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MERCERIAN THEOREMS FOR BEEKMANN MATRICES

Vladeta Vučković

To the memory of B. Martić

Abstract. A matrix $A = (a_{nk})$ is called *normal* if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all n. Such a matrix has a normal inverse $A^{-1} = (\alpha_{nk})$. If Ihe inverse A^{-1} of a normal and regular matrix A satisfies the conditions $\alpha_{nk} \leq 0$ for $k < n$ and $\alpha_{nn} > 0$ for all n, we call such a matrix a Beekmann matrix. Beekmann introduced those matrices and proved that for such a matrix A, the matrix $B = (I + \lambda A)/(1 + \lambda)$ is Mercerian for $\lambda > -1$. (I is the identity matrix.)

This paper extends Beekmann's theorem to the case of R_β-Mercerian matrices, $\beta > 0$.

1. Let $A = (a_{nk})$ be a *normal* matrix, i.e., such that

$$a_{nk} = 0 \text{ for } k > n \text{ and } a_{nn} \neq 0 \text{ for all } n. \tag{1.1}$$

Such a matrix has a normal inverse $A^{-1} = (\alpha_{nk})$, so that the transformations

$$y_n = \sum_{k=1}^{n} a_{nk} x_k \ldots, \quad n = 2, \ldots \tag{1.2}$$

and

$$x_n = \sum_{k=1}^{n} \alpha_{nk} y_k \ldots, \quad n = 1, 2, \ldots \tag{1.3}$$

are inverse one to the other.

If the inverse A^{-1} of a normal and regular matrix A satisfies the conditions

$$\alpha_{nk} \leq 0 \text{ for } k < n \text{ and } \alpha_{nn} > 0 \text{ for all } n, \tag{1.4}$$

we shall call such a matrix a *Beekmann matrix*.

Beekmann introduced those matrices in [1] and proved that for such a matrix A, the matrix $B = (I + \lambda A)/(1 + \lambda)$ is Mercerian for $\lambda > -1$. (I is the matrix.)

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The aim of this paper is to extend Beekmann's theorem to the case of R-Mercerian matrices, $\beta \geq 0$.

2. A sequence s is said to be *regularly varying* iff

$$\lim_{n\to\infty}(s_{[tn]}/s_n) = h(t) \tag{1.5}$$

exists for every $t > 0$. ($[x]$ = the greatest integer $\leq x$). Such sequences (and functions) were introduced by J. Karamata [2]; today they play an essential role in summability and probability. (1.5) implies that there is a real number β such that $h(t) = t^\beta$. The number β is called the *order* of s. In addition, a regularly varying sequence of "order 0" (i.e., for which the limit in (1.5) equals 1) is called a *slowly varying sequence.* It can be proved [2] that every regularly varying sequence s of order $\beta > 0$ can be written in the form

$$s_n = n^\beta L(n), \tag{1.6}$$

where L is a slowly varying sequence.

By R_β, $\beta > 0$, we denote the class of regularly varying sequences of order β, and by R_0 the class of slowly varying sequences.

At last, we say that a matrix A is *R_β-regular* ($\beta > 0$) iff for every $s \in R_\beta$ and any sequence r

$$r_n \sim s_n \text{ implies } \sum_{k=1}^{n} a_{nk} r_k \sim s_n, \quad n \to \infty \tag{1.7}$$

and it is called *R_β-Mercerian* iff

$$\sum k = 1^n a_{nk} r_k \sim s_n \text{ implies } r_n \sim s_n, \quad n \to \infty \tag{1.8}$$

(Obviously, a matrix A is *regular* iff $r_n \to L$ implies $\sum_{k=1}^n a_{nk} r_k \to L$, and *Mercerian* iff $\sum_{k=1}^n a_{nk} r_k \to L$ implies $r_n \to L$, $n \to \infty$).

3. The R_β-regularity theorems for matrices were first established by M. Vuilleumier in [6]. The first R_β-Mercerian theorems for regular, invertible triangular matrices were established by S. Zimering in [3].

Using their results, B. Martić [5] proved the following.

THEOREM M. *Let $A = (a_{nk})$ be normal, nonnegative (i.e. $a_{nk} \geq 0$) and regular matrix which, for some $\gamma > 0$, satisfies the condition.*

$$\sum_{k=1}^{n} a_{nk} k^{-\gamma} = O(n^{-\gamma}), \quad n \to \infty \tag{3.1}$$

Then the matrix $B = (I + \lambda A)/(1 + \lambda)$, where I is the unit triangular matrix, is R_0-Mercerian for $|\lambda| < 1$.

(Martić supposed $\sum_{k=1}^{n} a_{nk} = 1$, but his proof is valid also in case $\sum_{k=1}^{n} a_{nk} \to 1$). Since, in case of a Beekmann matrix A, the conditions (1.4) imply

$$a_{nk} \geq 0 \text{ for all } k < n \text{ and} a_{nn} > 0, \tag{3.2}$$

we can apply Martić's theorem and obtain

LEMMA. 3.1. *If a Beekmann matrix A satisfies the condition* (3.1) *for some $\gamma > 0$, then the matrix $B = (I + \lambda A)/(1 + \lambda)$ is R_0-Merceriun for $|\lambda| < 1$.*

Lemma 3.1 reduces the proof of a general R_0-Mercerian theorem for Beekmann matrices to the case $\lambda \geq 1$. However, a method used by Tanović-Miller [4] and based upon the relations

$$\beta_{nk} \leq 0 \text{ for } k < n \text{ and } \beta_{nn} > 0 \text{ for all } n, \tag{3.3}$$

$$\sum_{k=1}^{n} \beta_{nk} \to 1, \; n \to \infty \tag{3.4}$$

and

$$\sum_{k=1}^{n} |\beta_{nk}| k^{-\gamma} = O(n^{-\gamma}), \; n \to \infty \tag{3.5}$$

for the inverse B^{-1} of B above supplies readily the proof in this case. Tanović-Miller considered non-negative, normal, normalized ($\sum_{k=1}^{n} a_{nk} = 1$) matrices A, which satisfy the conditions

$$a_{n1} > 0, \; a_{n+1,i} a_{nk} \leq a_{ni} a_{n=1,k} \tag{3.6}$$

for $1 \leq k \leq i \leq n-1$ and the condition (3.1). and from these derived (3.3)-(3.5). Once one has (3.3)-(3.5), the proof is a straightforward application of Theorem 4.1 of M. Vuillemier in [6].

Thus, if we prove that for a Beekmann matrix A, which satisfies (3.1), the inverse B^{-1} of $B = (I + \lambda A)/(1 + \lambda)$ satisfies (3.3)-(3.5) for $\lambda > 1$, Lemma 3.1 will be completed for all $\lambda > -1$.

4. Our main result is contained in

THEOREM 4.1. *If A is a Beekmann matrix and $B = (I + \lambda A)/(1 + \lambda)$, then B is a Beekmann matrix for $\lambda > 0$.*

Proof. Let $A = (a_{nk})$, $A^{-1}(\alpha_{nk})$, $B = (b_{nk})$ and $B^{-1} = (\beta_{nk})$.

Let us remark that the transformations

$$y_n = \sum_{k=1}^{n} b_{nk} x_k \tag{4.1}$$

and

$$x_n = \sum_{k=1}^{n} \beta_{nk} y_k \tag{4.2}$$

are inverse.

Since $b_{nk} = \lambda a_{nk}/(I+\lambda)$ for $k < n$ and $b_{nn} = (1+\lambda a_{nn})/(1+\lambda)$, $b_{nk} = 0$ for $k > n$, B is normal and obviously regular. Thus B^{-1} exists and it is normal. Moreover, (4.1) and (4.2) are inverse and (1.2) and (1.3) are inverse.

The case $\lambda = 0$ being trivial, let $\lambda > 0$, and let $\varepsilon = (1+\lambda)/\lambda$. Obviously, $\varepsilon > 1$.

We have for any sequence x,

$$\sum_{k=1}^{n} a_{nk} x_k = \varepsilon b_{nk} - (\varepsilon - 1) x_n;$$

introducing the sequence y, defined by (4.1). this gives

$$\sum_{k=1}^{n} a_{nk} x_k = \varepsilon y_n - (\varepsilon - 1) x_n. \tag{4.3}$$

If in (1.2) we replace y_n by $\varepsilon y_n - (\varepsilon - 1)x_n$ and use (1.3), from (4.3) we obtain

$$x_n = \varepsilon \sum_{k=1}^{n} \alpha_{nk} y_k - (\varepsilon - 1) \sum_{k=1}^{n} a_{nk} x_k$$

which, using in the second sum on the right side formula (4.2), yields, after some elementary computations,

$$x_n = \sum_{k=1}^{n} \left\{ \varepsilon \alpha_{nk} - (\varepsilon - 1) \sum_{i=k}^{n} \alpha_{ni} \beta_{ik} \right\} y_k.$$

From this and (4.2) we obtain at once

$$\beta_{nk} = \varepsilon \alpha_{nk} - (\varepsilon - 1) \sum_{i=k}^{n} \alpha_{ni} \beta_{ik}, \tag{4.4}$$

and, in particular, for $k = 1, 2, \ldots, n$,

$$\beta_{kk} = \{\varepsilon / (1 + (\varepsilon - 1)\alpha_{kk})\} \alpha_{kk} \tag{4.5}$$

and for $k \geq 2$

$$\beta_{k,k-1} = \varepsilon \alpha; pha_{k,k-1} / 1(1 + (\varepsilon - 1) a_{kk})(1 + (\varepsilon - 1)\alpha_{k-1,k-1})\}. \tag{4.6}$$

Now, solving (4.4) for β_{nk} and using (4.5) we obtain, for $k = 1, 2, \ldots, n-2$

$$\beta_{nk} = \frac{\varepsilon}{(1 + (-1)\alpha_{nn})(1 + (\varepsilon - 1)\alpha_{kk})} \alpha_{nk} - \frac{\varepsilon - 1}{1 + (\varepsilon - 1)\alpha_{nn}} \sum_{i=k+1}^{n-1} \alpha_{ni} \beta_{ik}. \tag{4.7}$$

Since $\alpha_{kk} > 0$ and $\alpha_{k,k-1} \le 0$ we conclude from (4.5) and (4.6) (with $k = n$) that $\beta_{nn} > 0$ for all n and $\beta_{n,n-1} \le 0$, for $n \ge 2$. Then, from (4.7) we conclude: if $\beta_{k+1,k}, \beta_{k+2,k}, \dots, \beta_{n-1,k}$ are all ≤ 0 for $k < n$, then $\beta_{nk} \le 0$ too, for $k = 1, 2, \dots, n-2$, which completes the proof of the theorem.

COROLLARY. 4.1.1. *Let A be a Beekmann matrix which, for some $\gamma > 0$, satisfies the condition* (3.1). *Then B^{-1}, the inverse of $B = (I + \lambda A)/(1 + \lambda)$, satisfies the condition* (3.5) *for $\lambda \ge 0$.*

Proof. We use notations of Theorem 4.1. If D is any matrix, by $(D)_{nk}$ we denote its element in n-th row and k-th column. δ_n^k denotes the Kronecker symbol ($= 1$ if $k = n$, 0 otherwise).

Since

$$\sum_{i=1}^{n} b_{ni}\beta_{ik} = (BB^{-1})_{nk} = \delta_n^k,$$

we have, for $k < n$,

$$\sum_{i=1}^{n-1} b_{ni}\beta_{ik} = -b_{nn}\beta_{nk},$$

and, since $\beta_{nn} = 1/b_{nn}$,

$$-\beta_{nk} = \beta_{nn} \sum_{i=1}^{n-1} b_{ni}\beta_{ik}. \tag{4.8}$$

Taking into account the relations $\beta_{ik} \le 0$ for $i \ne k$ (B is Beekmann, by Theorem 4.1), $b_{ni} \ge 0$ and $\beta_{kk} > 0$, we obtain from (4.8), for $k < n$

$$-\beta_{nk} \le \beta_{nn} b_{nk} \beta_{kk} \le b_{nk}(1 + \lambda)^2. \tag{4.9}$$

since

$$\beta_{nn}\beta_{kk} = \frac{1+\lambda}{1+\lambda a_{nn}} \cdot \frac{1+\lambda}{1+\lambda a_{kk}} \le (1+\lambda)^2.$$

Using the relations between the elements of A and B, the fact that B is Beekmann, (3.1) and (4.9), we have:

$$\sim_{k=1}^{n} |\beta_{nk}| k^{-\gamma} = \sum_{k=1}^{n-1} -\beta_{nk} k^{-\gamma} + \beta_{mn} n^{-\gamma} \le (1+\lambda)^2 \sum_{k=1}^{n-1} b_{nk} k^{-\gamma} + \frac{1+\lambda}{1+\lambda a_{nn}} n^{-\gamma}$$

i.e.

$$\sum_{k=1}^{n} |\beta_{nk}| k^{-\gamma} \le \lambda(1+\lambda) \sum_{k=1}^{n-1} a_{nk} k^{-\gamma} + O(n^{-\gamma}),$$

which, by (3.1), gives (3.5).

COROLLARY 4.1.2. *The matrix B^{-1} of Theorem* 4.1 *satisfies* (3.4.).

Proof. From (4.9) follows

$$|\beta_{nk}| \le (1+\lambda)^2 b_{nk},\ k < n$$

i.e., (since B is regular) for every fixed k, $|\beta_{nk}| \to 0$, $n \to \infty$.

Also, by the same inequality and the fact that

$$\beta_{nn} = 1/b_{nn} = \frac{1+\lambda}{1+\lambda a_{nn}}, \quad \sum_{k=1}^{n} |\beta_{nk}| < (1+\lambda)^2 \sum_{k=1}^{n-1} b_{nk} + \frac{1+\lambda}{1+\lambda a_{nn}}$$

and since B is regular, there is $M > 0$ such that

$$\sum_{k=1}^{n} |\beta_{nk}| \le M. \tag{4.10}$$

Set now in (4.1) $x_k = 1$ for all k, so that $y_n = \sum_{k=1}^{n} b_{nk}$ Then, by (4.2)

$$1 = \sum_{k=1}^{n} \beta_{nk} y_k$$

and so

$$1 - \sum_{k=1}^{n} \beta_{nk} = \sum_{k=1}^{n} \beta_{nk}(y_k - 1).$$

Since $y_k - 1 \to 0$, $k \to \infty$, by (4.10) and the fact that, for fixed k, $|\beta_{nk} \to 0$, $n \to \infty$ follows $\lim\limits_{n\to\infty} \sum_{k=1}^{n} \beta_{nk} = 1$ in usual way.

Remark. A consequence of the content of Corollary 4.1.2 is that B^{-1} is a regular matrix. Contrary to this, A^{-1} does not need to be regular. For example, for the matrix $A = (1/n)_{k\le n}$ of arithmetic means, $\alpha_{nk} = 0$ for $k \le n-2$, $\alpha_{n,n-1} = -(n-1)$, $a_{nn} = n$ and $\sum_{k=1}^{\overline{n}} |a_{nk}| = 2n - 1$ is not bounded!

5. We are able now to prove the extensions of Beekmann's Mercerian Theorem to regularly varying functions.

THEOREM. 5.1. *Let A be Beekmann matrix, such that, for some $\gamma > 0$,*

$$\sum_{k=1}^{n} a_{nk} k^{-\gamma} = O(n^{-\gamma}),\ \ n \to \infty. \tag{5.1}$$

Then, for $\lambda > -1$, the matrix $B = (I + \lambda A)/(1+\lambda)$ is R_0-Mercerian.

Proof. Case $|\lambda| < 1$ by Lemma 3.1. For $\lambda \ge 1$, by Theorem 4.1 and its Corollaries, B^{-1}, the inverse of B, satisfies all the conditions (3.3) – (3.5). By the remark at the end of section 3, B is R_0-Mercerian.

Since every regularly varying sequence s of order $\beta > 0$, satisfies (1.6), applying Theorem 5.1 to the sequence $\{s_n/n^\beta\}$, we obtain, in a similar way as Martić in [5],

THEOREM. 5.2. *Let A be a Beekmann matrix such that there are two numbers α and β, $0 < \alpha < \beta$,for which*

$$\sum_{k=1}^{n} a_{nk} \left(\frac{k}{n}\right)^{\alpha} \to A_\alpha, \quad \textit{and} \quad \sum_{k=1}^{n} a_{nk} \left(\frac{k}{n}\right)^{\beta} \to A_\beta, \quad n \to \infty.$$

Then, for every λ such that $1 + \lambda A_\alpha > 0$ and $1 + \lambda A_\beta > 0$, the matrix $B_\beta = (I + \lambda A)/(1 + \lambda A_beta)$ is R_β-Mercerian.

One should remark that conditions $1 + \lambda A_\alpha > 0$ and $1 + \lambda A_\beta > 0$ imply one another, depending on the sign of λ.

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University of Notre Dame
Department of Mathematics
Notre Dame, Indiana 46556
U.S.A.

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