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A NOTE ON CERTAIN CLASS DEFINED BY RUSCHEWEYH DERIVATIVES

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Abstract. The object of this paper is to prove new some results about the class $M(n,\alpha)$ of analytic functions $f(z)$ in the unit disk, defined by Ruscheweyh derivatives $D^n f(z)$. That is, a property of the class $M(n,\alpha)$ and the subordination theorems for Ruscheweyh derivatives $D^n f(z)$ are shown.

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Introduction. Let A denote the class of functions of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \tag{1}$$

which are analytic in the unit disk $U = \{z : | z | < 1\}$. Let the functions

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n \qquad (j = 1, 2)$$

be in the class A; then we define the convolution product $f_1 * f_2(z)$ of $f_1(z)$ and $f_2(z)$ by

$$f_1 * f_2(z) = z + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n.$$

With the aid of the above convolution product, Ruscheweyh [**7**] has introduced a derivative $D^n f(z)$ of $f(z)$ by

$$D^n f(z) = z(1-z)^{-(n+1)} * f(z) \quad (n \in N_0 = \{0, 1, 2, \cdots\})$$

for $f(z) \in A$. Note that

$$D^n f(z) = z(z^{n-1} f(z))^{(n)}/n! \qquad (n \in N_0).$$

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By using the Ruscheweyh derivative $D^n f(z)$, Goel and Sohi [**2**] introduced a subclass $M(n,\alpha)$ of A consisting of functions $f(z)$ which satisfy the condition

$$Re\{D^{n+1}f(z)/z\} > \alpha \qquad (n \in N_0)$$

for some $\alpha(0 \leq \alpha < 1)$, and for all $z \in U$. We observe that the class $M(0,\alpha)$ when $n = 0$ is the subclass of A consisting of functions $f(z)$ satisfying the condition $Re\{f'(z)\} > \alpha$ for some $\alpha(0 \leq \alpha < 1)$, for all $z \in U$.

Let $f(z)$ and $g(z)$ be analytic un the unit disk U. Then a function $f(z)$ is said to be subordinate to $g(z)$ if there exists an analytic function $w(z)$ in the unit disk U satisfying $w(0) = 0$ and $\mid w(z) \mid < 1$ $(z \in U)$ such that $f(x) = g(w(z))$. We denote by $f(z) \prec g(z)$ this relation. If $g(z)$ is univalent in U, then the subordination $f(z) \prec g(z)$ iz equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

The concept of subordination can be traced back to Lindelöf [**3**], but Littlewood [**4**] and Rogosinski [**6**] have introduced the term and discovered the basic relatons.

2. A property of the class $M(n,\alpha)$. Let us recall the following lemma by Nehari [**5**]

LEMMA 1. *Let the function* $\Phi(z)$ *be analytic in the unit disk* U *such that* $\mid \Phi(x) \mid \leq 1$ *for* $z \in U$. *Then*

$$\mid \Phi'(z) \mid \leq (1- \mid \Phi(z) \mid^2)/(1- \mid z \mid^2) \qquad (z \in U).$$

With the aid of Lemma 1, we derive

THEOREM 1 *Let the function* $f(z)$ *defined by* (1) *be in the class* $M(n,\alpha)$ *with* $0 \leq \alpha \leq 1/2$ *and* $n \in N_0$. *Then, for* $z \in U$, *we have*

$$Re\left\{\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right\} \geq \frac{(n+2) - 2(n+3)(1-\alpha) \mid z \mid + (n+2)(1-2\alpha) \mid z \mid^2}{(n+2)(1- \mid z \mid)\{1-(1-2\alpha) \mid z \mid\}}$$

Proof. Since $f(z) \in M(n,\alpha)$ implies

$$D^{n+1}f(z)/z \prec (1 + (1 - 2(\alpha)z)/(1-z) \qquad (z \in U),$$

there exists an analytic function $w(z)$ in the unit disk U with $w(0) = 0$ and $\mid w(z) \leq 1 (z \in U)$ such that

(2) $$D^{n+1}f(z)/z = (1 + (1-2\alpha)w(z))/(1-w(z)).$$

Applying the Schwarz lemma, (2) can be written as

(3) $$D^{n+1}f(z)/z = (1 + (1-2\alpha)z\Phi(z))/(1-z\Phi(z)) \qquad (z \in U),$$

where $\Phi(z)$ is analytic in the unit disk U and satisfies $\mid \Phi(z) \mid \leq 1$ for $z \in U$. Making the logarithmic differentiations of both sides in (3), and using the identity

(4) $$z(D^{n+1}f(z))' = (n+2)D^{n+2}f(z) - (n+1)D^{n+1}f(z),$$

we obtain

$$\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} = 1 + \frac{2(1-\alpha)\{z^2\Phi'(z) + z\Phi(z)\}}{(1 - z\Phi(z))\{1 + (1-2\alpha)z\Phi(z)\}}, \quad \text{or}$$

$$\frac{D^{n+2}f(z)}{D^{n+1}f(z)} = 1 + \frac{2(1-\alpha)\{z^2\Phi'(z) + z\Phi(z)\}}{(n+2)(1 - z\Phi(z))\{1 + (1-2\alpha)z\Phi(z)\}}, \tag{5}$$

Therefore, from Lemma 1 and (5), it follows that

$$\begin{aligned} Re\left\{\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right\} &\geq 1 - \frac{2(1-\alpha)\{\mid z^2\Phi'(z)\mid + \mid z\Phi(z)\mid\}}{(n+2)(1-\mid z\Phi(z)\mid)\{1-(1-2\alpha)\mid z\Phi(z)\mid\}}, \\ &\geq 1 - \frac{2(1-\alpha)\mid z\mid(\mid z\mid + \mid \phi(z)\mid)}{(n+2)(1-\mid z\mid^2)\{1-(1-2\alpha)\mid z\Phi(z)\mid\}} \\ &\geq 1 - \frac{2(1-\alpha)\mid z\mid}{(n+2)(1-\mid z\mid)\{1-(1-2\alpha)\mid z\mid\}} \\ &= \frac{(n+2) - 2(n+3)(1-\alpha)\mid z\mid + (n+2)(1-2\alpha)\mid z\mid^2}{(n+2)(1-\mid z\mid)\{1-(1-2\alpha)\mid z\mid\}} \end{aligned}$$

which completes the assertion of the Theorem.

Taking $n = 0$ in Theorem 1, we have

COROLLARY 1. *Let the function $f(z)$, defined by* (1), *be in the class $M(0,\alpha)$ for $0 \leq \alpha \leq 1/2$. Then, for $z \in U$, we have*

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \frac{1 - 3(1-\alpha)\mid z\mid + (1-2\alpha)\mid z\mid^2}{(1-\mid z\mid)\{1-(1-2\alpha)\mid z\mid\}}.$$

3. Subordination Results. We need the following results by Eenigenburg, Miller, Mocanu and Reade [**1**].

LEMMA 2 *Let the function $p(z)$ and $h(z)$ be analytic in the unit disk U such that $p(0) = h(0) = 1$. Further, let $h(z)$ be a convex and univalent function in the unit disk U satisying the condition $Re\{\beta h(z) + \gamma\} > 0$ for complex numbers β, γ and for all $z \in U$. If $p(z), h(z), \beta$ and γ satisfy the Briot-Bouquet differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), \ \ \textit{then} \ \ p(z) \prec h(z) \quad (z \in U).$$

LEMMA 3. *Under the hypotheses of Lemma 2, if the Briot-Bouquet differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z) \qquad (q(0) = 1)$$

has a univalent solution, then $p(z) \prec q(z) \prec h(z)$. Furthermore, $q(z)$ is the best dominant

Applying the lemmas above, we derive

THEOREM 2. *Let a function $h(z)$ be convex and univalent in the unit disk U such that $h(0) = 1$ and $Re\{h(z)\} > 0$ for $z \in U$. For $f(z)$ belonging to A and $n \in N_0$, if*

$$D^{n+2}f(z)/z \prec h(z), \text{ then } D^{n+1}f(z)/z \prec h(z) \quad (z \in U).$$

Proof. Defining the function $p(z)$ by

$$p(z) = D^{n+1}f(z)/z, \tag{6}$$

we know that $p(z)$ is analytic in the unit disk U with $p(0) = 1$. Differentiating both sides of (6), and applying (4), we have

$$(n+2)D^{n+2}f(z)/z - n(n+1)D^{n+1}f(z)/z = p(z) + zp'(z),$$

that is

$$D^{n+2}f(z)/z = p(z) + zp'(z)/(n+2) \prec h(z).$$

Consequently, by taking $\beta = 0$ and $\gamma = n+2$ in Lemma 2, we complete the proof of Theorem 2.

Letting $n = 0$ in Theorem 2, we have

COROLLARY 2. *Under the hypothesis in Theorem 2,*

if

$$f'(z)xf''(z)/2 \prec h(z), \text{ then } f'(z) \prec h(z) \quad (z \in U).$$

Further, by putting $h(z) = \{1 + (1-2\alpha)z\}/(1-z)$ in Theorem 2, we have

COROLLARY 3. [**2**] *For $0 \le \alpha \le 1$ and $n \in N_0$, we have $M(n+1, \alpha) \subset M(n, \alpha)$.*

Finally, we prove

THEOREM 3. *Under the hypotheses of Theorem 2, if the Briot-Bouquet differential equation*

$$q(z) + zq'(z)/(n+2) = h(z) \qquad (q(0) = 1)$$

has a univalent solution, then

$$D^{n+1}f(z)/z \prec q(z) \prec h(z) \tag{7}$$

Furthermore, $q(z)$ is the best dominant.

Proof. If we replace $p(z$ by $D^{n+1}f(z)/z$ and take $\beta = 0$ and $\gamma = n+2$ in Lemmas 2 and 3, we see that the result follows from (7).

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