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ON σ-PERMUTABLE n-GROUPS

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Abstract. In this paper σ-permutable n-groups are defined and considered. An n-group (G, f) is called σ-permutable, where σ is a permutation of the set $\{1, \dots, n+1\}$, iff

$$f(x_{\sigma 1}, \dots, x_{\sigma n}) = x_{\sigma(n+1)} \Leftrightarrow f(x_1, \dots, x_n) = x_{n+1}$$

for all $x_1, \dots, x_{n+1} \in G$. Such n-groups are a special case of σ-permutable n-groupoids considered in [7] and also they represent a generalization of i-permutable n-groups from [6] and some other classes of n-groups. Examples of σ-permutable n-groups are given and some of their properties described. Necessary and sufficient conditions for an n-group to be σ-permutable are determined. Several conditions under which such n-groups are derived from a binary group are given.

1. Introduction. In [7] σ-permutable n-groupoids which represent a generalization of several classes of n-groupoids were considered. Some well known classes of n-groupoids are special cases of σ-permutable n-groupoids. Among them are (i, j)-commutative and commutative n-groupoids, totally symmetric n-quasigroups, medial n-groups [2], cyclic n-quasigroups from [12], [13] and [18], i-permutable n-groupoids from [15], alternating symmetric n-quasigroups ([14], [16]), parastrophy invariant n-quasigroups ([9], [8]) and some others. Binary groupoids which are σ-permutable for different values of σ are commutative groupoids, semisymmetric groupoids (satisfying the identity $(xy)x = y$), totally symmetric quasigroups, groupoids satisfying Sade's left "key's" law ($x(xy) = y$) and Sade's right "key's" law ($(xy)y = x$), [1]. In this paper we shall consider σ-permutable n-groups, but first we give some necessary definitions and notations.

2. Notation and Definitions. We shall use the following abreviated notation: $f(x_1, \dots, x_k, x_{k+1}, \dots, x_{k+s}, x_{k+s+1}, \dots, x_n) = f(x_1^k, \overset{(s)}{x}, x_{k+s+1}^n)$, whenever $x_{k+1} = x_{k+2} = \cdots = x_{k+s} = x$ (x_i^j is the empty symbol for $i > j$ and for $i > n$, also $\overset{(0)}{x}$ is the empty symbol).

To avoid repetitions we assume throughout the whole text that $n \geq 2$.

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An n-groupoid (G,f) is k-solvable, where $k \in \{1,\dots,n\} = N_n$ is fixed, iff the equation $f(a_1^{k-1}, x, a_{k+1}^n) = b$ has a solution $x \in G$ for all a_1^n, $b \in G$. If this equation has a unique solution for every $k \in N_n$ then (G,f) is called an n-quasigroup.

An n-groupoid (G,f) is (i,j)-associative, where $1 \le i < j \le n$, iff

$$f(x_1^{i-1},\, f(x_i^{n+i-1}),\, x_{n+i}^{2n-1}) = f(x_1^{j-1},\, f(x_j^{n+j-1}),\, x_{n+j}^{2n-1})$$

for all $x_1^{2n-1} \in G$. An n-groupoid is associative (i.e. it is an n-semigroup) iff it is (i,j)-associative for every pair (i,j), $i,\, j \in N_n$.

An associative n-quasigroup is an n-group. One can prove [10, p. 213] that an n-semigroup is an n-group iff it is k-solvable for $k = 1$ and $k = n$ or for some k other than 1 and n. On the other hand Sokolov proved (in [11], but this proof is not complete, cf. [4]) that an n-quasigroup is an n-group iff it is $(i, i+1)$-associative for some $i \in N_{n-1}$. A similar characterization of n-groups is given in [2] and [4].

An n-group is called medial (or abelian) iff it is $(1,n)$-commutative ([2], [3]).

By S_n we denote the symmetric group of degree n.

If $\sigma \in S_n$, then $x_{\sigma i}, x_{\sigma(i+1)}, \dots, x_{\sigma j}$ we denote by $x_{\sigma i}^{\sigma j}$. If $i > j$, then $x_{\sigma i}^{\sigma j}$ is considered empty.

By w we shall always denote the automorphism $x \mapsto x^{-1}$ of a commutative group.

If G is a set, by it we denote the identity mapping of G. If A is a set of integers, by gcd A we denote the greatest common divisor of all elements of A.

If an n-group (G,f) has the form

$$f(x_1^n) = x_1 \theta x_2 \theta^2 x_3 \dots \theta^{n-1} x_n b,$$

where $(G,\cdots)$ is a binary group, $b \in G$, θ an automorphism of $(G,\cdot)$, such that $\theta b = b$, $\theta^{n-1} x b = bx$ for all $x \in G$, then this n-group is called $\langle\theta,b\rangle$-derived from $(G,\cdot)$ and it is denoted by $\mathrm{der}_{\theta,b}(G,\cdot)$. If θ is the identity mapping, then this n-group is called b-derived from $(G,\cdot)$. If θ is the identity mapping and b is the neutral element of $(G,\cdot)$, then an n-group $\mathrm{der}_{\theta,b}(G,\cdot)$ is called derived from $(G,\cdot)$. In this case we say also that an n-group operation is a long product of the group operation (see [2] or [5]). By Hosszú theorem every n-group is $\langle\theta,b\rangle$-derived from some binary group.

Let (G,f) be an n-group, $a_2^{n-1} \in G$ be fixed and let $x \cdot y = f(x, a_2^{n-1}, y)$. Then the groupoid $(G,\cdot)$ is a group and it is called a (binary) retract of the n-group (G,f). As it is well known [19], if an n-group (G,f) is $\langle\theta,b\rangle$-derived from a group $(G,\cdot)$, then $(G,\cdot)$ is isomorphic to some retract of (G,f) and (since all retracts of a given n-group are isomorphic, [5]) it is isomorphic to every retract of (G,f).

3. σ-permutable n-groupoids. σ-permutable n-groupoids were introduced and investigated in [7]. Here we shall give some of the basic definitions and propositions from [7].

Definition 1. Let $\sigma \in S_{n+1}$. An n-groupoid (G,f) is called σ-permutable iff for all $x_1^{n+1} \in G$

$$f(x_1^n) = x_{n+1} \Leftrightarrow f(x_{\sigma 1}^{\sigma n}) = x_{\sigma(n+1)}.$$

An equivalent form of the preceding definition is the following.

Definition 2. Let $\sigma \in S_{n+1}$. If $\sigma i = n+1$ for some $i \in N_n$, then an n-groupoid (G, f) is σ-permutable iff for all $x_1^n \in G$

$$f(x_{\sigma 1}^{\sigma(i-1)}, f(x_1^n), x_{\sigma(i+1)}^{\sigma n}) = x_{\sigma(n+1)}.$$

If $\sigma(n+1) = n+1$, then (G, f) is σ-permutable iff for all $x_1^{n+1} \in G$

$$f(x_{\sigma 1}^{\sigma n}) = f(x_1^n).$$

Let $\sigma \in S_{n+1}$ and H be the subgroup of S_{n+1}, generated by σ. An n-groupoid (G, f) is σ-permutable iff it is τ-permutable for every $\tau \in H$. The set of all permutations $\varrho \in S_{n+1}$ such that

$$f(x_{\varrho 1}^{\varrho n}) = x_{\varrho(n+1)} \Leftrightarrow f(x_1^n) = x_{n+1}$$

for all $x_1^{n+1} \in G$ is a subgroup of S_{n+1}.

Let H be a subgroup of S_{n+1}. If an n-groupoid (G, f) is σ-permutable for every $\sigma \in H$, then (G, f) is called a H-permutable n-groupoid.

4. σ-permutable n-groups. A σ-permutable n-groupoid which is an n-group is called a σ-permutable n-group. Now we shall consider such n-groups, but first we give some examples.

1. n-groups satisfying the cyclic identity $(f(f(x_1^n)x_1^{n-1}) = x_n)$ from [17] are σ-permutable n-groups where $\sigma = (12\ldots n+1)$.

2. Commutative n-groups are H permutable where $H \simeq S_n$ and $\sigma(n+1) = n+1$ for every $\sigma \in H$.

3. Medial n-groups ([2], [3]) are σ-permutable, where $\sigma = (1n)$.

4. Let $(G, +)$ be an abelian group and let $f(x, y, z) = x - y + z$. Then (G, f) is a ternary group which is H-permutable, where H is the Klein's four group generated by the permutations (1 2) (3 4) and (1 3) (2 4), and which is not reducible to a binary group.

5. Let $G = Z_2 \times \cdots \times Z_2$ (k times), where $k \geq 3$ and Z_2, is a cyclic group of order 2. Then the mapping $\theta(x_1^k) = (x_3, x_2, x_1, x_4^k)$ is an automorphism of the group G such that $\theta^3 = \mathrm{id}$, and by Hosszù theorem the set G with the operation

$$f(y_1^{10}) = y_1\theta y_2\theta^2 y_3 y_4 \theta y_5 \theta^2 y_6 y_7 \theta y_8 \theta^2 y_9 y_{10}$$

is a non-commutative 10-group. It is non-reducible and it is H-permutable, where H is a subgroup of S_{11} generated by the transpositions (1 11), (4 11), (7 11) and (10 11).

In the sequel we shall consistently use the following abreviations: if $\sigma \in S_{n+1}$, then always $i = \sigma^{-1}(n+1)$ and $j = \sigma(n+1)$.

THEOREM 1. *If* $(G, f) = \mathrm{der}_{\theta,b}(G, \cdot)$ *is a σ-permutable n-group, then* $(G, \cdot)$ *is commutative,* $\theta^{n-1} = \mathrm{id}$ *and* (G, f) *is medial (abelian). Moreover, if* $\sigma(n+1) \neq n+1$, *then* $b = b^{-1}$ *and if* $x \cdot y = f(x, \overset{(n-2)}{a}, y)$ *for some* $a \in G$, *then*

$\theta x = f(a, x, e, \overset{(n-3)}{a})$ and when n is odd $b = a$, or when n is even $b = e$, where e is the unit of $(G, \cdot)$.

Proof. Let (G, f) be a σ-permutable n-group. By Hosszú theorem there exist a group $(G, \cdot)$, and an element $b\|G\|$ such that

$$f(x_1^n) = x_1 \theta x_2 \theta^2 x_3 \dots \theta^{n-1} x_n b,$$

where $\theta b = b$ and $\theta^{n-1} x = b x b^{-1}$ for all $x \in G$.

We shall consider two cases: 1. $\sigma(n+1) \neq n+1$ and 2. $\sigma(n+1) = n+1$.

1. Since (G, f) is σ-permutable, from Definition 2 we get the following identity

$$x_{\sigma 1} \theta x_{\sigma 2} \dots \theta^{i-2} x_{\sigma(i-1)} \theta^{i-1} (x_1 \theta x_2 \dots \theta^{n-1} x_n b) \theta^i x_{\sigma(i+1)} \dots \\ \dots \theta^{n-1} x_{\sigma n} b = x_{\sigma(n+1)}. \tag{1}$$

If we take $r \in N_n \setminus \sigma(n+1)$, then fixing by e all variables in (1) except x, we get $\theta^k x_r \theta^m x_r b^2 = e$ or $\theta^m x_r b \theta^k x_r b = e$, where $k = \sigma^{-1} r - 1$, $m = i + r - 2$. In either case, by letting $x_r = e$ we get $b^2 = e$ and so $\theta^k x_r \theta^m x_r = e$ or

$$\theta^m (x_r b) \theta^k (x_r b) = e, \ \text{ i.e. } \theta^{k-m} x = x^{-1}.$$

Hence the mapping $x \mapsto x^{-1}$ is an automorphism of $(G, \cdot)$ which means that $(G, \cdot)$ is commutative.

2. The σ-permutability of (G, f) implies the identity

$$x_1 \theta x_2 \theta^2 x_3 \dots \theta^{n-1} x_n = x_{\sigma 1} \theta x_{\sigma 2} \theta^2 x_{\sigma 3} \dots \theta^{n-1} x_{\sigma n}. \tag{2}$$

Fixing in (2) all variables by e, except one, we get that the identity $\theta^{k-1} x = \theta^{\sigma^{-1} k - 1} x$ is valid for every $k \in N_n$. σ is not the identity permutation, hence there exist k, $m \in N_n$ such that fixing in (2) all variables by e, except x_k, x_m, we get $\theta^{k-1} x_k \theta^{m-1} x_m = \theta^{\sigma^{-1} m - 1} x_m \theta^{\sigma^{-1} k - 1} x_k$, i.e. $(G, \cdot)$ is commutative.

So we have proved that in both cases $(G, \cdot)$ is commutative, which means that θ^{n-1} is the identity mapping. Thus $f(x_1^n) = f(x_n, x_2^{n-1}, x_1)$, i.e. (G, f) is medial.

From $xy = f(x, \overset{(n-2)}{a}, y) = x \theta a \theta^2 a \dots \theta^{n-2} a y b$ it follows that

$$\theta a \theta^2 \dots \theta^{n-2} a = b.$$

Applying θ to the preceding equality and multiplying by θa, we get $a = \theta a$. Henee $b = a^{n-2}$ and

$$f(a, x, e, \overset{(n-3)}{a}) = \theta x.$$

Putting in (1) $x_{\sigma 1} = a$ and $x_m = e$ for all $m \in N_n \setminus \{\sigma 1\}$ is follows $a^2 = e$. So, if n is odd then $b = a$, if n is even then $b = e$.

In the sequel only σ-permutable n-groups such that $\sigma(n+1) \neq n+1$ will be considered and we shall assume that this condition is always satisfied without stating it explicitely.

THEOREM 2. *An n-group $(G, f) = \mathrm{der}_{\theta,b}(G, \cdot)$ is σ-permutable iff $(G, \cdot)$ is a commutative group, $b = b^{-1}$ and θ is such that $\theta^{n-1} = \theta^{i+j-2} = \omega$ for all $k \in N_n \setminus \{i\}$.*

Proof. Let $(G, f) = \mathrm{der}_{\theta,b}(G, \cdot)$ be a σ-permutable n-group. From Theorem 1 it follows that $(G, \cdot)$ is commutative, $b^2 = e$, $\theta^{n-1} = \mathrm{id}$ and the σ-permutability of (G, f) implies (1). If in (1) we fix by e all variables except x_j, we get $\theta^{i+j-2} = \mathrm{id}$. Fixing in (1) all variables by e except x_k, $k \in N_n \setminus \{j\}$, we get $\theta^{\sigma^{-1}k-1} x_k \theta^{i+k-2} x_k = e$, i.e. $\theta^{\sigma^{-1}k-k-i+1} = \omega$. But every σ-permutable n-group is also σ^{-1}-permutable, applying the preceding to σ^{-1} we get that $\theta^{\sigma k-k-j+1} = \omega$, for all $k \in N_n \setminus \{i\}$.

The converse part of the theorem follows by a straightforward computation.

COROLLARY 1. *If $(G, f) = \mathrm{der}_{id,b}(G, \cdot)$ is σ-permutable, then $(G, \cdot)$ is boolean.*

COROLLARY 2. *If n is even and $(G, f) = \mathrm{der}_{\theta,b}(G, \cdot)$ is a σ permutable n-group, then $(G, \cdot)$ is boolean and there exists a boolean group $(G, *)$ such that b is the unit of $(G, *)$ and $(G, f) = \mathrm{der}_{\theta,b}(G, *)$.*

Proof. By Theorem 2 ω is a power of θ and since $\theta^{n-1} = \mathrm{id}$ it follows $\omega^{n-1} = \mathrm{id}$. Since n is even we get $\omega = \mathrm{id}$.

If we define a new operation by $x * y = xyb$, then $(G, *)$ is a boolean group and b is the unit of this group. Since $b^2 = e$ and n is even we have

$$\begin{aligned} f(x_1^n) &= x_1\theta x_2\theta^2 x_3 \ldots \theta^{n-2} x_{n-1} x_n b = x_1 \theta x_2 b \theta^2 x_3 b \ldots \theta^{n-2} x_{n-1} b x_n b \\ &= x_1 * \theta x_2 * \theta^2 x_3 * \cdots * \theta^{n-2} x_{n-1} * x_n. \end{aligned}$$

It is not difficult to see that θ is an automorphism of the group $(G, *)$ and the corollary is proved.

Remark. If $(G, \cdot)$ is a boolean group, b fixed element from G, $f(x_1^n) = x_1 x_2 \ldots x_n b$, then (G, f) is a σ-permutable n-group for every σ. Since every finite boolean group is of order 2^k, k nonegative integer, and for every such k there exists a boolean group of order 2^k, we get that for every even n there exists a nontrivial σ-permutable n-group of order g iff $g = 2^k$, $k \in N$. (An n-group (G, f) is called nontrivial iff $|G| > 1$.)

Definition 3. If $\sigma \in S_{n+1}$, then

$$\begin{aligned} d(\sigma) &= \gcd\{n-1, i+j-2, 2(\sigma k - k - j + 1)\}_{k \in N_n \setminus \{i\}}, \\ d'(\sigma) &= \gcd\{n-1, i+j-2, \sigma k - k - j + 1\}_{k \in N_n \setminus \{i\}}. \end{aligned}$$

LEMMA. *If $(G, f) = \mathrm{der}_{\theta,b}(G, \cdot)$ is a σ-permutable n-group, then $\theta^{d(\sigma)} = \mathrm{id}$, $\theta^{d'(\sigma)} = \omega$.*

Proof. If we express $d(\sigma)$ and $d'(\sigma)$ as linear combinations of elements of the corresponding sets, by Theorem 2 we get that $\theta^{d(\sigma)} = \mathrm{id}$ and $\theta^{d'(\sigma)} = \omega$.

THEOREM 3. *Let (G, f) be a σ-permutable n-group.*

(*i*) *If* $d(\sigma) = 1$, *then* (G, f) *is* b*-derived from a commutative group.*

(*ii*) *If* $d'(\sigma) = 1$, *then* (G, f) *is* $\langle\theta, b\rangle$*-derived from a commutative group, where* $\theta = \omega$.

(*iii*) *If* $d(\sigma) = d'(\sigma)$, *then* (G, f) *is* $\langle\theta, b\rangle$*-derived from a boolean group.*

(*iv*) *If* $d(\sigma) = 1$ *and* n *is even, then* (G, f) *is a long product of a boolean group.*

Proof. Folows from Theorem 2, Corollary 2 and the Lemma.

COROLLARY 3. *Let* (G, f) *be a* σ*-prrmutable* n*-semigroup. If* $i + j - 2$ *and* $n - 1$ *are relatively prime, then* (G, f) *is an* n*-group which is* b*-derived from a boolean group.*

Proof. Since $i + j - 2$ and $n - 1$ are relatively prime, then at least one of the integers i, j belongs to the set $\{2, \dots, n-1\}$. Hence by Proposition 5 from [7], this n-semigroup is an n-group, which is by Theorem 3 b-derived from a boolean group.

The following theorem gives a more convenient expression for $d(\sigma)$ and $d'(\sigma)$.

THEOREM 4. *If* $\sigma \in S_{n+1}$, *then*

$$d'(\sigma) = \gcd\{n-1, i-1, j-1, \sigma k - k\}_{k \in N_n \setminus \{i\}},$$
$$d(\sigma) = \gcd\{n-1, i+j-2, 2d'(\sigma)\}.$$

Proof. To prove the first equality it suffices to prove that $j \equiv 1 \pmod{d'(\sigma)}$. Let $c_m = md'(\sigma) + 1$, $m = 0, 1, \dots, s$, where $s = (n-1)/d'(\sigma)$. Then $c_m \leq n$ for all m. If for some m $c_m = i$, then $md'(\sigma) = i - 1$, i.e. $i \equiv 1 \pmod{d'(\sigma)}$, which implies that $j \equiv 1 \pmod{d'(\sigma)}$. If for every $m = 0, 1, \dots, s$ $c_m \neq i$, then since $\sigma k - k - j + 1 = 0 \pmod{d'(\sigma)}$ for every $k \neq i$, $n+1$, we get that for all $m = 0, 1, \dots, s \sigma c_m = j \pmod{d'(\sigma)}$. If $a = \min_m \sigma c_m$, then for all m $a \leq \sigma c_m \leq n$, hence for all $m \sigma c_m - a \leq n - a$. We have obtained that each of $s + 1$ different nonegative integers $\sigma c_m - a$,, which are all congruent modulo $d'(\sigma)$, is not greater then $n - a$, hence $n - a \geq sd'(\sigma) = n - 1$. So $a = 1$, i.e. there exists m such that $\sigma c_m = 1$. Hence $j \equiv 1 \pmod{d'(\sigma)}$.

The second equality follows directly from the definition of $d'(\sigma)$ and $d(\sigma)$.

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REFERENCES

[1] J.Dénes, A.D. Keedwell, *Latin Squares and their Applications*, Akadémiai Kiado, Budapest and Academic Press, New York, 1974.

[2] W.A. Dudek, *Remarks on n-groups*, Demonstratio Math. **13** (1980), 165–181.

[3] W.A. Dudek, *On medial n-groups*, (to appear).

[4] W.A. Dudek, K. Glazek, B. Gleichgewicht, *A note on the axioms of n-groups*, Coll. Math. Soc. J. Bolyai, **29.** Universal Algebra (Esztergom) (1977), 195–202.

[5] W.A. Dudek, J. Michalski, *On a generalization of Hosszú theorem*, Demonstratio Math. **15** (1982), 783–805.

[6] W.A. Dudek, Z. Stojaković, *On permutable n-groups*, (to appear).

[7] W.A. Dudek, Z. Stojaković, *On a permutable n-groupoids*, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. **15** (1985), No. 1, 189–198.

[8] D.G. Hoffman, *On the spectrum of n-quasigroups with given conjugate invariant subgroup*, J. Combin. Theory Ser. A, **35** (1983), 98–99.

[9] Đ. Paunić, Z. Stojaković, *Parastrophy invariant n-quasigroups*, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak., Ser. Mat. **13** (1983), 251–257.

[10] E. Post, *Polyadic groups*, Trans. Amer. Math. Soc. **48** (1940), 208–350.

[11] E.I. Sokolov, *On the theorem of Gluskin-Hosszú on Dörnte n-groups* (Russian), Mat. Issled. **39** (1976), 187–189.

[12] Z. Stojaković, *Cyclic n-quasigroups*, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak., Ser. Mat. **12** (1982), 407–415.

[13] Z. Stojaković, *A generalization of Mendelsohn triple systems*, Ars Combinatoria **18** (1984), 131–138.

[14] Z. Stojaković, *Alternating symmetric n-quasigroups*, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak., Ser. Mat. **13** (1983), 259–272.

[15] Z. Stojaković, W.A. Dudek, *Permutable n-groupoids*, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. **14** (1984), No. 2, 155–166.

[16] Z. Stojaković, R. Madaras, *On tetrahedral quadruple systems*, Utilitas Math., **29** (1986), 19–26.

[17] Z. Stojaković, Đ. Paunić, *On a class of n-groups*, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak., Ser. Mat. **14** (1984), No. 2, 147–154.

[18] Z. Stojaković, Đ. Paunić, *Self-orthogonal cyclic n-quasigroups*, Aequationes Math. **30** (1986), 252–257.

[19] J. Timm, *Zur gruppentheoretischen Beschreibung n-stelliger structuren*, Publ. Math. Debrecen **17** (1970), 183–192.

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