

COMPLETENESS THEOREM FOR A MONADIC LOGIC
WITH BOTH FIRST-ORDER AND PROBABILITY QUANTIFIERS

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Abstract. We prove a completeness theorem for a logic with both probability and first-order quantifiers in the case when the basic language contains only unary relation symbols.

Let $\mathcal{A} \subseteq HC$ be an admissible set which contains infinite ordinals and let L be a nonempty \mathcal{A} -recursive language which contains only unary relation symbols; HC denotes, as usual, the set of hereditarily countable sets.

Definition 1. The set of formulas of $(L_{\omega P \exists})_{\mathcal{A}P}$ is the least set such that: (i) each atomic formula of first-order logic without equality symbol is a formula of $(L_{\omega P \exists})_{\mathcal{A}P}$; (ii) if φ is a formula, then $\neg\varphi$ is a formula; (iii) if $\Phi \in \mathcal{A}$ is a set of formulas, then $\bigwedge\Phi$ is a formula; (iv) if φ is a finite formula, then $(\exists v_n)\varphi$ is a formula; (v) if φ is a formula and $r \in \mathcal{A} \cap [0, 1]$, then $(P\mathbf{x} \geq r)\varphi$ is a formula.

Abbreviations $(P\mathbf{x} \leq r)$, $(P\mathbf{x} = r)$ and $(\forall v_n)$ are introduced as usual.

Definition 2. A probability structure for L is a structure (\mathfrak{A}, μ) where \mathfrak{A} is a first-order structure for L (with universe A), and μ is a σ -additive probability measure on A such that each relation of \mathfrak{A} is σ -measurable.

We can define in the usual way satisfaction relation in a probability structure; here μ^n denotes the n -fold product of μ 's.

Thus: $(\mathfrak{A}, \mu) \models (P\mathbf{x} \geq r)\varphi(\mathbf{x}, \mathbf{a})$ iff $\mu^n\{\mathbf{b} \in A^n \mid (\mathfrak{A}, \mu) \models \varphi(\mathbf{b}, \mathbf{a})\} \geq r$.

The axioms for $(L_{\omega P \exists})_{\mathcal{A}P}$ are the axioms A1–A6 and B1–B6 from [K] with the usual first-order axioms. The rules of inference are the rules R1–R3 from [K] with the usual first-order generalization added.

SOUNDNESS THEOREM. *If the set Φ of sentences of $(L_{\omega P \exists})_{\mathcal{A}P}$ has a model, then it is consistent.*

LEMMA 1. *Each $(L_{\omega P \exists})_{\mathcal{A}P}$ sentence is $(L_{\omega P \exists})_{\mathcal{A}P}$ -equivalent to a σ -Boolean combination of finite sentences.*

Proof. The proof can be obtained in the similar way as the proof of the Normal Form Theorem from [H2]. So we omit it.

The notion of a weak structure $(\mathfrak{A}, \mu_n)_{n \in \omega}$ can be introduced as in [H1].

LEMMA 2. *A sentence of $(L_{\omega P \exists})_{AP}$ is consistent if and only if it has a weak model in which each theorem of $(L_{\omega P \exists})_{AP}$ is true.*

Proof. Hoover's modification of Henkin's argument (see [H1]) would work.

LEMMA 3. *Let $(\mathfrak{A}, \mu_n)_{n \in \omega}$ be a weak structure, $\varphi(\mathbf{x}, \mathbf{y})$ a finite $(L_{\omega P \exists})_{HCP}$ -formula and $\mathbf{b} \in A^m$. Then there is a quantifier free formula $\Phi(\mathbf{x})$ such that: $(\mathfrak{A}, \mu_n)_{n \in \omega} \models (\forall \mathbf{x})(\varphi(\mathbf{x}, \mathbf{b}) \iff \Phi(\mathbf{x}))$.*

Proof. We use induction on the complexity of φ . If φ is atomic the statement is trivial. The inductive step when φ is a propositional combination of formulas of smaller rank is also trivial. Suppose now that φ is of the form $(P\mathbf{z} \geq r)\psi(\mathbf{x}, \mathbf{z})$. By the inductive assumption we may assume that ψ is a finite quantifier free formula. Further, suppose that \mathbf{x} is (x_0, x_1, \dots, x_n) and that all relational symbols which occur in ψ are R_0, R_1, \dots, R_k . Now define: $\Gamma(v) = \{\bigwedge \{R_i^{f(i)}(v) \mid 0 \leq i \leq k\} \mid f \in 2^{k+1}\}$.

Let $\Sigma(\mathbf{x})$ be the set of all formulas of the form

$$\bigwedge \{\Phi_i(x_i) \mid 0 \leq i \leq n\} \quad \Phi_i(x_i) \in \Gamma(x_i)$$

for which there exists $a_0, a_1, \dots, a_n \in A$ with: $(\mathfrak{A}, \mu_n)_{n \in \omega} \models \Phi_i(a_i)$ for $0 \leq i \leq n$, and $(\mathfrak{A}, \mu_n)_{n \in \omega} \models (P\mathbf{z} \geq r)\psi(\mathbf{a}, \mathbf{z})$. Finally let $\Phi(\mathbf{x})$ be the formula $\bigvee \Sigma(\mathbf{x})$. It is straightforward to check that the following holds:

$$(\mathfrak{A}, \mu_n)_{n \in \omega} \models (\forall \mathbf{x})(\varphi(\mathbf{x}, \mathbf{b}) \iff \Phi(\mathbf{x})).$$

The case when φ is of the form $(\exists \mathbf{z})\psi(\mathbf{x}, \mathbf{z})$ can be dealt with in the same way as the previous one, so the claim of the lemma is established.

COROLLARY 1. *Let $(\mathfrak{A}, \mu_n)_{n \in \omega}$ be a weak probability structure.*

- (a) *If $B \subseteq A^n$ is definable by a finite formula, with parameters from A , then B is ${}^n\mu_1$ measurable; here by ${}^n\mu_1$ we denote the finitely additive n -product of μ_1 's.*
- (b) *If $B \subseteq A^n$ is definable by a formula, possibly infinite with parameters from A , and μ_n is σ -additive then B is μ_1^n -measurable.*

Thus, the corollary allows us to identify (\mathfrak{A}, μ_1) with $(\mathfrak{A}, \mu_n)_{n \in \omega}$ when only finite formulas are considered.

COROLLARY 2. *Let $(\mathfrak{A}, \mu_n)_{n \in \omega}$ be a weak probability structure. Then for every finite $(L_{\omega P \exists})_{HCP}$ -formula $\varphi(\mathbf{x}, \mathbf{y})$ with parameters from A , the set $\{{}^n\mu_1\{\mathbf{b} \in A^n \mid (\mathfrak{A}, \mu_1) \models \varphi(\mathbf{b}, \mathbf{a})\} \mid \mathbf{a} \in A^m\}$ is finite.*

COMPLETENESS THEOREM. *A sentence φ of $(L_{\omega P \exists})_{AP}$ is consistent if and only if it has a probability model.*

Proof. The nontrivial part is to prove that $\not\models \varphi$ implies $\not\models \varphi$, so suppose $\not\models \varphi$. By Lemma 2 there is a weak structure $(\mathfrak{A}, \mu_n)_{n \in \omega}$ which is a model for $\neg \varphi$ and every axiom. By Lemma 1 it is enough to find a probability structure (\mathfrak{B}, ν) which is a model for all finite $(L_{\omega P \exists})_{AP}$ sentences which hold in $(\mathfrak{A}, \mu_n)_{n \in \omega}$. To do that we will use Rašković's method from [R]. Let $K = L \cup C$ ($(K_{\omega P \exists})_{AP}$) be the language (logic) introduced in Hoover's construction [H1], where C is a countable set of new constant symbols and $C \in \mathcal{A}$.

Now, we introduce a language M with three sorts of variables. Let X, Y, Z, \dots be variables for sets, x_0, x_1, \dots variables for urelements and r, s, \dots variables for reals from $[0, 1]$. We suppose that predicates of our language are $E_n(x_0, x_1, \dots, x_{n-1}, X)$ for $n \geq 1$ (with a canonical meaning $(x_0, x_1, \dots, x_{n-1}) \in X$) and $\mu(X, r)$ (with a meaning $\mu(X) = r$). For each finite $(K_{\omega P \exists})_{HCP}$ -formula we have a constant symbol A_φ for a set, for each real number $r \in [0, 1]$ a constant symbol \mathbf{r} , and a set D of new constant symbols of the cardinality of the continuum. Functional symbols are $+$ and \cdot for reals.

Let T be the first order theory with the following list of axioms:

- (1) $(\forall X) \bigwedge_{n < m} \neg(\exists \mathbf{x}, \mathbf{y})(E_m(\mathbf{x}, \mathbf{y}, X) \wedge E_n(\mathbf{x}, X))$, where $\{\mathbf{x}\} \cap \{\mathbf{y}\} = \emptyset$.
- (2) Axioms of extensionality: $(\forall \mathbf{x})(E_n(\mathbf{x}, X) \iff E_n(\mathbf{x}, Y)) \iff X = Y$.
- (3) Axioms of satisfaction:
 - (a) $(\forall \mathbf{x})(E_n(\mathbf{x}, A_\varphi) \iff \bigwedge_{\psi \in \Phi} E_n(\mathbf{x}, A_\psi))$ for φ is $\bigwedge \Phi$, Φ finite;
 - (b) $(\forall \mathbf{x})(E_n(\mathbf{x}, A_\varphi) \iff \neg E_n(\mathbf{x}, A_\psi))$ for φ in $\neg \psi$.
 - (c) $(\forall \mathbf{x})(E_n(\mathbf{x}, A_\varphi) \iff (\exists \mathbf{y}) E_n(\mathbf{x}, \mathbf{y}, A_\psi))$ for φ is $(\exists \mathbf{y}) \psi$;
 - (d) $(\forall \mathbf{x})(E_n(\mathbf{x}, A_\varphi) \iff (\exists_1 X)(\mu(X, r_1^\varphi) \vee \mu(X, r_2^\varphi) \vee \dots \vee \mu(X, r_n^\varphi) \wedge (\forall \mathbf{y})(E_{n+m}(\mathbf{x}, \mathbf{y}, A_\psi) \iff E_m(\mathbf{y}, X))))$ for φ is $(P\mathbf{x} \geq r)\psi$ where $r_1^\varphi, r_2^\varphi, \dots, r_k^\varphi$ are all reals from the set
$$\{^n \mu_1 \{ \mathbf{b} \in A^n \mid (\mathfrak{A}, \mu_1) \models \psi(\mathbf{b}, \mathbf{a}) \} \mid \mathbf{a} \in A^m \} \quad (*)$$

(4) Axioms of additivity:

- (a) $(\forall X)(\exists_1 r)\mu(X, r)$
- (b) $(\forall X)(\forall Y)(\neg(\exists \mathbf{x})(E_n(\mathbf{x}, X) \wedge E_n(\mathbf{x}, Y)) \implies (\exists Z)((\exists \mathbf{x}) E_n(\mathbf{x}, Z) \wedge (\forall \mathbf{x})(E_n(\mathbf{x}, Z) \iff (E_n(\mathbf{x}, X) \vee E_n(\mathbf{x}, Y)) \wedge \mu(Z, r + s)))$ for $n \in \omega$.

(5) Axioms which are transformations of finite axioms of $(K_{\omega P \exists})_{HCP}$: $(\forall \mathbf{x}) E_n(\mathbf{x}, A_\varphi)$ where φ is a finite axiom.

(6) Sets of axioms which ensures σ -additivity of extended measure:

$$\{E_n(\mathbf{d}, A_\varphi)\} \cup \{\neg E_n(\mathbf{d}, A_{\varphi_m}) \mid m \in \omega\}$$

where $\{\varphi_m \mid m \in \omega\}$ is a sequence of finite formulas, \mathbf{d} is a tuple of different constant symbols from D and all such tuples for a different sequences of formulas are pairwise disjoint, $\{\{\mathbf{a} \in A^n \mid (\mathfrak{A}, \mu) \models \varphi_m(\mathbf{a})\} \mid m \in \omega\}$ is a monotone increasing sequence of subsets of A^n , $(\mathfrak{A}, \mu) \models (\forall \mathbf{x})(\varphi_m(\mathbf{x}) \implies \varphi(\mathbf{x}))$ and

$$\begin{aligned} & \mu(\{\mathbf{a} \in A^n \mid (\mathfrak{A}, \mu) \models \varphi(\mathbf{a})\}) \\ & > \sup\{\mu(\{\mathbf{a} \in A^n \mid (\mathfrak{A}, \mu_1) \models \varphi_m(\mathbf{a})\})\} \quad m \in \omega \quad (**) \end{aligned}$$

(7) Axioms of a field (for real numbers) with a diagram for $+$ and \cdot .

Let a standard structure for the first order logic for M be the structure $\mathfrak{M} = (M, B, F, E_n^{\mathfrak{M}}, \mu^{\mathfrak{M}}, +, \cdot, d^{\mathfrak{M}}, A_\varphi^{\mathfrak{M}}, r)_{n \geq 1, \varphi \in S, r \in F, d \in D}$ (for short, $\mathfrak{M} = (M, B, F, A_\varphi)_S$), where $B \subseteq \bigcup_{n \geq 1} \mathcal{P}(M^n)$, $F = F' \cap [0, 1]$, $F' \subseteq R$ a field, $E_n^{\mathfrak{M}} \subseteq M^n \times B$, $\mu^{\mathfrak{M}} : B \rightarrow F$, $+, \cdot : F^2 \rightarrow F$, $d^{\mathfrak{M}} \in M$, $A_\varphi^{\mathfrak{M}} \in B$ and $S \subseteq \{\varphi \in (L_{\omega P \exists})_{HCP} \mid \varphi \text{ is finite}\}$.

We claim that T is consistent. To prove the claim it is enough, by compactness, to show that all finite subtheories of T are consistent.

First, note that a weak structure can be transformed to a standard structure by taking:

$$A_\varphi^{\mathfrak{M}} = \{\mathbf{a} \in M^n \mid (\mathfrak{A}, \mu) \models \varphi(\mathbf{a})\}, \quad B = \{A_\varphi \mid \varphi \in (K_{\omega P \exists})_{HCP} \text{ is finite}\},$$

and arbitrarily interpreting constants from D , we may get a model for a fixed finite subtheory of T .

Let T' be a finite subtheory of T and let $\varphi, \{\varphi_n \mid n \in \omega\}$ be as in the axiom 6. Pick some $m \in \omega$ such that $\neg E_n(\mathbf{d}, A_{\varphi_n}) \in T'$ for all $k \geq m$. By (**) we may choose $\mathbf{d}^{\mathfrak{M}} \in \{\mathbf{a} \in M^n \mid (\mathfrak{A}, \mu_1) \models \varphi(\mathbf{a})\} \setminus \bigcup_{i < m} \{\mathbf{a} \in M^n \mid (\mathfrak{A}, \mu_1) \models \varphi_i(\mathbf{a})\}$. Thus we get a model for T' .

Since every finite subtheory $T' \subseteq T$ has a model, by compactness, we conclude that T has a model, say \mathfrak{M} . Now we can transform our model \mathfrak{M} to a probability structure with a first order part \mathfrak{B} . For a relational symbol R of the language L we define relation $R^{\mathfrak{B}} = \{x \in M \mid E_1^{\mathfrak{M}}(x)\}$, and a finitely additive measure $\bar{\mu}$ on the ring $\{A_\varphi \mid \varphi \text{ is finite}\}$ with: $\bar{\mu}(A_\varphi) = r$ iff $\mu(A_\varphi, r)$ holds in $\mathfrak{M} = (M, \dots)$.

Note that axiom 3d ensures $\bar{\mu}$ to map $\{A_\varphi \mid \varphi \text{ is finite}\}$ into the reals. Axiom 6 allows us to apply Karatheodory's Theorem to the measure $(\{A_\varphi \mid \varphi \text{ is finite}\}, \bar{\mu})$. Thus $\bar{\mu}$ can be extended to a σ -additive measure ν on the σ -ring which extends $\{A_\varphi \mid \varphi \text{ is finite}\}$. Let ν be the σ -additive extension of $\bar{\mu}$. It is straightforward to check that (\mathfrak{B}, ν) is a probability structure which satisfies the same finite $(L_{\omega P \exists})_{AP}$ sentences as (\mathfrak{A}, μ_1) does. That finishes a proof of the theorem.

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