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ON THE NUMBERS OF POSITIVE AND NEGATIVE EIGENVALUES OF A GRAPH

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Abstract. We consider simple connected graphs with a fixed number of negative eigenvalues (including their multiplicities). We show that these graphs have uniformly bounded numbers of positive eigenvalues, and the last numbers run over a set $[m] = \{1, 2, \dots, m\}$.

Throughout this paper we consider only finite connected graphs without loops or multiple edges. The vertex set of a graph G is denoted by $V(G)$, and its order (the number of its vertices) by $|G|$. If H and G are graphs, relation $H \subseteq G$ will always mean that H is an induced subgraph of the graph G.

The spectrum of a graph G is the spectrum of its 0-1 adjacency matrix. The number of its positive and the number of its negative eigenvalues (including their multiplicities) are denoted by $n^+(G)$ and $n^-(G)$ respectively. For a positive integer n, $P(n)$ will denote the set of all connected nonisomorphic graphs with the property $n^-(G) = n$.

If G is a graph, consider the following equivalence relation α on the vertex set $V(G)$: two vertices $x, y \in V(G)$ are in relation α if and only if they are nonadjacent and they have the same neighbours in G. This means that x and y are related if and only if the corresponding rows (columns) of the adjacency matrix are equal.

The corresponding quotient graph g is called the *canonical graph* of G. It is also connected. The graph G is called *canonical* if $g = G$, that is if G has no two equivalent vertices. If, for instance, G is an arbitrary complete p-partite graph, then its canonical graph is the complete graph K_p with p vertices. The path P_m with m vertices ($m \geq 2$) is a canonical graph if and only if $m \neq 3$.

PROPOSITION 1 [**6**]. *For an arbitrary graph G and its canonical graph g, the following equalities hold*:

$$n^+(G) = n^+(g), \qquad n^-(G) = n^-(g).$$

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Consequently, in the investigation of relations between the numbers of positive and negative eigenvalues of graphs, we can consider only canonical graphs.

Next, let $P_c(n)$ be the class of all nonisomorphic canonical graphs from the class $P(n)$. An important property of this class has been proved in [**6**].

THEOREM A [**6**]. *For each positive integer n, the class $P_c(n)$ is finite.*

Consequently, we have that the number

$$A_n := \sup\{n^+(G) \mid G \in P_c(n)\}$$

is finite, for every positive integer n.

Next, we need the notion of minimal graphs from the class $P(n)$. A graph $G \in P(n)$ is called *minimal* if no of its proper induced subgraphs is in the class $P(n)$. The set of all nonisomorphic minimal graphs from the class $P(n)$ is denoted by $M(n)$. We obviously have that $M(n) \subseteq P_c(n)$ for every positive integer n. By Theorem A we also find that class $M(n)$ is finite for every n.

PROPOSITION 2. *For every positive integer n, the numbers $\{|H| : H \in M(n)\}$ run over the set $\{n+1, n+2, \dots, 2n\}$.*

Proof. Since each graph $H \in M(n)$ has exactly n negative and at least one positive eigenvalue, we obviously have that $|H| \geq n+1$. The fact that all graphs $H \in M(n)$ have at most $2n$ vertices, will be proved by induction on n.

As is known, the class $M(1)$ contains exactly one graph K_2, while the class $M(2)$ contains exactly two graphs — K_3 and P_4 (see e.g. [**6**]). Hence, this statement is true for $n = 1, 2$.

Next, assume that, for a positive integer k, each graph $H \in M(k)$ has at least $k+1$ and at most $2k$ vertices. Let the graph G runs over the class $M(k)$ and S runs over the all nonempty subsets from the set $V(G)$. Form a graph Gx by adding a new vertex x to G and by connecting it with vertices from S. If the graph Gx has just $k+1$ negative eigenvalues, define $G_S = Gx$. If Gx has k negative eigenvalues, define $G_S = Gxy$ to be the graph obtained from Gx by adding a new vertex y adjacent only to x. By a result of [**5**] we then have

$$M(k+1) = \{G_S \mid G \in M(k),\ S \subseteq V(G) \setminus \{\varnothing\}\}.$$

Consequently, we find that all graphs from the class $M(k+1)$ have at most $2k+2 = 2(k+1)$ vertices. By induction on k we get $n+1 \leq |H| \leq 2n$, for every graph $H \in M(n)$ and every positive integer n.

Next, let X_{pq} ($p \geq 0$, $q \geq 1$) be the graph obtained by identification of a point in the graph K_{p+2} with an endpoint of the path P_{2q-1}. In particular, we have that $X_{p1} = K_{p+1}$ and $X_{0q} = P_{2q}$. Proposition 5 of [**6**] provides that

$$n^-(X_{pq}) = p + q, \qquad n^+(X_{pq}) = q. \qquad (p \geq 0,\ q \geq 1).$$

In particular, consider the graphs $X_k = X_{n-k,k}$ ($k = 1, \dots, n$). We have that $n^-(X_k) = n$, $n^+(X_k) = k$, and it is not difficult to see that all the graphs $X_1, \dots, X_n$ belong to the class $M(n)$. Since $|X_k| = n + k$ ($k = 1, \dots, n$), our proposition is completely proved. □

By Proposition 2 and the graphs $X_1, \dots, X_n$ we have the following result.

COROLLARY 1. *For every positive integer n, the numbers $\{n^+(H) \mid H \in M(n)\}$ run over the set $[n] = \{1, 2, \dots, n\}$.*

Now, we are able to prove the main result of the paper.

THEOREM 1. *For every positive integer n, the numbers $\{n^+(G) \mid G \in P_c(n)\}$ run over the set $[A_n] = \{1, 2, \dots, A_n\}$.*

Proof. Corollary 1 provides that the mentioned numbers cover the set $[n] = \{1, 2, \dots, n\}$. Consequently, we find that $A_n \geq n$ for every n. Next, we only have to prove that these numbers also cover the set $\{n+1, n+2, \dots, A_n\}$.

Let T be an arbitrary graph from the class $P_c(n)$ such that $n^+(T) = A_n$. Let H be an arbitrary minimal graph of the graph T $(H \subseteq T)$. Since $H \subseteq T$ and both H and T are connected graphs, it is easy to see that there is a sequence of connected graphs $F_i \subseteq T$ $(i = 0, 1, \dots, r)$, such that

$$H = F_0 \subseteq F_1 \subseteq \dots \subseteq F_r = T$$

and $|F_{i+1}| = |F_i| + 1$ $(i = 0, 1, \dots, r-1)$. Since by the known interlacing theorem [**2**, p. 19] we have $n^-(H) = n \leq n^-(F_i) \leq n^-(T) = n$, we find that $n^-(F_i) = n$; thus $F_i \in P(n)$ $(i = 0, 1, \dots, r)$. By the same theorem, we also find $n^+(F_{i+1}) - n^+(F_i) \in \{0, 1\}$ $(i = 0, 1, \dots, r-1)$. This shows that the numbers

$$\{n^+(F_i) \mid i = 0, 1, \dots, r\} \tag{1}$$

run over the set $\{n^+(H), n^+(H) + 1, \dots, n^+(T)\} = [n^+(H), A_n]$.

On the other hand, by Corollary 1 we have that $n^+(H) \leq n$. This proves that the sequence (1) covers the set $[n+1, A_n]$. Finally, taking into account the canonical graphs f_i of the graphs F_i $(i = 0, 1, \dots, r)$ completes the proof. □

By Theorem 1, any estimate of growth of the function $n \mapsto A_n$ can be of a great importance. By the corresponding results in the papers [**4**] and [**8**], we know that $A_1 = 1$, $A_2 = 3$, $A_3 = 6$. But, so far, we have no information about this function in the general case.

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