

ON ISOMORPHISMS OF L^1 SPACES OF ANALYTIC FUNCTIONS ONTO l^1

Miroslav Pavlović

Abstract. It is proved that an L^1_φ space of analytic functions in the unit disc, with the weight $\varphi'(1 - |z|)$, is isomorphic to the Lebesgue sequence space l^1 only if φ is “normal”. The converse is known from the papers of Shields and Williams [13] and Lindenstrauss and Pelczynski [4]. The key of our proof are three classical results: Paley’s theorem on lacunary series, Pelczynski’s theorem on complemented subspaces of l^1 and Lindenstrauss-Pelczynski’s theorem on the equivalence of unconditional bases in l^1 .

Throughout the paper we assume that φ is a quasi-normal function [7]. This means that φ is defined, increasing and continuously differentiable on the interval $(0, 1]$, $\varphi(0+) = 0$, and that

(Q) there is a constant β such that
 $\varphi(t)/t^\beta$ is almost decreasing for $0 < t < 1$.

(A function ψ is almost decreasing if $\psi(t_2) \leq C\psi(t_1)$ for $t_2 > t_1$, where C is a constant, see [1, 15]. If, in addition,

(N) there is a constant $\alpha > 0$ such that
 $\varphi(t)/t^\alpha$ is almost increasing for $0 < t < 1$,

then φ is said to be *normal* [13, 15].

For a (complex-valued) harmonic function f on the unit disc U , we define the quantities

$$\|f\|_{1,\varphi} = \frac{1}{\pi} \int_U |f(z)|\varphi'(1 - |z|) dm(z)$$

(dm denotes the Lebesgue measure, and φ' the derivative of φ) and

$$\|f\|_{\infty,\varphi} = \sup\{|f(z)|\varphi(1 - |z|) : z \in U\}.$$

In this note we are concerned with the following spaces:

$$A^p(\varphi) = \{f : \|f\|_{p,\varphi} < \infty, f \text{ analytic in } U\} \quad (p = 1, \infty),$$

$$A_0(\varphi) = \{f \in A^\infty(\varphi) : |f(z)|\varphi(1 - |z|) = o(1), z \rightarrow 1^-\}.$$

Analogous spaces of harmonic functions are denoted by $h^p(\varphi)$ and $h_0(\varphi)$.

It is known that each of the harmonic spaces is isomorphic, via a multiplier transform, to a space of Lipschitz functions on the unit circle [8, 9]. On the other hand, it was shown by Shields and Williams [14] that $h^p(\varphi)$ is isomorphic to l^p , and $h_0(\varphi)$ is isomorphic to c_0 , the space of null-sequences (see Remarks at the end). And it follows from the results of Lindenstrauss and Pelczynski [4] and Shields and Williams [13] that h can be replaced by A if φ is normal. Our aim here is to prove the converse. More precisely, we have the following result.

THEOREM. *If one of the following assertions (i), (ii) or (iii) holds, then the function φ is normal:*

- (i) $A^1(\varphi)$ is isomorphic to l^1 ;
- (ii) $A^\infty(\varphi)$ is isomorphic to l^∞ ;
- (iii) $A_0(\varphi)$ is isomorphic to c_0 .

Before discussing the case of $A^1(\varphi)$ we note that the other two cases are contained implicitly in [14, Theorem 7] and [15]. Namely, if $A^\infty(\varphi)$ is isomorphic to l^∞ , then $A^\infty(\varphi)$ is complemented in $h^\infty(\varphi)$, because l^∞ is complemented in every space containing it (see [5, p. 105]). Then, by using a method of Rudin [13], we conclude that $A^\infty(\varphi)$ is complemented by the “analytic” projection, i.e. that $h^\infty(\varphi)$ is self-conjugate and hence, by [15], φ is normal.

If $A_0(\varphi)$ is isomorphic to c_0 , we can use a theorem of Sobczyk that asserts that c_0 is complemented in every separable space containing it. We can also reduce the problem to the case of A^∞ by using the fact, proved by Rubel and Shields [11], that the second dual of $A_0(\varphi)$ is isometrically isomorphic to $A^\infty(\varphi)$.

In order to discuss the case of $A^1(\varphi)$ we use three famous theorems.

PALEY THEOREM [6]. *If $\{m_n\}_0^\infty$ is a lacunary sequence of positive integers, then there is a constant $C < \infty$ such that*

$$\sum |\hat{f}(m_n)|^2 \leq C \|f\|_1^2, \quad f \in H^1,$$

where $\|\cdot\|_1$ stands for the norm in the Hardy class H^1 .

PELCZYNSKI THEOREM [10]. *Every infinite dimensional complemented subspace of l^1 is isomorphic to l^1 .*

LINDENSTRAUSS-PELCZYNSKI THEOREM [3]. *Every normalized unconditional basis of l^1 is equivalent to the canonical basis of l^1 .*

Proof of the Theorem. Assume that $A^1(\varphi)$ is isomorphic to l^1 . Let Y denote the subspace spanned by the elements $h_n(z) = z^{2^n}$, and $Pf(z) = \sum \hat{f}(2^n)z^{2^n}$, $f \in A^1(\varphi)$. By the Paley theorem,

$$(1) \quad M_1(r, Pf) \leq M_2(r, Pf) \leq CM_1(r, f), \quad 0 < r < 1,$$

where C is independent of r, f . (Here we use the usual notation for the integral means of f [2]). Multiplying inequality (1) by $\varphi'(1-r) dr$ and then integrating from 0 to 1, we conclude that P is a bounded projection of $A^1(\varphi)$ onto Y . Hence, by Pelczynski's theorem, Y is isomorphic to l^1 . Using (1) again we see that

$$M_1(r, f) \asymp M_2(r, f), \quad f \in Y, 0 < r < 1,$$

which implies that $\{h_n\}$ is an unconditional basis of Y . Hence, by Lindenstrauss-Pelczynski's theorem, there is a constant $c > 0$ such that

$$(2) \quad \left\| \sum a_n h_n \right\|_Y \geq c \sum |a_n| \|h_n\|_Y$$

for scalar sequences $\{a_n\}$.

To deduce from (2) that φ is normal, we need some calculation. We have

$$\|h_n\|_Y = \int_0^1 \varphi'(1-r)r^{2^n} dr,$$

whence, by Lemma 3.2 of [7],

$$(3) \quad \|h_n\|_Y \asymp \varphi(2^{-n}).$$

Extend φ so that

$$(4) \quad \varphi(t)\varphi(1/t) = \varphi(1)^2, \quad t > 1,$$

and let F denote the inverse function. In view of condition (Q), there is a constant $m \geq 2$ such that $F(mt)/F(t) \geq 4$ for $t > 0$. It follows that the interval

$$E_t := [\log F(t), \log F(mt)] \quad (\log = \log_2)$$

contains at least two integers. For a fixed t with $\log F(t) > 0$ define $\{a_n\}$ in the following way:

$$a_n = \begin{cases} \varphi(2^n) & \text{for } n \in E_t, \\ 0, & \text{otherwise,} \end{cases}$$

and let $f_t = \sum a_n h_n$. It follows from (2), by (3) and (4), that

$$(5) \quad \|f_t\|_Y \geq c(\log F(mt) - \log F(t)),$$

where $c > 0$ is independent of t . On the other hand, by Parseval's relation,

$$M_1^2(r, f_t) \leq M_2^2(r, f_t) = \sum |a_n|^2 r^{2^{n+1}} \leq \sum \varphi(2^n)^2 r^{2F(t)} \quad (n \in E_t).$$

Since $\varphi(2^n) \leq \varphi(F(mt)) = mt$ for $n \in E_t$, we see that

$$M_1(r, f_t) \leq mtr^{F(t)}(\log F(mt) - \log F(t) + 1)^{1/2}.$$

Multiply thus by $\varphi'(1-r) dr$, then integrate and use the estimate

$$\int_0^1 r^x \varphi'(1-r) dr \asymp \varphi(1/x), \quad x > 1$$

(see [7, Lemma 3.2]). As a result we obtain

$$\|f_t\|_Y \leq C(\log F(mt) - \log F(t))^{1/2}.$$

From this and (5) we find that $F(2t) \leq KF(t)$ for $t > 0$, where K is a constant. As a consequence, $F(t)/t^\beta$ is almost decreasing for $t > 0$, where $\beta = \log_2 K$. Hence, φ satisfies the condition (N) with $\alpha = 1/\beta$, which was to be proved.

Remarks. Shields and Williams [14] proved that $h_0(\varphi)$ is isomorphic to c_0 if φ satisfies some regularity conditions. However, it follows from Theorems 1 and 7 of [14] and the relation $h_0(\varphi)^* = h^1(\varphi)$ [7] that $h_0(\varphi)$ is isomorphic to c_0 whenever φ is quasi-normal. It would be interesting to construct an explicit isomorphism between $h_0(\varphi)$ and c_0 , or between $h^1(\varphi)$ and l^1 . In the case of $A^1(\varphi)$, where φ is a power function, an explicit isomorphism onto l^1 was constructed by Wojtaszczyk [17].

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Matematički fakultet
11001 Beograd, p.p. 550
Yugoslavia

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