PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 40 (54), 1986, pp. 181–185

SPECTRAL TYPE OF SOME TRANSFORMATIONS OF CERTAIN STOCHASTIC PROCESSES

S. Mitrović

Abstract. We introduce a stochastic process with multiplicity equal to one which satisfies certain conditions and consider spectral type of the derivative process and of the non-anticipative integral transformations for the given process.

0. The technique used in the paper is the same as in [1] or [3].

Let the process $x(t)$ be given by Cramer representation:

$$x(t) = \int_a^t g(t,u)dz(u), \tag{1}$$

$u \leq t$, $t \in T = (a,b)$ where $z(u)$ is a process of orthogonal increments such that $Ez(u) = 0$ and $Ez^2(u) = F(u)$ and $g(t,u)$ is a nonrandom function for $u \leq t$ from $L^2(dF)$ space. We suppose that the second order process $x(t)$ is continuous to the left and purely nondeterministic.

Let us introduce the following conditions for $g(t,u)$ and $z(u)$:

(R_1) The functions $g(t,u)$ and $g'_t(t,u)$ are continuous and bounded for $u \leq t$, $u,t \in T$.

(R_2) $g(t,t) = 0$ for all $t \in T$.

(R_3) The function $F(u) = Ez^2(u)$ is absolutely continuous and not identically constant and $f(u) = F'(u)$ has at most a finite number of discontinuity points in any finite subinterval of T.

In [4] we proved the following theorem: the process $x(t)$, $t \in T$ given by (1) and satisfying (R_1), (R_2), (R_3) has multiplicity $N = 1$. Further more we suppose that $x(t)$ satisfies conditions above.

1. As it is well known a form of correlation function for $x(t)$ is:

$$r(s,t) = \int_a^{\min(s,t)} g(s,u) \cdot g(t,u) f(u) du,$$

AMS Subject Classification (1980): Primary 60G07, 60G12.

and $r(s,t)$ is continuous everywhere in the interval $T \times T$. By the condition R_1, $r(s,t)$ has partial derivatives r'_s and r'_t which are continuous everywhere except perhaps on the diagonal $s = t$. But we have:

$$\lim_{\substack{s\to t\\ s\geq t}} \frac{r(s,t) - r(t,t)}{s-t} = \int_a^t g(t,u)g'_t(t,u)f(u)du,$$

$$\lim_{\substack{s\to t\\ s\leq t}} \frac{r(s,t) - r(t,t)}{s-t} = \int_a^t g(t,u)g'_t(t,u)f(u)du + g^2(t,t)\cdot f(t).$$

By the condition R_2 this two limiting values will be equal. Hence r'_s and r'_t are continuous at every point. Similarly the partial derivative $r''_{s,t}$ is a continuous function for s, t and its form is:

$$r''_{s,t} = \int_a^{\min(s,t)} g'_s(s,u)g'_t(t,u)f(u)du,$$

$s, t \in T$. The expression above is the correlation function of the derivative preocess:

$$x'(t) = \int_a^t g'_t(t,u)dz(u), \qquad u \leq t,\ \ u,t \in T. \tag{2}$$

THEOREM 1. *The derivative process $x'(t)$ given by* (2) *is continous and has the same spectral type as the process $x(t)$.*

Proof. Continuity of $x'(t)$ follows from the fact that its correlation function is continuous. By the theorem from [2] and from the form of $r(s,t)$, for $x'(t)$ it is sufficient to show that $g'_t(t,u)$, $u \leq t$, $u,t \in T$ is complete in $L^2(dF)$. If $\int_a^s g'_s(s,u)\psi(u)f(u)du = 0$ for all $s \in (a,t]$, where $\psi(u)$ is any function from $L^2(dF)$ space, and t is any point from T, then for all $s \in (a,t]$ the following holds: $\left(\int_a^s g(s,u)\psi(u)f(u)du\right)'_s = 0$. That means $\int_a^s g(s,u)\psi(u)f(u)du = 0$ for all $s \in (a,t]$. Since $g(t,u)$ is complete in $L^2(dF)$, $u,t \in T$ then $\psi(u) = 0$ almost ewerywhere related to the measure dF and that is what we hand to show. The spectral measure for $x'(t)$ is dF, multiplicity equal to one and the expression (2) is Cramer representation of $x'(t)$.

Example 1. Let $x(t) = \int_a^t (P(t) - P(u)) \cdot dz(u)$, $u \leq t$, $u,t \in (a,b)$ be a process with absolutely continuous $F(u)$, where $P(t)$ is a polynomial of any degree $n \geq 1$. If $g(t,u) = P(t) - P(u)$, $u \leq t$, $u,t \in (a,b)$ is complete in $L^2(dF(u))$, then the process $x'(t)$ exists, has multiplicity one and its spectral measure is dF.

Example 2. The same fact holds for the process $x(t) = \int_a^t Q(t-u)\cdot dz(u)$, $u \leq t$, $u,t \in T$, where $F(u)$ is absolutely continuous, $Q(t)$ is a polynomial of degree $n \geq 1$, and $Q(0) = 0$.

2. Let us introduce now $y(t)$, $t \in T$ as a nonanticipative integral transformation of $x(t)$ given by (1):

$$y(t) = \int_a^t \varphi(t,u)x(u)du, \tag{3}$$

$u \leq t,\ u, t \in T$. The function $\varphi(t,u)$ is such that for each $t \in T$ the quadratic mean integral from (3) exists. It is easy to see that:

$$\begin{aligned} y(t) &= \int_a^t \varphi(t,u) \left(\int_a^u g(u,\nu) dz(\nu) \right) du \\ &= \int_a^t \left(\int_\nu^t \varphi(t,u) g(u,\nu) du \right) dz(\nu), \quad t \in T. \end{aligned}$$

Let us denote $\int_\nu^t \varphi(t,u)g(u,\nu)du$ by $G(t,\nu)$ where $a < \nu \leq u \leq t < b$.

LEMMA. *The functions $G(t,\nu)$ and $G'_t(t,\nu)$ are continuous if $\varphi(t,u)$, $\varphi'_t(t,u)$, $g(t,u)$ are continuous on t and u, $u \leq t,\ u,t \in T$.*

Proof. The continuity of the function $G(t,\nu)$on t for all ν follows from:

$$\begin{aligned} |G(t_2,\nu) - G(t_1,\nu)| \leq \int_\nu^{t_1} &|\varphi(t_2,u) - \varphi(t_1,u)| \cdot |g(u,\nu)| du \\ &+ \int_{t_1}^{t_2} |\varphi(t_2,u)| \cdot |g(u,\nu)| du, \end{aligned}$$

when $\nu_1 \to \nu_2$ and $\nu_1 \leq \nu_2$. In a similar way we can show that by conditions of lemma the function $G'_t(t,\nu)$ is continuous for t and ν. Here is:

$$G'_t(t,\nu) = \int_\nu^t \varphi'_t(t,u) g(u,\nu) du + \varphi(t,t) g(t,\nu), \quad \nu \leq u \leq t, \ \ \nu, t \in T.$$

THEOREM 2. *The nonanticipative integral transformation $y(t)$ defined by (3) has the same spectral type as $x(t)$ from (1) if the functions $\varphi(t,u)$ and $\varphi'_t(t,u)$ are continuous and bounded for $u,t \in T$, $u \leq t$.*

Proof. From the continuity and the limitation of $\varphi(t,u)$ and $\varphi'_t(t,u)$ on t and u, and from the fact that $x(t)$ satisfies the condition R_1 it follows by lemma that R_1 holds for $G(t,u)$, $u \leq t,\ u,t \in T$. The condition: $G(t,t) = 0$ for all $t \in T$ is valid too. Finally since all of the three conditions R_1, R_2, R_3 hold for the process $y(t)$ then $y(t)$ has multiplicity equal to one. The spectral measure dF is the same as for $x(t)$. That implies the same spectral type. (See theorem 5.2 in [1], and the remark in [4]).

Example 3. The process $y(t) = \int_a^t x(u)du$, $u,t \in T$ has the same spectral type as $x(t)$. Here is $\varphi(t,u) \equiv 1$ for $u \leq t$, $G(t,\nu) = \int_\nu^t g(u,\nu)du$ and $G'_t(t,\nu) = g(t,\nu)$ where $a < \nu \leq u \leq t < b$.

Example 4. The process $x(t) = z(t)$, $t \in [0,\tau] = T$ with the absolutely continuous function $F(u)$, $u \in T$ has multiplicity $N = 1$. The process $y(t)$ from (3) has the same spectral type as $x(t)$ if $\varphi(t,u)$ and $\varphi'_t(t,u)$ are continuous on t and u, $t,u \in T$. They are bounded because T is compact. Here $G(t,\nu) = \int_\nu^t \varphi(t,u)du$, and $G'_t(t,\nu) = \int_\nu^t \varphi'_t(t,u)du + \varphi(t,t)$, where $0 \leq \nu \leq u \leq t \leq \tau$.

Remark. If we want to prove that the process $y(t)$ has multiplicity equal to one when multiplicity of $x(t)$ is unknown, then we may ommit the assumptions that $g_t'(t,u)$ is continuous and bounded for $u,t \in T$ and $g(t,t) = 0$ for all $t \in T$. Namely the next theorem is valid.

THEOREM 3. *Let the process $x(t)$ be given by expression (1), let $g(t,u)$ be a continuous and bounded function on t and u, $u,t \in T$ and let the condition R_3 be satisfied. Then the process $y(t)$ given by (3) has multiplicity equal to one if the functions $\varphi(t,u)$ and $\varphi_t'(t,u)$ are continuous and bounded for $u,t \in T$.*

Proof. Since the conditions R_1, R_2, R_3 hold for $y(t)$, $t \in T$ then the statement is valid [1, 5.2].

Example 5. The process which has multiplicity equal to two, while its nonanticipative integral transformation has multiplicity equal to one: Let $x(t)$ be represented by $x(t) = w_1(t) + h(t) \cdot w_2(t)$, where w_1 and w_2 are two independent Wiener processes for $t \geq 0$. A function $h(t)$ is absolutely continuous with $h'(t) > 0$, so that $h'(t)$ does not belong to $L^2((l,m))$ for any open interval $(l,m) \subset [0,\infty)$ but does belong to $L^1([0,t))$ for any $t > 0$. By [5] multiplicity of this process is two and the spectral type is $dt \leq dt$. Let us define the nonanticipative integral transformation of $x(t)$ as above in which $\varphi(t,u) \equiv 1$, $u \leq t$, $u,t \in [0,\infty)$. That means:

$$\begin{aligned} y(t) &= \int_0^t x(u)du = \int_0^t (w_1(u) + h(u) \cdot w_2(u))du \\ &= \int_0^t w_1(u)du + \int_0^t h(u)w_2(u)du \\ &= \int_0^t \int_0^u dw_1(\nu)du + \int_0^t h(u) \int_0^u dw_2(\nu)du \\ &= \int_0^t \left(\int_\nu^t du \right) dw_1(\nu) + \int_0^t \left(\int_\nu^t h(u)du \right) dw_2(\nu) \end{aligned}$$

where $0 \leq \nu \leq u \leq t < \infty$. The functions

$$G(t,\nu) = (G_1(t,\nu),\, G_2(t,\nu)) = \left(t - \nu,\, \int_\nu^t h(u)du \right) \quad \text{and } G_t'(t,\nu) = (1, h(t))$$

are continuous and bounded for t, ν, $0 \leq \nu \leq t < \infty$. It is easy to see that $\int_0^t G^2(t,\nu)d\nu < \infty$ holds for $t \in [0,\infty)$ and the nonanticipative transformation exists in the quadratic mean. Hence by the last theorem, multiplicity of $y(t)$ is one and its spectral measure is dt.

REFERENCES

[1] H. Cramer, *Structural and Statistical Problems for a Class of Stochastic processes*, Princeton University Press, Princeton, New Jersey, 1971.

[2] Z. Ivković, Yu A. Rozanov, *A characterization of Cramer representation of stochtastic processes*, Publ. Inst. Math. (Beograd) **14** (**28**) (1973), 69–74.

[3] Z. Ivković, J. Bulatović, J. Vukmirović, S. Živanović, *Application of Spectral Multiplicity in Separable Hilbert Space to Stochastic Processes*, Math. Inst. Beograd, Posebna izdanja, **12** (1974).

[4] S. Mitrović, *On a class of processes with multiplicity* $N = 1$, Publ. Inst. Math. (Beograd) **38** (**52**) (1985), 117–119.

[5] M. Hitsuda, *Multiplicity of some classes of Gaussian processes*, Nagoya Math. J. **52** (1973), 39–46.

[6] T. Hida, *Canonical representation of Gaussian processes and their applications*, Mem. Coll. Sci. Univ. Kyoto, Ser. A (Math) **33** (1960), 109–155.

Šumarski Fakultet
Kneza Višeslava 1
11000 Beograd
Jugoslavija

(Received 11 18 1985)