PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 47 (61), 1990, 61–67

LINEAR COMBINATIONS OF REGULAR FUNCTIONS OF ORDER ALPHA WITH NEGATIVE COEFFICIENTS

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Abstract. Let $f(z) = a_p z^p - \sum_{n=1}^{\infty} a_{n+k} z^{n+k}$, $k \geq p \geq 1$, with $a_p > 0$, $a_{n+k} \geq 0$ be regular in $U = \{z : |z| < 1\}$ and $F(z) = (1-\lambda)f(z) + \lambda p^{-1} z f'(z)$, $z \in U$, where $\lambda \geq 0$.

The radius of p-valent starlikeness of order δ, $0 \leq \delta < p$, of $F(z)$ as $f(z)$ varies over a certain subclass of p-valent regular functions of order α, $0 \leq \alpha < p$, in U is determined. All the results are sharp.

1. Introduction. Let $U = \{z : |z| < 1\}$ be the unit disc and $H = \{w : w$ is regular in U such that $w(0) = 0$, $|w(z)| < 1$, $z \in U\}$. Let $P_p(A, B, \alpha)$ denote the class of functions regular in U which are of the form

$$\frac{p + [pB + (A - B)(p - \alpha)]w(z)}{1 + Bw(z)}, \qquad -1 \leq A < B \leq 1, \quad 0 \leq \alpha < p, \quad w \in H.$$

Let T_p be the class of functions $f(z) = a_p z^p - \sum_{n=1}^{\infty} a_{n+k} z^{n+k}$, $k \geq p \geq 1$, $a_p > 0$ and $a_{n+k} \geq 0$, regular in U. Let

$$S_p^*(A, B, \alpha) = \{f \in T_p : zf'(z)/f(z) \in P_p(A, B, \alpha)\} \quad \text{and}$$
$$K_p(A, B, \alpha) = \{f \in T_p : 1 + zf''(z)/f'(z) \in P_p(A, B, \alpha)\}.$$

We note that $S_p^*(A, B, \alpha)$ and $K_p(A, B, \alpha)$ are subclasses of T_p consisting of p-valently starlike functions of order α, and p-valently convex functions of order α, $0 \leq \alpha < p$, respectively. Further if $f(z) \in S_p^*(A, B, \alpha)$ and $z = re^{i\theta}$, $r < 1$,

$$\frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} \frac{zf'(z)}{f(z)}\, d\theta = \frac{(p - \alpha)}{2\pi} \int_0^{2\pi} \operatorname{Re} \frac{1 + Aw(z)}{1 + Bw(z)}\, d\theta + \frac{\alpha}{2\pi} \int_0^{2\pi} d\theta = p,$$

AMS Subject Classification (1985): Primary 30 C 45 since $\operatorname{Re}\dfrac{1+Aw(z)}{1+Bw(z)}$ is a harmonic function in U with $w(0)=0$. This argument shows the p-valence of $f(z)$ in $S_p^*(A,B,\alpha)$. Similarly $f(z)\in K_p(A,B,\alpha)$ is p-valently convex of order α, $0\le\alpha<p$ in U. Define $P^*(A,B,\alpha)=\{f\in T_1 : f'(z)\in P_1(A,B,\alpha),\ a_1=1\}$.

In this paper we determine the radius of p-valence of the function $F(z)+(1-\lambda)f(z)+\lambda p^{-1}zf'(z)$, $\lambda\ge 0$, under the assumption that $0<B\le 1$, when $f(z)$ is in $S_p^*(A,B,\alpha)$, $K_p(A,B,\alpha)$ and $P^*(A,B,\alpha)$.

Throughout this paper we assume that $0<B\le 1$ and $\lambda\ge 0$.

2. Main Results. We use the following notations for the sake of brevity. $n+k=m$, $(m-p)(B+1)+(B-A)(p-\alpha)=C_m$ and $\sum_{m=k+1}^{\infty}=\sum$. We begin by proving the following:

LEMMA 1. *Let $f(z)\in T_p$. Then $f(z)\in S_p^*(A,B,\alpha)$ if and only if*

$$\sum C_m a_m \le (B-A)(p-\alpha)a_p. \tag{2.1}$$

Proof. Suppose $f(z)\in S_p^*(A,B,\alpha)$. Then

$$\frac{zf'(z)}{f(z)}=\frac{p+[pB+(A-B)(p-\alpha)]w(z)}{1+Bw(z)},$$
$$-1\le A<B\le 1,\quad 0<B\le 1,\quad 0\le\alpha<p,\quad w(z)\in H,\quad z\in U.$$

That is,

$$w(z)=\frac{p-zf'(z)/f(z)}{Bzf'(z)/f(z)-[pB+(A-B)(p-\alpha)]},\quad w(0)=0$$

and

$$\begin{aligned}|w(z)|&=\left|\frac{zf'(z)-pf(z)}{Bzf'(z)-[pB+(A-B)(p-\alpha)f(z)]}\right|\\&=\left|\frac{\sum(m-p)a_mz^m}{(B-A)(p-\alpha)a_pz^p-\sum[(m-p)B+(B-A)(p-\alpha)]a_mz^m}\right|<1.\end{aligned}$$

Thus

$$\operatorname{Re}\left\{\frac{\sum(m-p)a_mz^m}{(B-A)(p-\alpha)a_pz^p-\sum[(m-p)B+(B-A)(p-\alpha)]a_mz^m}\right\}<1. \tag{2.2}$$

Take $z=r$ with $0<r<1$. Then, for sufficiently small r, the denominator of the left-hand member of (2.2) is positive and so it is positive for all r, $0<r<1$, since $w(z)$ is regular for $|z|<1$. Then (2.2) gives

$$\sum(m-p)a_mr^m<(B-A)(p-\alpha)a_pr^p-\sum[(m-p)B+(B-A)(p-\alpha)]a_mr^m,$$

that is,

$$\sum[(m-p)(B+1)+(B-A)(p-\alpha)]a_m r^m < (B-A)(p-\alpha)a_p r^p,$$

that is $\sum C_m a_m r^m < (B-A)(p-\alpha)a_p r^p$, and (2.1) follows on letting $r \to 1$.

Conversely, for $|z| = r$, $0 < r < 1$, since $r^m < r^p$, by (2.1) we have $\sum C_m a_m r^m < r^p \sum C_m a_m < (B-A)(p-\alpha)a_p r^p$. Using this inequality we have

$$\begin{aligned}\Big|\sum(m-p)a_m z^m\Big| &\le \sum(m-p)a_m r^m\\ &< (B-A)(p-\alpha)a_p r^p - \sum[(m-p)B+(B-A)(p-\alpha)]a_m r^m\\ &\le \Big|(B-A)(p-\alpha)a_p z^p - \sum[(m-p)B+(B-A)(p-\alpha)a_m z^m]\Big|.\end{aligned}$$

This proves that $zf'(z)/f(z)$ is of the form

$$\frac{p+[pB+(A-B)(p-\alpha)]w(z)}{1+Bw(z)}, \qquad w \in H.$$

Therefore $f(z) \in S_p^*(A,B,\alpha)$ and the proof is complete.

Remark on Lemma 1. For $\alpha = 0$, Lemma 1 reduces to Lemma 1 in [**3**].

Corollary 1. *Let $f(z) \in T_p$. Then $f(z) \in S_p^*(-1,1,\alpha) = S_p^*(\alpha)$, the class of p-valent starlike functions of order α, $0 \le \alpha < p$, if and only if $\sum(m-\alpha)a_m \le (p-\alpha)a_p$.*

Remarks on Corollary 1. (1) For $k = p = 1$, Corollary 1 reduces to Corollary 1 in [**3**]. (2) For $k = p = 1$ and $a_1 = 1$, Corollary 1 reduces to Theorem 1 in [**5**].

Theorem 1. *Let $f(z) \in S_p^*(A,B,\alpha)$ and $F(z) = (1-\lambda)f(z) + \lambda p^{-1}zf'(z)$, $z \in U$. Then $F(z)$ is p-valently starlike of order δ, $0 \le \delta < p$, for*

$$|z| < r_1 = \inf_m \left[\frac{p(p-\delta)}{(m-\delta)(p+\lambda(m-p))}\cdot\frac{C_m}{(B-A)(p-\alpha)}\right]^{1/(m-p)}, \quad m = k+1, k+2, \dots .$$

The result is sharp.

Proof. We have

$$\begin{aligned}F(z) &= (1-\lambda)f(z) + \lambda p^{-1}zf'(z)\\ &= a_p z^p - \sum p^{-1}\big(p+\lambda(m-p)\big)a_m z^m,\\ \frac{zF'(z)}{F(z)} &= \frac{pa_p z^p - \sum mp^{-1}\big(p+\lambda(m-p)\big)a_m z^m}{a_p z^p - \sum p^{-1}\big(p+\lambda(m-p)\big)a_m z^m}.\end{aligned}$$

Now it sufficies to show that $|zF'(z)/F(z) - p| \le (p-\delta)$ for $|z| < r_1$. Now

$$\left|\frac{zF'(z)}{F(z)} - p\right| = \left|\frac{-\sum(m-p)p^{-1}\big(p+\lambda(m-p)\big)a_m z^m}{a_p z^p - \sum p^{-1}\big(p+\lambda(m-p)\big)a_m z^m}\right|$$

$$\leq \frac{\sum (m-p)p^{-1}\big(p+\lambda(m-p)\big)a_m|z|^{m-p}}{\big|a_p - \sum p^{-1}\big(p+\lambda(m-p)\big)a_m|z|^{m-p}\big|}. \tag{2.3}$$

Consider the values of z for which $|z| < r_1$, so that

$$|z|^{m-p} \leq \frac{p(p-\delta)}{(m-\delta)\big(p+\lambda(m-p)\big)} \cdot \frac{C_m}{(B-A)(p-\alpha)}$$

holds. Then

$$\begin{aligned}\sum \left(\frac{p+\lambda(m-p)}{p}\right) a_m |z|^{m-p} &\leq \sum \frac{(p-\delta)}{(m-\delta)} \cdot \frac{C_m}{(B-A)(p-\alpha)} a_m \\ &\leq \frac{C_m}{(B-A)(p-\alpha)} a_m < a_p,\end{aligned}$$

which is true by (2.1). Thus, the expression within the modulus sign in the denominator of the right hand side of (2.3) for the considered values of z is positive and so we have

$$\left|\frac{zF'(z)}{F(z)} - p\right| \leq \frac{\sum (m-p)p^{-1}\big(p+\lambda(m-p)\big)a_m|z|^{m-p}}{a_p - \sum p^{-1}\big(p+\lambda(m-p)\big)a_m|z|^{m-p}} \leq (p-\delta)$$

if

$$\sum (m-\delta)p^{-1}\big(p+\lambda(m-p)\big)a_m|z|^{m-p} \leq (p-\delta)a_p,$$

that is, if

$$\sum \frac{(m-\delta)p^{-1}\big(p+\lambda(m-p)\big)a_m|z|^{m-p}}{(p-\delta)a_p} \leq 1. \tag{2.4}$$

By Lemma 1, we have $f(z) \in S_p^*(A, B, \alpha)$ if and only if $\sum \dfrac{C_m a_m}{(B-A)(p-\alpha)a_p} \leq 1$.

Hence (2.4) is true if

$$\frac{(m-\delta)p^{-1}\big(p+\lambda(m-p)\big)}{(p-\delta)}|z|^{m-p} \leq \frac{C_m}{(B-A)(p-\alpha)},$$

that is, if

$$|z| \leq \left[\frac{p(p-\delta)}{(m-\delta)\big(p+\lambda(m-p)\big)} \cdot \frac{C_m}{(B-A)(p-\alpha)}\right]^{1/(m-p)}.$$

To see the p-valence of $F(z)$ in $|z| < r_1$, we observe that $zF'(z)/F(z)$ is regular in $|z| < r_1$ and hence $\operatorname{Re}\big(zF'(z)/F(z)\big)$ is harmonic in that disc. For $r < r_1$ and $z = re^{i\theta}$,

$$\frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\frac{zF'(z)}{F(z)}\, d\theta = p,$$

showing that $F(z)$ is p-valent in $|z| < r_1$. Hence the proof follows. The extremal function is given by

$$f(z) = a_p z^p - (B-A)(p-\alpha)a_p C_m^{-1} z^m, \qquad m = k+1, k+2, \ldots .$$

Remarks on Theorem 1. (1) For $k = p = 1$, $A = -1$, $B = 1$ and $a_1 = 1$, Theorem 1 reduces to Theorem 2 in [**1**]. (2) For $\alpha = 0$, $m = n + p$, $n = 1, 2, 3, \dots$, $a_p = 1$ and $\delta = p\delta'$, Theorem 1 reduces to Theorem 2 in [**2**].

COROLLARY 2. *If $f(z) \in S_p^*(A, B, \alpha)$, then $f(z)$ is p-valently starlike of order δ, $0 \le \delta < p$, in*

$$|z| < \inf_m \left[\frac{(p-\delta)}{(m-\delta)} \cdot \frac{C_m}{(B-A)(p-\alpha)} \right]^{1/(m-p)}, \quad m = k+1, k+2, \dots .$$

The result is sharp.

Proof. Put $\lambda = 0$ in Theorem 1.

COROLLARY 3. *If $f(z) \in S_p^*(A, B, \alpha)$, then $f(z)$ is p-valently convex of order δ, $0 \le \delta < p$, in*

$$|z| < \inf_m \left[\frac{p}{m} \cdot \frac{(p-\delta)}{(m-\delta)} \cdot \frac{C_m}{(B-A)(p-\alpha)} \right]^{1/(m-p)}, \quad m = k+1, k+2, \dots .$$

The result is sharp.

Proof. Put $\lambda = 1$ in Theorem 1 and note that $zf'(z)/p \in S_p^*(A, B, \alpha)$ if and only if $f(z) \in K_p(A, B, \alpha)$.

COROLLARY 4. *If $f(z) \in S_p^*(A, B, \alpha)$ and $c > -p$, then $F(z) = \big(z^c f(z)\big)' \times z^{-(c-1)}/(p+c)$, for $z \in U$, is p-valently starlike of order δ, $0 \le \delta < p$, in*

$$|z| < \inf_m \left[\frac{(p-\delta)(p+c)}{(m-\delta)(m+c)} \cdot \frac{C_m}{(B-A)(p-\alpha)} \right]^{1/(m-p)}, \quad m = k+1, k+2, \dots .$$

The result is sharp.

Proof. Put $\lambda = p/(p+c)$, $c > -p$, in Theorem 1.

THEOREM 2. *Let $f(z) \in K_p(A, B, \alpha)$ and $F(z) = (1-\lambda)f(z) + \lambda p^{-1} z f'(z)$ for $z \in U$. Then $F(z)$ is p-valently close-to-convex of order zero and type α, $0 \le \alpha < p$, in U if $\lambda < \dfrac{p(1+B)}{(B-A)(p-\alpha)}$ and $F(z)$ is p-valently convex of order δ, $0 \le \delta < p$, in $|z| < r_1$, where r_1 is as in Theorem 1. The result is sharp.*

Proof. We have $F'(z) = (1-\lambda)f'(z) + \lambda p^{-1}\{zf'(z)\}'$. Therefore

$$\operatorname{Re}\left\{\frac{F'(z)}{f'(z)}\right\} = 1 - \lambda + \frac{\lambda}{p} \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\}. \tag{2.5}$$

Since $f(z) \in K_p(A, B, \alpha)$, we can easily prove

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \ge \frac{p + [pB + (A-B)(p-\alpha)]}{1+B}. \tag{2.6}$$

By using (2.6) in (2.5) we have

$$\operatorname{Re}\left\{\frac{F'(z)}{f'(z)}\right\} \geq 1-\lambda+\frac{\lambda}{p}\,\frac{p+[pB+(A-B)(p-\alpha)]}{1+B} \geq 1-\lambda\frac{(B-A)(p-\alpha)}{p(1+B)}.$$

Now,

$$\operatorname{Re}\left\{\frac{F'(z)}{f'(z)}\right\} > 0 \quad \text{if} \quad 1-\lambda\frac{(B-A)(p-\alpha)}{p(1+B)} > 0 \quad \text{or if} \quad \lambda < \frac{p(1+B)}{(B-A)(p-\alpha)}.$$

Hence $F(z)$ is p-valently close-to-convex of order zero and type α, $0 \leq \alpha < p$, in U if $\lambda < \dfrac{p(1+B)}{(B-A)(p-\alpha)}$.

We now prove that $F(z)$ is p-valently convex of order δ, $0 \leq \delta < p$, in $|z| < r_1$, where r_1 is as given in Theorem 1. We have

$$\frac{zF'(z)}{p} = (1-\lambda)\frac{zf'(z)}{p} + \frac{\lambda}{p} z\left\{\frac{zf'(z)}{p}\right\}' \quad \text{for } z \in U. \tag{2.7}$$

Since $f(z) \in K_p(A,B,\alpha)$, it follows that $zf'(z)/p \in S_p^*(A,B,\alpha)$.

Applying Theorem 1 with $zf'(z)/p$ in place of $f(z)$, it follows from (2.7) that $zF'(z)/p$ is p-valently starlike of order δ, $0 \leq \delta < p$, in $|z| < r_1$, equivalently, $F(z)$ is p-valently convex of order δ, $0 \leq \delta < p$, in $|z| < r_1$. The extremal function is given by

$$f(z) = a_p z^p - \frac{p(B-A)(p-\alpha)a_p}{mC_m} z^m, \quad m = k+1, k+2, \ldots .$$

Remarks on Theorem 2. (1) For $k = p = 1$, $A = -1$, $B = 1$ and $a_1 = 1$, Theorem 2 reduces to Theorem 3 in [**1**]. (2) For $\alpha = 0$, $m = n+p$, $n = 1, 2, 3, \ldots$, $a_p = 1$ and taking $\delta = p\delta'$, $0 \leq \delta' < 1$, Theorem 2 reduces to theorem 3 in [**2**].

LEMMA 2. *Let $f(z) \in T_1$, $a_1 = 1$. Then $f(z) \in P^*(A,B,\alpha)$ if and only if*

$$\sum_{m=2}^{\infty} m(B+1)a_m \leq (B-A)(1-\alpha). \tag{2.8}$$

Proof. Proof of Lemma 2 is similar to the proof of Lemma 1 and is hence omitted.

Remarks on Lemma 2. (1) For $\alpha = 0$, Lemma 2 reduces to Lemma 2 in [**3**]. (2) For $A = -1$, $B = 1$, Lemma 2 reduces to Theorem 1 (ii) in [**4**].

THEOREM 3. *Let $f(z) \in P^*(A,B,\alpha)$ and $F(z) = (1-\lambda)f(z) + \lambda z f'(z)$ for $z \in U$. Then $\operatorname{Re} F'(z) > \delta$, $0 \leq \delta < 1$, for*

$$|z| < r_2 = \inf_m \left[\frac{(1-\delta)}{1+(m-1)\lambda} \cdot \frac{B+1}{(B-A)(1-\alpha)}\right]^{1/(m-1)}, \qquad m \geq 2.$$

The result is sharp.

Proof. It suffices to show that $|F'(z) - 1| \leq 1 - \delta$ for $|z| < r_2$. Since $f(z) \in P^*(A, B, \alpha)$, using Lemma 2, we see that (2.8) holds. Since $F'(z) = 1 - \sum_{m=2}^{\infty} m\big(1 + (m-1)\lambda\big) a_m z^{m-1}$, using (2.8), we see that $|F'(z) - 1| \leq \sum_{m=2}^{\infty} m\big(1 + (m-1)\lambda\big) a_m |z|^{m-1} \leq 1 - \delta$ provided

$$\frac{m(1 + (m-1)\lambda)}{1 - \delta} |z|^{m-1} \leq \frac{m(B+1)}{(B-A)(1-\alpha)}. \tag{2.9}$$

Now (2.9) holds if

$$|z| \leq \left[\frac{1-\delta}{1 + (m-1)\lambda} \cdot \frac{(B+1)}{(B-A)(1-\alpha)} \right]^{1/(m-1)}, \qquad m \geq 2$$

and the proof follows. The extremal function is

$$f(z) = z - \frac{(B-A)(1-\alpha)}{m(B+1)} z^m, \qquad m = 2, 3, \dots .$$

Remarks on Theorem 3. (1) For $\alpha = 0$, Theorem 3 reduces to Theorem 3 in [**3**]. (2) For $A = -1$, $B = 1$, Theorem 3 reduces to Theorem 4 in [**1**].

REFERENCES

[**1**] S. S. Bhoosnurmath, S. R. Swamy, *Analytic functions with negative coefficients*, Indian J. Pure Appl. Math. **12** (**6**) (1981), 738–742.

[**2**] Dashrath and V. Kumar, *Radii of p-valence of certain analytic functions with negative coefficients*, Comm. Fac. Sci. Ankara Sér. A_1 Math. **31** (1982), No. 17, 145–153 (1984).

[**3**] G. Lakshma and P. S. Padmanabhan, *Linear combinations of regular functions with negative coefficients*, Publ. Inst. Math. (Belgrade) **34** (**48**) (1983), 109–116.

[**4**] S. M. Sarangi and B. A. Uralegaddi, *The radius of convexity and starlikeness for certain classes of analytic functions with negative coefficients I*, Acad. Naz. Lincei, Rend. **65** (1978), 38–42.

[**5**] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. **51** (1975), 109–116.

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(Received 02 05 1989)