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CONVOLUTIONS OF MEROMORPHIC UNIVALENT FUNCTIONS WITH POSITIVE COEFFICIENTS

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Abstract. Let $f(z) = 1/z + \sigma a_n$, $a_n \geq 0$ and $g(z) = 1/z + \sigma b_n$, $b_n \geq 0$. We investigate certain properties of the convolution $1/z+\sigma a_n b_n$ where $f(z)$ and $g(z)$ are meromorphically starlike.

1. Introduction. Let Σ be the class of functions $f(z) = 1/z + \sigma a_n$ which are regular in the punctured disk $E = \{z : 0 < |z| < 1\}$ with a simple pole at $z = 0$. Σ_s be the subclass of Σ consisting of functions which are univalent in E. A function $f(z)$ in Σ is said to be starlike of order α $(0 \leq \alpha < 1)$ if $\operatorname{Re} zf'(z)/f(z) < -\alpha$ for $|z| < 1$. This class is denoted by $\Sigma^*(\alpha)$. It is well known that $\Sigma^*(\alpha) \subset \Sigma_s$. Let σ be the subclass of Σ consisting of functions of the form

$$f(z) = 1/z + \Sigma a_n z^n, \quad a_n \geq 0 \tag{1}$$

Set $\sigma_s = \Sigma_s \cap \sigma$ and $\sigma^*(\alpha) = \Sigma^*(\alpha) \cap \sigma$.

Let $f(z)$ be given by (1). Then

$$\sum (n + \alpha)a_n \leq 1 - \alpha \tag{2}$$

is a necessary and sufficient condition for the function $f(z)$ to be in $\sigma^*(\alpha)$ [3]. Since $\sigma^*(\alpha) \subset \sigma_s$, a sufficient condition for functions of the form (1) to be univalent is that

$$\sum na_n \leq 1. \tag{3}$$

Also the condition (3) is necessary for univalence, because $f'(r) = -1/r^2 + \sum na_n r^{n-1} = 0$ for some r (< 1) if $\sum na_n > 1$. Hence functions of the form (1) are univalent if and only if they are starlike. Thus $\sigma^*(0) = \sigma_s$.

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*) If otherwise not stated, $\sum$ means $\sum_{n=1}^{\infty}$

The covolution or Hadamard product of two functions

$$f(z) = 1/z + \sigma a_n \quad \text{and} \quad g(z) = 1/z + \sigma b_n$$

is defined by $f(z)^* g(z) = 1/z + \sigma a_n b_n$. In [1] Robertson proved the following result. Let $f(z) = 1/z + \sigma a_n$ and $g(z) = 1/z + \sigma b_n$ be univalent in $0 < |z| < 1$. Then the convolution $1/z + \sigma a_n b_n$ is also univalent in $0 < |z| < 1$ and even starlike. The above convolution property can be easily obtained for the class $\sigma^*(\alpha)$, by using the coefficient inequality (2).

Let $f(z) = 1/z + \sigma a_n$, $a_n \geq 0$ and $g(z) = 1/z + \sigma b_n$, $b_n \geq 0$ be starlike in $0 < |z| < 1$. In this paper we obtained some properties of convolution $1/z + \sigma a_n b_n$.

In [2] Schield and Silverman obtained some properties of convolutions of univalent function of the form $f(z) = z - \sum_2^\infty |a_n| z^n$.

2. Convolution properties

THEOREM 1. *Let $f(z) = 1/z + \sigma a_n$, $a_n \geq 0$ and $g(z) = 1/z + \sigma b_n$, $b_n \geq 0$ be in $\sigma^*(\alpha)$. Then $f(z)^* g(z) \in \sigma^*(2\alpha/(1+\alpha^2))$.*

Proof. From (2) we have

$$\sum (n+\alpha) a_n \leq 1 - \alpha \quad \text{and} \quad \sum (n+\alpha) b_n \leq 1 - \alpha.$$

In view of (2) we have to find the largest $\beta = \beta(\alpha)$ such that $\sum (n+\beta) a_n b_n \leq 1 - \beta$. We have to show that

$$\to n + \alpha 1 - \alpha a_n \quad \text{and} \quad \to n + \alpha 1 - \alpha b_n \tag{4}$$

imply that

$$\to n + \beta 1 - \beta a_n b_n \quad \text{for all} \quad \beta = \beta(\alpha) = 2\alpha/(1+\alpha^2).$$

From (4) we obtained by means of Cauchy-Schwarz inequality

$$\to n + \alpha 1 - \alpha \sqrt{a_n}\sqrt{b_n}.$$

Hence it suffices to show that

$$\frac{n+\beta}{1-\beta} a_n b_n \leq \frac{n+\alpha}{1-\alpha}\sqrt{a_n}\sqrt{b_n}, \quad \beta = \beta(\alpha) = \frac{2\alpha}{1+\alpha^2},\ n = 1, 2, \ldots$$

or $\sqrt{a_n}\sqrt{b_n} \leq \dfrac{n+\alpha}{n+\beta}\left(\dfrac{1-\beta}{1-\alpha}\right)$ for each n. Hence it suffices to show that

$$\frac{1-\alpha}{n+\alpha} \leq \frac{n+\alpha}{n+\beta}\frac{1-\beta}{1-\alpha}.$$

That is $\beta \leq 1 - \dfrac{(n+1)(1-\alpha)^2}{(n+\alpha)^2 + (1-\alpha)^2}$.

Since the right-hand side of the above inequality is an increasing function of n, taking $n = 1$ we get the result. The results is sharp with equality for

$$f(z) = g(z) = \frac{1}{z} + \frac{1-\alpha}{1+\alpha} z.$$

COROLLARY 1. *For $f(z)$ and $g(z)$ as in Theorem 1, we have $h(z) = 1/z + \sigma\sqrt{a_n}\sqrt{b_n} \in \sigma^*(\alpha)$.*

The result follows from the inequality (5). It is sharp for the same functions as in Theorem 1.

THEOREM 2. *Let $f(z) \in \sigma^*(\alpha)$ and $g(z) \in \sigma^*(\beta)$; then $h(z) = f(z)^* g(z) \in \sigma^*((\alpha+\beta)/(1+\alpha\beta))$.*

The proof is similar to that of Theorem 1.

COROLLARY 2. *Let $f(z) \in \sigma^*(\alpha)$, $g(z) \in \sigma^*(\beta)$ and $h(z) \in \sigma^*(\Gamma)$; then $f(z)^* g(z)^* h(z) \in \sigma^*((\alpha+\beta+\mathit{\Gamma}+\alpha\beta\mathit{\Gamma})/(1+\alpha\beta+\beta\mathit{\Gamma}+\mathit{\Gamma}\alpha))$.*

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