by (\Diamond) , and $\Box A$ is defined as $\neg\Diamond\neg A$. In these systems we cannot prove $\Box A \rightarrow A$. Otherwise, in the system $S5_H$ we could prove $\neg\neg\Box\neg\neg A \rightarrow A$, and it is easy to construct an $H\Box$ model with R_\Box reflexive, transitive and Euclidean such that $\neg\neg\Box\neg\neg A \rightarrow A$ is falsified in this model (cf. [6], [8]).

5. First-order modal logic. It is not difficult to find along the lines of [3], higher-level sequent formulations of first-order predicate logics corresponding to $S5_H$ and $S4_H$. If O is now the language of the first-order predicate calculus with the constants $\rightarrow, \wedge, \vee, \perp, \top, \forall, \exists$ (and eventually $=$), we can proceed exactly as in [3] to obtain first-order sequent-systems corresponding to $DS5_H$ and $DS4_H$. The additional rules are Substitution for individual variables and two double-line rules for quantifiers:

$$\left. \begin{array}{l} (\forall) \frac{\Gamma \vdash^1 \Delta + \{A(x)\}}{\Gamma \vdash^1 \Delta + \{\forall x A(x)\}} \\ (\exists) \frac{\Gamma + \{A(x)\} \vdash^1 \Delta}{\Gamma + \{\exists x A(x)\} \vdash^1 \Delta} \end{array} \right\} \begin{array}{l} \text{provided } x \text{ is not free} \\ \text{in } \Gamma \text{ or } \Delta \end{array}$$

(and eventually an additional double-line rule for identity), where none of these rules is subject to horizontalization. The restrictions made upon D to obtain first-order sequent-systems corresponding to $DS5_H$ and $DS4_HH$ are as in Section 1. It is a straightforward matter to obtain Hilbert-style systems corresponding to these first-order sequent-systems.

It is easy to show that the Barcan Formula $\forall x \Box A \rightarrow \Box \forall x A$ is provable in the first-order sequent-system corresponding to $DS5_H$: since the formulae $\neg \Box \neg \Box A \rightarrow A$ and $A \rightarrow \Box \neg \Box \neg A$ are provable in $DS5_H$ (the first of these schemata can replace $\Box 5$ for the axiomatization of $S5_H$), we have

$$\frac{\frac{\frac{\frac{\forall x \Box A \rightarrow \Box A}{\neg \Box \neg \forall x \Box A \rightarrow \neg \Box \neg \Box A}}{\neg \Box \neg \forall x \Box A \rightarrow A}}{\neg \Box \neg \forall x \Box A \rightarrow \forall x A}}{\Box \neg \Box \neg \forall x \Box A \rightarrow \Box \forall x A}}{\forall x \Box A \rightarrow \Box \forall x A}.$$

A fortiori, the Barcan Formula is provable in the first-order sequent-system corresponding to $DS5$. This formula is not provable in the first-order sequent-system corresponding to $DS4$, and hence it cannot be provable in the first-order sequent-system corresponding to $DS4_H$.

Considerations concerning the uniqueness of characterization of \Box from [3] apply in the present context too.

REFERENCES

- [1] M. Božić, K. Došen, *Models for normal intuitionistic modal logics*, *Studia Logica* **43** (1984), 217–245.
- [2] R. A. Bull, *Some modal calculi based on IC*, in J. N. Crossley and M. A. E. Dummett (eds), *Formal Systems and Recursive Functions*, North-Holland, Amsterdam, 1965, 3–7.
- [3] K. Došen, *Sequent-systems for modal logic*, *J. Symbolic Logic* **50** (1985), 149–168.
- [4] K. Došen, *Models for stronger normal intuitionistic modal logics*, *Studia Logica* **44** (1985), 39–70.
- [5] J. M. Font, *Monadicity in topological pseudo-Boolean algebras*, in M. M. Richter et al. (eds), *Models and Sets: Logic Colloquium '83*, Lecture Notes in Mathematics, 1103, Springer, Berlin, 1984, 169–192.
- [6] J. M. Font, *Modality and possibility in some intuitionistic modal logics*, University of Barcelona, Faculty of Mathematics preprint no **26** (1984).
- [7] G. Gentzen, *Untersuchungen über das logische Schliessen*, *Math. Z.* **39** (1934–35), 176–210, 405–431 [Engl. transl. in: *The Collected Papers of Gerhard Gentzen*, North-Holland, Amsterdam, 1969].
- [8] M. Mihajlova, *Reduction of modalities in several intuitionistic modal logics*, *Comptes rendus de l'Académie bulgare des Sciences* **33** (1980), 743–745.

- [9] H. Ono, *On some intuitionistic modal logics*, Publ. Res. Inst. Math. Sci. (Kyoto) **13** (1977), 687–722.
- [10] V. H. Sotirov, *Modal theories with intuitionistic logic* in: *Mathematical Logic Proceedings of the Conference on Mathematical Logic dedicated to Markov*, Bulgarian Academy of Sciences, Sofia, 1984, 139–171.

Matematički institut
Knez Mihajlova 35
11000 Beograd
Jugoslavija

(Received 25 07 1985)