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A NOTE ON THE GRAPH EQUATION $C(L(G)) = L(C(G))$

Slobodan K. Simić

Abstract. We find all solutions to the graph equation from the title. The same equation was already treated in the literature, but solved only partially.

We will consider only finite, undirected graphs, without loops or multiple lines. All definitions, not given here, may be found in [3]. An intersection graph of a nonempty family of nonempty sets is a graph whose points are in one-to-one correspondence with the members of the family, with two points being adjacent if, and only if, the corresponding sets have a nonempty intersection. $L(G)$ (line graph of G) is an intersection graph of the family of lines of G, whereas $C(G)$ (clique graph of G) is an intersection graph of a family of cliques (maximal complete subgraphs) of G.

The clique graphs of line graphs have been studied in the literature by several authors in different contexts. The following result is given in [4].

THEOREM 1. *If G is a connected graph containing no triangles and at least three points, then $C(L(G))$ is a graph obtained from G by deleting the points of degree one.*

The natural generalization of this theorem refers to graphs which are not necessarily triangle-free.

THEOREM 2. *If G ia a connected graph with at least three points, then $C(L(G))$ is a graph obtained from G as follows:*

(a) *all points of degree less than two are deleted from G, and also any point of degree two, if it belongs to a triangle;* (b) *to every triangle of G a new point is added which is adjacent to all points of a triangle that are not deleted by* (a); (c) *if two triangles of G have a line in common, then the corresponding points, added by* (b), *are adjacent.*

Proof. The points of $C(L(G))$ are, in fact, the cliques of $L(G)$. By a theorem of Krausz (see [3], p. 74), the lines of $L(G)$ can be partitioned into complete

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subgraphs (or c-subgraphs) in such a way that each line meets at most two c--subgraphs; namely, two different c-subgraphs have at most one point in common, while no three of them meet the same point. Moreover, this partition, due to a theorem of Whitney (see also [3], p. 72), is esentially unique; the only exception appears when G equals $K_{1,3}$ or K_3, i.e. $L(G)$ equals K_3. If G is one of the graphs $K_{1,3}+x$, K_4-x and K_4 (see [7]), the partitions are not unique, but are determined up to automorphisms of $L(G)$. In all these cases the theorem can be easily verified. For all other possibilities, the partitions are unique. The c-subgraphs with three or more points cannot be extended; they are just the cliques of $L(G)$. The c-subgraphs with two points are not necessarily the cliques. Namely, some c-subgraphs on two points can be extended but only to a triangle. In that case, each line of this triangle belongs to a different c-subgraph. So we can distingush in $L(G)$ two kind of cliques: those representing the c-subgraphs (type *one*) and the other being the extensions of c-subgraphs (type *two*). Having in mind how $L(G)$ is obtained from G, we can say that each clique of type *one* corresponds (in one-to-one manner) to a point of G whose degree is at least two and which is not a point of degree two belonging to a triangle; the cliques of type *two* are in the same correspondence with the triangles of G.

Let us now form an intersection graph of all these cliques. The intersection graph on a subfamily containing only the cliques of type *one* is just the graph obtained from G using (a). Each clique of type *two* corresponds to a point added by (b), while two of these points are adjacent in accordance with (c).

This completes the proof.

Remark. The theorem above is already contained in [7]. Actually the same theorem is given in [1], but stated as a characterization theorem for CL graphs (clique graphs on line graphs); in [5] it is stated as an algorithm (polynomial of order $O(n^3)$) for constructing the clique graph of a line graph. It is interesting to note that the author of [5] was not aware of results from [1].

In what follows, we will focus our attention on graph equations; see [2], for more details. The graph equations involving clique graphs and line graphs are already encountered in the literature. For example, in [6] the graph equation $C(L(G)) = G$ has been completely solved. The equation

$$L(C(G)) = C(L(G)), \tag{1}$$

i.e. the equation from the title, is already treated in [1], but as we will now exibit, solved only partially. Actually, in [1] only connected solutions were found, and besides, only those disconnected solutions which could be immediately derived from the former ones.

To simplify our investigations, we first note that if G_1 and G_2 are solutions for (1), then their union $G_1 \cup G_2$ is also a solution. So, we may restrict ourselves only to solutions which cannot be obtained as a union of some other solutions. As in [8], we call them fundamental solutions.

THEOREM 3. *All fundamental solutions to the graph equation* (1) *are the following graphs:*

1° *path* P_n, *where* $n \geq 3$ (Fig. 1.a); 2° *cycle* C_n, *where* $n \geq 4$ (Fig. 1.b);
3° *the graph of* Fig. 1.c; 4° *the graph of* Fig. 1.d.

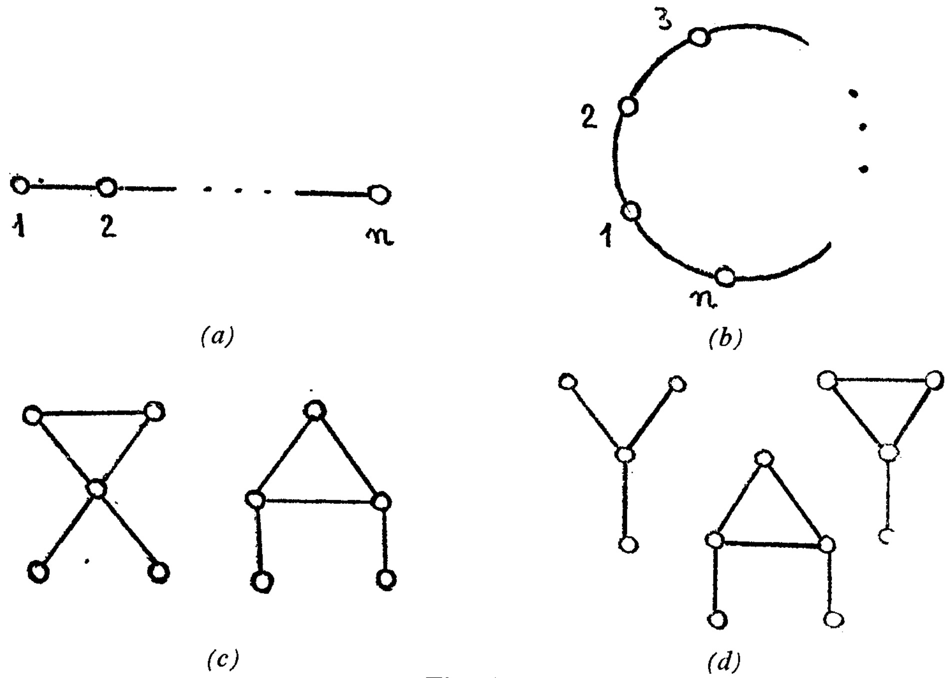

Fig. 1.

Proof. We first deduce that the ten graphs of Fig. 2 are forbidden in G as induced subgraphs. Actually, each of them gives some essential information about the clique structure of G.

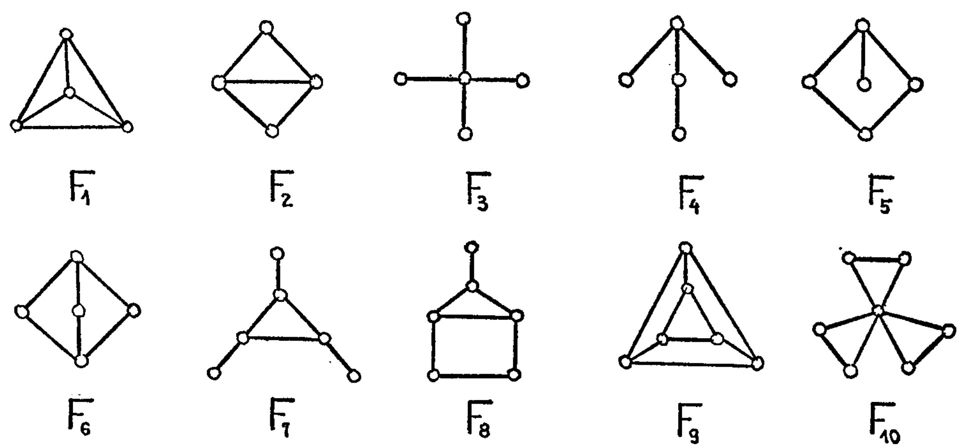

Fig. 2.

(i) F_1 is forbidden: By Theorem 2, to each triangle of $F_1(= K_4)$ there corresponds in $C(L(G))$ a point adjacent to all points of that triangle. Thus any such point together with K_4 induces $K_5 - x$ in $C(L(G))$, implying that $K_5 - x$ appears in $L(C(G))$ as an induced subgraph. The latter contradicts a theorem of Beineke (see [3], p. 74).

(ii) F_2 is forbidden: By Theorem 2, if $F_2(= K_4 - x)$ is a component of G, then K_4 is a component of $C(L(G))$, and of $L(C(G))$ as well. But then $K_{1,4}$ is a component of $C(G)$, which in turn implies that the central point of $K_{1,4}$ corresponds to a clique of G with at least four points. The latter contradicts (i). So, if F_2 is an induced subgraph of G, there must exist at least one point of G, say u, adjacent to at least one point of F_2. If u is adjacent to some point of F_2 whose degree (in F_2) is two, then, by Theorem 2, $K_5 - x$ appears again in $C(L(G))$. Otherwise, by Theorem 2, it follows that $C(L(G))$ is as given in Fig. 3.a, where $k \geq 4$. Since the latter graph needs to be equal to $L(C(G))$, its root graph $(= C(G))$ has the structure as shown in Fig. 3.b. If $k > 4$, we are done; the point w corresponds in G to a clique with at least four points. The same follows for $k = 4$ if p and q are nonadjacent. Otherwise, if p and q are adjacent, there must exist a point, say s, in $C(L(G))$ (see Fig. 3.a) adjacent to ν_1 and ν_2, but not to ν_i $(i > 2)$.

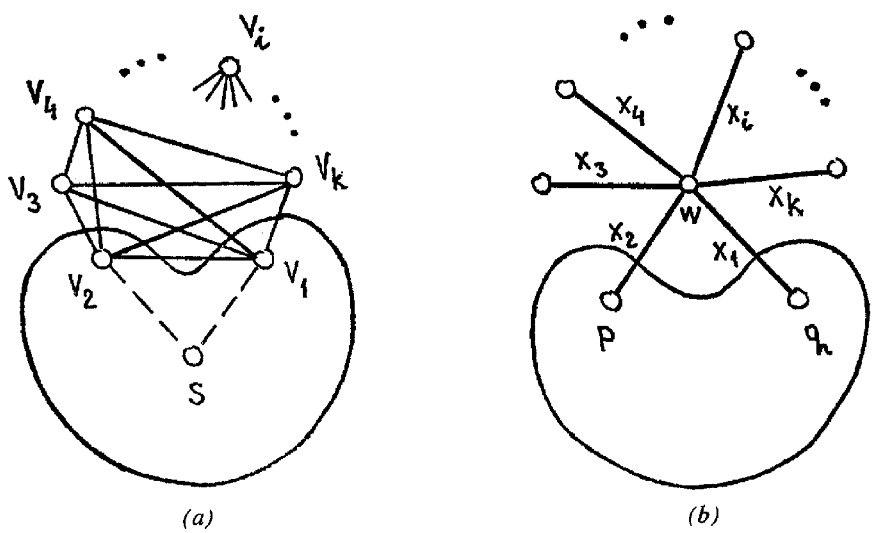

Fig. 3.

Now, regarding s as a point of $C(L(C))$, it follows that it must be a point already existing in G; otherwise, by Theorem 2, it would be adjacent to ν_1 $(i > 2)$. Also, as already seen, a point such that as s must be of degree two in G and adjacent only to ν_1 and ν_2. But then, by Theorem 2, it cannot be a point of $C(L(G))$.

At this moment, assuming (i) and (ii) we have: all cliques of G are either lines or triangles; any two of them have at most one point in common.

(iii) F_3 is forbidden: Suppose not, i.e. $F_3 (= K_{1,4})$ appears in G. Then K_4 is a subgraph of $C(G)$, while $L(K_4)$ (octahedron) is an induced subgraph of $L(C(G))$, and also of $C(L(G))$. Thus, in particular, $K_4 - x$ is an induced subgraph of $C(L(G))$. The latter is impossible as can easily be seen by using Theorem 2. Indeed, we now get that either G contains $K_4 - x$, or that a point added by (b) is adjacent to three points of G which are not on the triangle.

Taking into acount (iii), we now get: at most three cliques of G have a point in common. Next we proceed to show a somewhat stronger claim: any clique of G meets at most two cliques.

(iv) F_4 is forbidden: Suppose not, i.e. F_4 appears in G. Then all lines of F_4 belong to different cliques of G. Therefore, $K_{1,3} + x (= C(F_3))$ is an induced subgraph of $C(G)$, while $K_4 - x (= L(K_{1,3} + x))$ is an induced subgraph of $L(C(G))$. This implies that $K_4 - x$ must be an induced subgraph of $C(L(G))$. The latter is impossible, as pointed in (iii).

(v) F_5 is forbidden: The arguments are the same as for (iv).

(vi) F_6 is forbidden: See (v).

We now notice that if some clique of size two (line) meets three other cliques, then one of the graphs as in (iv)-(vi) appears in G as an induced subgraph. This proves the first "half" of our claim.

(vii) F_7 is forbidden: Suppose not, i.e. F_7 appears in G. By Theorem 2, K_4 is now an induced subgraph of $C(L(G))$. Let t be the point added to a triangle of F_7. Since t has no more neighbours except those on the triangle, there is a point in $C(G)$ (root graph of $L(C(G))$), say u, of degree four with just one hanging line. Let ν be the other endpoint of this hanging line, while w_1, w_2, w_3 the remaining points adjacent to u. Now the cliques of G that correspond to u and ν, together with two cliques which correspond to an appropriately chosen pair of points among w_1, w_2, w_3 induce F_4 in G.

(viii) F_8 is forbidden: The arguments are the same as for (vii).

(ix) F_9 is forbidden: See (viii).

From (vii)-(ix) it follows that the second "half" of our claim also holds.

(x) F_{10} is forbidden: Using Theorem 2, we easily get that $K_{1,3}$ is an induced subgraph of $C(L(G))$ and of $L(C(G))$ as well. The latter is impossible for line graphs.

Now let $G_1, G_2, \ldots, G_k$ be mutually different components of G which satisfy

$$C(L(G_1)) = L(C(G_2)),\ C(L(G_2)) = L(C(G_3)), \ldots, C(L(G_k)) = L(C(G_1)).$$

The following cases can now be observed:

Case 1: No three cliques of G_i $(i = 1, \ldots, k)$ have a point in common. Then all graphs $L(C(G_i))$ are either paths or cycles. By Theorem 2, the same holds for any graph $C(L(G_i))$. So we can easily deduce that $k = 1$, and that G itself is a path or a cycle of appropriate length (see 1° and 2°).

Case 2: Suppose now that there exists a component among $G_1, G_2, \ldots, G_k$, say G_1, having a point which meets just three cliques. If so, there are no more cliques in G_1; otherwise, some of these cliques would meet at least three other cliques. By (x), at most two among three cliques are triangles. It is now a matter of routine to deduce that G is one of the graphs given by 3° or 4°.

This proves the theorem.

REFERENCES

[1] R.C. Brigham, P.Z. Chinn, R.D. Dutton, *On the clique graph of the line graph*, Proc. Fifteen Southeast Conf. Graph Theory and Computing (Baton Rouge, La., 1984); Congressus Numerantium **43** (1984), pp. 175–189.

[2] D.M. Cevtković, S.K. Simić, *A bibliography on graph equations*, J. Graph Theory **3** (1979), 311-324.

[3] F. Harary, *Graph Theory*, Addison-Wesley, Reading, Massachusets, 1972.

[4] S. Hedetniemi, P.J. Slater, *Line graphs of triangless graphs and iterated clique graphs*, in: *Graph theory and Applications*, Springer-Verlag, Berlin-Heidelberg-New York, 1972, pp. 139–147.

[5] B. Hedman, *A polynomial algorithm for constructing the clique graph of a line graph*, Discrete Appl. Math. **15** (1986), 61–66.

[6] I. Sato, *On a few clique-graph equations*, TRU Mathematics **14-2** (1978), 31–36.

[7] S.K. Simić, *Contributions to the investigations of graph operations*, Ph. D. Thesis (in Serbo-Croation), Beograde, 1979.

[8] S.K. Simić, *Graph equations for line graphs and n-th distance graphs*, Publ. Inst. Math. (Beograd), **33** (**47**) (1983), 203–216.

[9] H. Whitney, *Congruent graphs and the connectivity of graphs*, Amer. J. Math. **54** (1932), 150–168.

Elektroterhnički fakultet
11000 Beograd
Jugoslavija

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