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DISCRETE APPOXIMATION IN THE INNOVATION THEORY OF SECOND-ORDER CONTINUOUS PROCESSES

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Abstract. A simple test for the multiplicity of a given process is proposed. The consistency of a discrete approximation of this test is proved. A statistical approach is also proposed.

Introduction. Let $\{X(t), 0 \leq t \leq 1\}$ be a real second-order continuous process, $\mathbf{E}X(t) = 0$ and let $\mathcal{L}_2(X;t)$ be the linear closure (in the mean square convergence) of $\{X(u), u \leq t\}$. $\mathcal{L}_2(X) = \mathrm{Cl}\big(\bigcup_t \mathcal{L}_2(X;t)\big)$ is the separable Hilbert space with the inner product $\langle X, Y\rangle = \mathbf{E}XY$. Assume that $\{X(t)\}$ is purely--nondeterministic process, i.e., $\bigcap_t \mathcal{L}_2(X;t) = 0$.

The Cramer representation of $\{X(t)\}$, [**1**], is

$$X(t) = \sum_{k=1}^{N} \int_0^t g_k(t,u)\, dY_k(u),\ N \leq \infty, \tag{1}$$

where: 1. The so-called innovation processes $\{Y_k(t), 0 \leq t \leq 1\}$, $k = 1, \dots, N$, are mutually orthogonal wide-sense martingales for which $\mathcal{L}_2(X;t) = \oplus \sum_{k=1}^{N} \mathcal{L}_2(Y_k;t)$; 2. The measures $dF(t) = d\|Y_k(t)\|^2$, $k = 1, \dots, N$, are ordered by the absolute continuity $dF_1 \geq dF_2 \geq \cdots \geq dF_N$. Let ρ_k be the class of all measures equivalent (by the absolute continuity) to dF_k: The chain

$$\rho_1 \geq \rho_2 \geq \cdots \geq \rho_N \tag{2}$$

is called the spectral type of $\{X(t)\}$ and N is the multiplicity of $\{X(t)\}$. The multiplicity function $N(t)$, $0 \leq t \leq 1$ is the number of $F_k(s)$, $0 \leq s \leq 1$ having $s = t$ as the increasing point, $N = \sup_t N(t)$. The representation (1) is not unique, but the spectral type (2) is uniquely determined by the correlation function $\gamma(s,t) = \mathbf{E}X(s)X(t)$. It was shown in [**1**] that for any chain (2) there exists a continuous process having (2) as its spectral type. We may suppose that $\{X(t)\}$ is a Gaussian process, because we are in the frame of the correlation theory.

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Let P_t be the projection operator onto $\mathcal{L}_2(X;t)$. Consider the process $\{Z_1(t),\ 0 \le t \le 1\}$ defined by $Z_1(t) = P_tX(1) = \sum_{k=1}^N \int_0^t g_k(1,u)dY_k(u)$. It is evident that $\{Z_1(t)\}$ is the wide-sense martingale and that $\mathcal{L}_2(Z_1;t)$ reduces $\{P_s,\ 0 \le s \le 1\}$. Also the measure dG_1 generated by $G_1(t) = \|Z_1(t)\|^2 = \sum_{k=1}^N \int_0^t g^2(1,u)\,dF_k(u)$ belongs to the maximal class ρ_1 in (2). Using $\{Z_1(t)\}$ as one innovation process we rewrite the Cramer representation of $\{X(t)\}$ by $X(t) = \sum_{k=1}^N \int_0^t h_k(t,n)\,dZ_k(n)$. Let Q_t be the projection operator onto $\mathcal{L}_2(Z_1;t)$. Consider

$$\delta(t) = X(t) - Q_tX(t) = \sum_{k=2}^N \int_0^t h_k(t,n)dZ_k(n) \quad \text{and}$$

$$d^2(t) = \|\delta(t)\|^2 = \sum_{k=2}^N \int_0^t h_k^2(t,n)dG_k(n), \quad G_k(t) = \|Z_k(t)\|^2.$$

Evidently: *If $d^2(t) > 0$ for some $0 < t < 1$, than $N \ge 2$. If $d^2(t) = 0$ for some $0 < t < 1$ then the spectral function $N(s) = 1$ for $0 \le s \le t$. If $N(0) = N$ then the condition $d^2(t) > 0$ for all $0 \le t \le 1$, is also necessary for $N \ge 2$.*

Discrete approximation and its consistency. In this section we find one discrete approximation $\delta(t,n)$ of $\delta(t)$ such that $\|\delta(t;n) - \delta(t)\|^2 \to 0$, $n \to \infty$ for each $t > 0$. In [**2**, §8], we find the motivation for such approximation.

Consider for $n = 1,2,\dots$, the partition of $[0,1]$ by the points $k2^{-n}$, $k = 1,\dots,2^n$. Let $\mathcal{L}_2(X;t;n)$ be the linear closure over $\{X(j2^{-n}), j2^{-n} \le t\}$. We conclude that $\mathcal{L}_2(X;t) = \mathrm{Cl}\big(\bigcup_n \mathcal{L}_2(X;t;n)\big)$ by the separability of $\mathcal{L}_2(X)$ and $\mathcal{L}_2(X;t;1) \subset \mathcal{L}_2(X;t;2) \subset \dots$. Denote by P_{tn} the projection operator onto $\mathcal{L}_2(X;t;n)$ and consider the process $\{Z_{1n}(t),\ 0{\le}t{\le}1\}$ defined by $Z_{1n}(t) = P_{tn}X(1)$. Evidently, $\|Z_{1n}(t) - Z_1(t)\|^2 \to 0$, $n \to \infty$ for fixed t.

Example. Let $\phi(t)$, $0 \le t \le 1$, $\phi(1) = 1$ be a non-constant continuous function such that at $t = t_0$, $0 < t_0 < 1$, $\phi(t_0) \ne 1$

$$[\phi(t_0) - \phi(t_0 - h)]^2/h \to \infty, \quad h \downarrow 0. \tag{3}$$

Let $\{X(t),\ 0 \le t \le 1\}$ be defined by

$$X(t) = W_1(t) + \phi(t)W_2(t), \tag{4}$$

where $\{W_i(t),\ 0 \le t \le 1\}$, $i = 1,2$, are independent standard Wiener processes.

The multiplicity of more general processes of this form was studied in [**3**]. Consider the projection $Z_{1n}(t_0)$ of $X(1)$ onto $\mathcal{L}_2(X;t_0;n)$, $t_0 = k_0 2^{-n}$. It is easy to see that, $(h = 2^{-n})$

$$\langle X(1) - [aX(t_0) + bX(t_0 - h)], X(u)\rangle = 0$$

for all $u \le t_0 - h$ if

$$a = \big(1 - \phi(t_0 - h)\big)/\Delta\phi_0, \quad b = \big(\phi(t_0) - 1\big)/\Delta\phi_0, \quad \Delta\phi_0 = \phi(t_0) - \phi(t_0 - h).$$

Rewrite

$$Z_{1n}(t_0) = aX(t_0) + bX(t_0 - h) = X(t_0) + [1 - \phi(t_0)][X(t_0) - X(t_0 - h)]/\Delta\phi_0.$$

There exists, under the assumption (3), the mean-square limit

$$[X(t_0) - X(t_0 - h)]/\Delta\phi_0 \to X'_\phi(t_0),\ h \downarrow 0 \text{ and } X'{}_\phi(t_0) = W_z(t_0). \tag{5}$$

Indeed,

$$\begin{aligned} &\|[X(t_0) - X(t_0 - h)]/\Delta\phi_0 - W_2(t_0)\|^2 \\ &\quad = 1/(\Delta\phi_0)^2 \cdot \|W_1(t_0) - W_1(t_0 - h) + \phi_0(t_0 - h)[W_2(t_0) - W_2(t_0 - h)]\|^2 \\ &\quad = h/(\Delta\phi_0)^2 \cdot [1 - \phi^2(t_0 - h)] \to 0, \quad h \downarrow 0. \end{aligned}$$

So the innovation process $\{Z_1(t)\}$ at $t = t_0$ is

$$Z_1(t_0) = \lim_{n\to\infty} Z_{1n}(t) = X(t_0) + [1 - \phi(t_0)]X'{}_\phi(t_0) = W_1(t_0) + W_2(t_0). \tag{6}$$

We conclude, from (5) and (6), that $W_1(t_0)$ and $W_2(t_0)$ belong to $\mathcal{L}_2(X; t_0)$. If we state $Z_2(t_0) = W_1(t_0) - W_2(t_0)$ we have $Z_2(t_0) \in \mathcal{L}_2(X; t_0)$ and $Z_1(t_0) \perp Z_2(t_0)$. From (4) we obtain

$$X(t_0) = [1 + \phi(t_0)]/2 \cdot Z_1(t_0) + [1 - \phi(t_0)]/2 \cdot Z_2(t_0). \tag{7}$$

Finally, we have from $Q(t_0)X(t_0) = [1 + \phi(t_0)]/2 \cdot Z_2(t_0)$ that

$$d^2(t_0) = \|\delta(t_0)\|^2 = \|[1 - \phi(t_0)]/2 \cdot Z_2(t_0)\|^2 = [1 - \phi(t_0)]^2/2 \cdot t_0 > 0.$$

We conclude that the multiplicity N of $\{X(t)\}$ is greater than one. Actually (7) is the Cramer representation of $\{X(t)\}$ at the point $t = t_0$, but we may not conclude that $\{Z_1(t)\}$ and $\{Z_2(t)\}$, $Z_2(t) = W_1(t) - W_2(t)$ are the innovation processes of $\{X(t)\}$. We do not even know whether $G_1(t)$ is continuous.

We assume in the rest of the paper that $G_1(t) = \|Z_1(t)\|^2$, $0 \le t \le 1$, is a continuous function. Under this assumption the satement that pointwise convergence $\|Z_{1n}(t) - Z_1(t)\| \to 0$, $n \to \infty$ becomes uniform, is easily proved.

Let Q_{tn} be the projection operator onto $\mathcal{L}_2(Z_{1n}; t)$.

PROPOSITION. *For fixed t $\|Q_{tn}X(t) - Q_tX(t)\| \to 0$, $n \to \infty$.*

Proof. For arbitrary $\varepsilon > 0$ there exists a finite partition $\{\Delta_i : i = 1, \dots, M(t)\}$ of $[0, t]$, such that $\|Q_tX(t) - Q_t^\Delta X(t)\| < \varepsilon$, where Q_t^Δ is the projection operator onto $\{Z_1(\Delta_i) : i = 1, \dots, M(t)\}$, ($\Delta = [\alpha, \beta]$, $Z(\Delta) = Z(\alpha) - Z(\beta)$). Denote $a = \min_i \|Z_1(\Delta_i)\| > 0$, $\eta_i = Z_1(\Delta_i)/\|Z_1(\Delta_i)\|$, $\eta_{in} = Z_{1n}(\Delta)/\|Z_{1n}(\Delta)\|$. From $Z_{1n}(t) \rightrightarrows Z_1(t)$ follows that for each $\varepsilon' > 0$ and all $n \geq n'(\varepsilon)$: $\|Z_1(\Delta_i) - Z_{1n}(\Delta_i)\| < \varepsilon'$ or $\big\| \|Z_1(\Delta_i)\|\eta_i - \|Z_{1n}(\Delta_i)\|\eta_{in} \big\| < \varepsilon'$. So

$$\|Z_{1n}(\Delta_i)\| = \|Z_1(\Delta_i)\| + \theta_i, \quad |\theta_i| \leq \varepsilon', \quad \text{and}$$
$$\big\| \|Z_1(\Delta_i)\|(\eta_i - \eta_{in}) - \theta_i \eta_{in} \big\| \leq \varepsilon'.$$

Finally, $\|\eta_i - \eta_{in}\| < 2\varepsilon'/\|Z_1(\Delta_i)\| \leq 2\varepsilon'/a$ for all $n \geq n(\varepsilon')$. Since, $Q_t^\Delta X(t) = \sum_{i=1}^{M(t)} \langle X(t), \eta_i \rangle \eta_i$ and $Q_{tn}X(t) = \sum_{i=1}^{M(t)} \langle X(t), \eta_{in} \rangle \eta_{in}$ we have

$$\begin{aligned} &\|Q_t^\Delta X(t) - Q_t X(t)\| \\ &\leq \sum_{i=1}^{M(t)} \Big[\big\| (\langle X(t), \eta_i \rangle - \langle X(t), \eta_{in} \rangle) \eta_{in} \big\| + \big\| \langle X(t), \eta_{in} \rangle (\eta_i - \eta_{in}) \big\| \Big] \\ &\leq \sum_{i=1}^{M(t)} 2\|X(t)\| \, \|\eta_i - \eta_{in}\| \leq 4\eta X(t)(M(t)/a) \cdot \varepsilon'. \end{aligned}$$

This way $\|Q_t X(t) - Q_{tn} X(t)\| \leq \varepsilon + 4\|X(t)\|(M(t)/a) \cdot \varepsilon'$. For any $\varepsilon_0 > 0$ we choose, say, $\varepsilon = \varepsilon_0/2$ and we find $\{\Delta_i\}, M(t), a$. Then we have for sufficiently small $\varepsilon' = \varepsilon'(\varepsilon_0, M(t), a)$, that $4\|X(t)\|(M(t)/a) \cdot \varepsilon' \leq \varepsilon_0/2$ for all $n \geq n_1(\varepsilon')$. Finally $\|Q_t X(t) - Q_{tn} X(t)\| \leq \varepsilon_0$ for all $n \geq n_2(\varepsilon_0)$.

One statistical approach. Let t, $0 \leq t \leq 1$, be fixed, say, $t = 1/2$. Consider $d^2 = d^2(1/2)$ and $d_n^2 = \|X(t) - Q_{1/2,n}X(1/2)\|^2$. Then $e_n^2 = d_n^2 - d^2 = \|Q_{1/2,n}X(1/2) - Q_{1/2}X(1/2)\|^2$ is the square error of the approximation.

We consider the following admissible family $\mathcal{X}$ of the processes $\{X(t)\}$: The multiplicity function satisfies $N(0) = N$. If the multipliplicity $N = 1$ (i.e. $d^2 = 0$) for $\{X(t)\} \in \mathcal{X}$ then $e_{n'}^2 < e_n^2$, $n' > n \geq n_0$. If $N \geq 2$ (i.e. $d^2 > 0$) then the error e_n^2 is considerably smaller than d^2 i.e. $d_n^2/d^2 \approx 1$ for $n \geq n_0$.

Starting from one sample $X^{(i)}(2^{-n}), X^{(i)}(2 \cdot 2^{-n}), \dots, X^{(i)}(s), \dots, X^{(i)}(1)$, $i = 1, \dots, m$, $m > 2^n$, $n \geq n_0$, we estimate $Z_{1n}(s)$ as the linear regression of $X(1)$ on $X(2^{-n}), \dots, X(s)$ for $s = 2^{-n}, \dots, 1$. Let $Z_{1n}^*(s)$ be this estimation. Then, considering $Z_{in}^*(2^{-n}), \dots, Z_{in}^*(2^{-1})$, $i = 1, \dots, m$, as the sample of $\{Z_{1n}(t)\}$ we estimate $Q_{1/2,n}X(1/2)$ as the linear regression of $X(1/2)$ on $Z_{1n}(2^{-n}), \dots, Z_{1n}(2^{-1})$. Let S_n^2 be the estimation of the mean square error d_n^2 of this regression. Then mS_n^2/d_n^2 has χ^2-distribution with $m - 2^{n-1} - 1$ degrees of freedom.

Let the null hypothesis be $H_0(N \geq 2)$ and the alternative hypothesis be $H_1(N = 1)$. Consider two partitions $n(2)$ and $n(1)$, $n(2) > n(1) \geq n_0$. In our case of the admissible family $\mathcal{X}$ testing $H_0(N \geq 2)$ against $H_1(N = 1)$ becomes testing $H_0(d_{n(2)} = d_{n(1)})$ against $H_1(d_{n(2)} < d_{n(1)})$. Using two independent samples of the sizes $m(2) > 2^{n(2)}$ and $m(1) > 2^{n(1)}$, we proceed with the standard Fisher F-test.

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