

ON THE RATE OF CONVERGENCE FOR MODIFIED  
SZÁSZ-MIRAKYAN OPERATORS ON FUNCTIONS  
OF BOUNDED VARIATION

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**Abstract.** Recently Gupta and Agrawal [4] gave an estimate of the rate of convergence for modified Szász-Mirakyan operators for functions of bounded variation, using probabilistic approach. Because of some mistakes in their paper, we were motivated to correct and improve their results.

**1. Introduction.** Let  $f$  be a function defined on  $[0, \infty)$ . The Szász-Mirakyan operator  $S_n$  applied to  $f$  is

$$S_n(f; x) = \sum_{k=0}^{\infty} p_k(nx) f(k/n),$$

where

$$p_k(t) = e^{-t} t^k / k!$$

Kasana et al. [5] proposed modified Szász-Mirakyan operators to approximate functions integrable on  $[0, \infty)$ :

$$M_n(f; x) = n \sum_{k=0}^{\infty} p_k(nx) \int_0^{\infty} p_k(nt) f(t) dt.$$

We shall study  $M_n(f; x)$  for functions of bounded variation on every finite subinterval of  $[0, \infty)$ , giving quantitative estimates of the rate of convergence. Results of this type for Fourier series were obtained by Bojanić [2].

**2. Auxiliary results.** We shall need the following lemmas in the proof of our theorem.

LEMMA 1. [1, pp. 104, 110] and [3, p. 304]: Berry–Essen Theorem. *Let  $X_1, X_2, \dots, X_n$  be  $n$  independently and identically distributed (i.i.d.) random*

variables with zero mean and a finite absolute third moment. If  $\rho_2 \equiv E(X_1^2) > 0$ , then  $\sup_{x \in \mathbb{R}} |F_n(x) - \phi(x)| \leq (0.41)l_{3,n}$ , where  $F_n(x)$  is the distribution function of  $(X_1 + X_2 + \dots + X_n)(n\rho_2)^{-1/2}$ ;  $\phi(x)$  is the standard normal distribution function, and  $l_{3,n}$  is the Liapounov ratio  $l_{3,n} = \rho_3\rho_2^{-3/2}n^{-1/2}$ ,  $\rho_3 = E|X_1|^3$ .

We may note here that Gupta and Agrawal (1991) mistook  $\beta_3 = E(X_1^3)$  for  $\rho_3$  and since  $\rho_3 > \beta_3$  for all nondegenerate distribution functions of i.i.d.  $X_i$ 's, this led to an error in the estimate, e.g., in their Lemma 1, it should have read  $p_k(nx) \leq 5/2(\sqrt{nx})$ .

LEMMA 2. For every  $x \in (0, \infty)$ , we have  $p_k(nx) \leq \phi(x)/\sqrt{n}$ , where  $\phi(x) = 2x^{-1/2}(4x^2 + 3x + 1)$ .

*Proof.* Let  $\{\xi_k\}$  be a sequence of i.i.d. random variables all having the same Poisson distribution with parameter  $x$ . Let  $\eta_n = \sum_{k=1}^n \xi_k$ ; then

$$p(\eta_n = k) = e^{-nx}(nx)^k/k! = p_k(nx),$$

wherefrom

$$\begin{aligned} \rho_2 = x; \quad \rho_3 = E|\xi_1 - x|^3 &\leq E(\xi_1^3) + 3xE(\xi_1^2) + 3x^2E(\xi_1) + x^3 \\ &= 8x^3 + 6x^2 + x, \quad \text{and} \end{aligned}$$

$$l_{3,n} = (8x^3 + 6x^2 + x)/\sqrt{n}(x^{3/2}).$$

So, we have

$$p_k(nx) = P(k-1 < \eta_n \leq k) = P\left(\frac{k-1-nx}{\sqrt{nx}} < \frac{\eta_n - nx}{\sqrt{nx}} \leq \frac{k-nx}{\sqrt{nx}}\right).$$

By using Lemma 1 we have

$$\left| p_k(nx) - \frac{1}{\sqrt{2\pi}} \int_{(k-1-nx)/\sqrt{nx}}^{(k-nx)/\sqrt{nx}} e^{-t^2/2} dt \right| < 2(0.41) \frac{8x^2 + 6x + 1}{\sqrt{nx}} < \frac{8x^2 + 6x + 1}{\sqrt{nx}}.$$

Also

$$\frac{1}{\sqrt{2\pi}} \int_{(k-1-nx)/\sqrt{nx}}^{(k-nx)/\sqrt{nx}} e^{-t^2/2} dt \leq \frac{1}{\sqrt{2knx}} \leq \frac{1}{\sqrt{nx}}.$$

Therefore,

$$p_k(nx) \leq \frac{8x^2 + 6x + 1}{\sqrt{nx}} + \frac{1}{\sqrt{nx}} < \frac{\phi(x)}{\sqrt{n}}.$$

LEMMA 3. For  $n \geq 2$ , we have  $2x/n \leq M_n((t-x)^2; x) \leq (2x+1)/n$ .

*Proof.* It is easily verified that:  $M_n(1; x) = 1$ ,  $M_n(t; x) = (1+nx)/n$  and  $M_2(t^2; x) = (x^2n^2 + 4nx + 2)/n^2$ . Hence,  $M_n((t-x)^2; x) = 2x/n + 2/n^2$ , which leads to the assertion of the lemma.

LEMMA 4. Let  $k_n(x; t) = n \sum_{k=0}^{\infty} p_k(nx)p_k(nt)$ . Then

(i) For  $0 \leq y < x$ , we have

$$\int_0^y k_n(x, t) dt \leq \frac{2x + 1}{n(x - y)^2} \tag{2.1}$$

(ii) For  $x < z < \infty$ , we have

$$\int_z^\infty k_n(x, t) dt \leq \frac{2x + 1}{n(z - x)^2} \tag{2.2}$$

*Proof.* The results follow from straightforward application of Lemma 3 and from the fact that  $(t - x)^2 \geq (x - y)^2$  in (i) and  $(t - x)^2 \geq (z - x)^2$  in (ii).

LEMMA 5. For  $x \in (0, \infty)$ , we have

$$n \int_x^\infty p_k(nt) dt = \sum_{j=0}^k p_j(nx).$$

*Proof.* The above equality can be obtained by repeated integration of its left-hand side by parts. Also,

$$n \int_0^x p_k(nt) dt = 1 - n \int_x^\infty p_k(nt) dt = \sum_{j=k+1}^\infty p_j(nx).$$

**3. Main Result.** Our main result may be stated as follows.

THEOREM. Let  $f$  be a function of bounded variation on every finite subinterval of  $[0, \infty)$  and let  $f(t) = O(e^{\alpha t})$  for some  $\alpha > 0$  as  $t \rightarrow \infty$ . If  $x \in (0, \infty)$ , then for  $n$  sufficiently large we have

$$\begin{aligned} |M_n(f; x) - \frac{1}{2}\{f(x+) + f(x-)\}| &\leq \frac{(x^2 + 6x + 3)x^{-2}}{n} \sum_{k=1}^n V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}} g_x \\ &+ \frac{(4x^2|3x + 1|)x^{-1/2}}{\sqrt{n}} |f(x+) - f(x-)| + O(1)e^{-\alpha x} \frac{(2x + 1)x^{-2}}{n}; \end{aligned} \tag{3.1}$$

Where  $V_a^b(g_x)$  is total variation of  $g_x$  on  $[a, b]$  and

$$g_x \equiv g_x(t) = \begin{cases} f(t) - f(x+), & x < t < \infty; \\ 0, & t = x; \\ f(t) - f(x-), & 0 \leq t < x. \end{cases}$$

*Proof.* First,

$$\begin{aligned} |M_n(f; x) - \frac{1}{2}(f(x+) + f(x-))| \\ \leq |M_n(g_x; x)| + \frac{1}{2}|f(x+) - f(x-)| |M_n(\text{Sign}(t - x); x)| \end{aligned} \tag{3.2}$$

Thus to estimate the left-hand-side, we need estimates for  $M_n(g_x; x)$  and  $M_n(\text{Sign}(t-x); x)$ . Now,

$$\begin{aligned} M_n(\text{Sign}(t-x); x) &= \int_0^\infty \text{Sign}(t-x)k_n(x; t)dt \\ &= \int_x^\infty k_n(x; t)dt - \int_0^x k_n(x; t)dt \\ &= A_n(x) - B_n(x), \text{ say.} \end{aligned}$$

Using Lemma 5, we have

$$\begin{aligned} A_n(x) &= \int_x^\infty k_n(x, t)dt \\ &= n \sum_{k=0}^\infty p_k(nx) \int_x^\infty p_k(nt)dt = \sum_{k=0}^\infty p_k(nx) \sum_{j=0}^k p_j(nx) \\ &= p_0^2 + p_1(p_0 + p_1) + p_2(p_0 + p_1 + p_2) + \dots \\ &= p_0^2 + p_1^2 + p_2^2 + \dots + p_0(p_1 + p_2 + \dots) + p_1(p_2 + p_3 + \dots) + \dots \end{aligned}$$

Let  $1 = (p_0 + p_1 + p_2 + \dots)(p_0 + p_1 + p_2 + \dots)$ . Then  $1 - A_n(x) = p_0(p_1 + p_2 + \dots) + p_1(p_2 + p_3 + \dots) + \dots$ . Further,  $A_n(x) - (1 - A_n(x)) = 2A_n(x) - 1 = p_0^2 + p_1^2 + p_2^2 + \dots$ , and  $A_n(x) + B_n(x) = 1$ ; so we get

$$|A_n(x) - B_n(x)| = |2A_n(x) - 1| = \sum_{k=0}^\infty p_k^2(nx) \leq \frac{\phi(x)}{\sqrt{n}} \sum_{k=0}^\infty p_k(nx) = \frac{\phi(x)}{\sqrt{n}}.$$

To estimate  $M_n(g_x; x)$ , we first decompose  $[0, \infty)$  into three parts, as follows

$$\begin{aligned} M_n(g_x; x) &= n \sum_{k=0}^\infty p_k(nx) \int_0^\infty p_k(nt)g_x(t)dt = \int_0^\infty g_x(t)k_n(x; t)dt \\ &= \int_0^{x-x/\sqrt{n}} g_x(t)k_n(x; t)dt + \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} g_x(t)k_n(x; t)dt \\ &\quad + \int_{x+x/\sqrt{n}}^\infty g_x(t)k_n(x; t)dt \\ &= \left( \int_{I_1} + \int_{I_2} + \int_{I_3} \right) g_x(t)k_n(x; t)dt \\ &= E_1 + E_2 + E_3, \text{ say.} \end{aligned}$$

First, we estimate  $E_2$ . For  $t \in I_2$ , we have

$$|g_x(t)| = |g_x(t) - g_x(x)| \leq V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x),$$

and so

$$\begin{aligned} E_2 &= \left| \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} g_x(t)k_n(x; t)dt \right| \leq V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x) \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} k_n(x; t)dt \\ &= V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x) \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} d_t \lambda_n(x; t); \text{ where, } \lambda_n(x; t) = \int_0^t k_n(x; u)du. \end{aligned}$$

Since  $\int_a^b dt \lambda_n(x, t) \leq 1$  for all  $[a, b] \subseteq [0, \infty)$ , we have

$$|E_2| = V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x) \leq \frac{1}{n} \sum_{k=0}^n V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x). \quad (3.3)$$

Let us estimate  $E_1$ . By using Lebesgue-Stieltjes integration by parts, with  $y = x - x/\sqrt{n}$ , we get

$$E_1 = g_x(y+) \lambda_n(x; y) - \int_0^y \lambda_n(x; t) d_t g_x(t).$$

Since,  $|g_x(y+)| = |g_x(y+) - g_x(x)| \leq V_{y+}^x(g_x)$ , it follows that

$$|E_1| \leq V_{y+}^x(g_x) \lambda_n(x, y) + \int_0^y \lambda_n(x, t) d(-V_t^x g_x).$$

By (2.1) of Lemma 4, we have

$$|E_1| \leq V_{y+}^x(g_x) \frac{2x+1}{n(x-y)^2} + \frac{2x+1}{n} \int_0^y \frac{1}{(x-t)^2} d_t(-V_t^x(g_x)).$$

Integration by parts leads to the following

$$\int_0^y \frac{1}{(x-t)^2} d_t(-V_t^x(g_x)) = \frac{-V_{y+}^x(g_x)}{(x-y)^2} + \frac{V_0^x(g_x)}{x^2} + 2 \int_0^y \frac{\hat{V}_t^x(g_x)}{(x-t)^3} dt,$$

where  $\hat{V}_t^x(g_x)$  is the normalized form of  $V_t^x(g_x)$  and  $\hat{V}_t^x(g_x) = V_t^x(g_x)$ . Consequently, we have

$$\begin{aligned} |E_1| &\leq V_{y+}^x(g_x) \frac{2x+1}{n(x-y)^2} + \frac{2x+1}{n} \left[ \frac{-V_{y+}^x(g_x)}{(x-y)^2} + \frac{V_0^x(g_x)}{x^2} + 2 \int_0^y \frac{V_t^x(g_x)}{(x-t)^3} dt \right] \\ &= \frac{2x+1}{n} \left[ \frac{V_0^x(g_x)}{x^2} + 2 \int_0^{x-x/\sqrt{n}} V_t^x(g_x) \frac{1}{(x-t)^3} dt \right]. \end{aligned}$$

Substituting  $x - x/\sqrt{t}$  for the variable  $t$  in the integral above, we get

$$\begin{aligned} |E_1| &\leq \frac{2x+1}{nx^2} (V_0^x(g_x) + \int_1^n V_{x-x/\sqrt{t}}^x(g_x) dt) \\ &\leq \frac{2x+1}{nx^2} (V_0^x(g_x) + \sum_{k=1}^n V_{x-x/\sqrt{k}}^x(g_x)) \\ &\leq \frac{2(2x+1)}{nx^2} \sum_{k=1}^n V_{x-x/\sqrt{k}}^x(g_x). \end{aligned} \quad (3.4)$$

Finally, we evaluate  $E_3$ . Setting,  $Z = x + x/\sqrt{n}$ , we obtain

$$E_3 = \int_z^\infty g_x(t) k_n(x; t) dt = \int_z^\infty g_x(t) d_t \lambda_n(x; t).$$

We define  $Q_n(x; t)$  on  $[0, 2x]$  as

$$\begin{aligned} Q_n(x, t) &= 1 - \lambda_x(x; t-), \quad 0 \leq t < 2x \\ &= 0, \quad t = 2x, \end{aligned}$$

So

$$\begin{aligned} E_3 &= \int_z^{2x} g_x(t) d_t Q_n(x; t) - g_x(2x) \int_{2x}^{\infty} k_n(x; u) du + \int_{2x}^{\infty} g_x(t) d_t \lambda_n(x; t) \\ &= E_{31} + E_{32} + E_{33}; \text{ say.} \end{aligned} \quad (3.5)$$

Now, using partial integration for the first term, we get

$$E_{31} = g_x(z-)Q_n(x, z-) + \int_z^{2x} \hat{Q}_n(x; t) d_t g_x(t),$$

where  $\hat{Q}_n(x; t)$  is the normalized form of  $Q_n(x; t)$ . Since  $Q_n(x; z-) = Q_n(x; z)$  and  $|g_x(z-)| \leq V_x^{z-}(g_x)$ , we have

$$|E_{31}| \leq V_x^{z-}(g_x)Q_n(x; z) + \int_z^{2x} \hat{Q}_n(x; t) d_t V_x^t(g_x).$$

Further, by using Lemma 4(ii) and the fact that  $\hat{Q}_n(x; t) \leq Q_n(x; t)$  on  $[0, 2x]$ , we have

$$\begin{aligned} |E_{31}| &\leq V_x^{z-}(g_x) \frac{(2x+1)}{n(x-z)^2} + \frac{(2x+1)}{n} \int_z^{2x-} \frac{1}{(t-x)^2} d_t V_x^t(g_x) \\ &\quad + \frac{1}{2} \left( V_{2x-}^{2x}(g_x) \int_{2x}^{\infty} k_n(x; u) du \right) \\ &\leq V_x^{z-}(g_x) \frac{(2x+1)}{n(x-z)^2} + \frac{(2x+1)}{n} \int_z^{2x-} \frac{1}{(t-x)^2} d_t V_x^t(g_x) \\ &\quad + \frac{1}{2} \left( V_{2x-}^{2x}(g_x) \frac{2x+1}{nx^2} \right) \\ &\leq V_x^{z-}(g_x) \frac{(2x+1)}{n(z-x)^2} + \frac{(2x+1)}{n} \int_z^{2x} \frac{1}{(x-t)^2} d_t V_x^t(g_x). \\ &= V_x^{z-}(g_x) \frac{(2x+1)}{n(z-x)^2} + \frac{(2x+1)}{n} \left\{ \frac{V_x^{2x}(g_x)}{x^2} - \frac{V_x^{z-}(g_x)}{(z-x)^2} \right. \\ &\quad \left. + 2 \int_z^{2x} \frac{1}{(t-x)^3} V_x^t(g_x) dt \right\}. \end{aligned}$$

Thus

$$|E_{31}| \leq \frac{(2x+1)}{n} \left\{ \frac{V_x^{2x}(g_x)}{x^2} + 2 \int_z^{2x} \frac{1}{(t-x)^3} V_x^t(g_x) dt \right\}.$$

Replacing the variable in the last integral by  $x + x/\sqrt{u}$ , we have

$$\begin{aligned} \int_{x+x/\sqrt{n}}^{2x} V_x^t(g_x) \frac{dt}{(t-x)^3} &= \frac{1}{2x^2} \int_1^n V_x^{x+x/\sqrt{u}}(g_x) du \\ &\leq \frac{1}{2x^2} \sum_{k=1}^n V_x^{x+x/\sqrt{k}}(g_x). \end{aligned}$$

Therefore

$$\begin{aligned} |E_{31}| &\leq \frac{(2x+1)}{nx^2} \left\{ V_x^{2x}(g_x) + \sum_{k=1}^n V_x^{x+x/\sqrt{k}}(g_x) \right\} \\ &\leq \frac{2(2x+1)}{nx^2} \sum_{k=1}^n V_x^{x+x/\sqrt{k}}(g_x). \end{aligned} \quad (3.6)$$

Further, for evaluating  $E_{32}$ , using Lemma 4(ii) we obtain

$$|E_{32}| \leq g_x(2x) \frac{(2x+1)}{nx^2} \leq \frac{(2x+1)}{nx^2} \sum_{k=1}^n V_x^{x+x/\sqrt{k}}(g_x). \quad (3.7)$$

Finally, using Lemma 4(ii) and the assumption that  $f(t) = O(e^{\alpha t})$ , ( $\alpha > 0$ ) as  $t \rightarrow \infty$ , we find that for  $n$  sufficiently large,

$$\begin{aligned} |E_{33}| &\leq Mn \sum_{k=0}^{\infty} p_k(nx) \int_{2x}^{\infty} e^{\alpha t} \frac{e^{-nt} (nt)^k}{k!} dt \\ &= nM \sum_{k=0}^{\infty} p_k(nx) \int_{2x}^{\infty} e^{-(n-\alpha)t} \frac{(nt)^k}{k!} dt \\ &= mM \sum_{k=0}^{\infty} e^{-\alpha x} \cdot p_k(mx) \int_{2x}^{\infty} p_k(mt) \cdot \left(\frac{n}{m}\right)^{2k+1} dt, \text{ where } m = n - \alpha. \quad (3.8) \\ &\leq M \cdot e^{-\alpha x} \cdot m \sum_{k=0}^{\infty} \left(1 + \frac{\alpha}{m}\right)^{2k+1} p_k(mx) \int_{2x}^{\infty} p_k(mt) dt \\ &\leq M' \cdot \frac{(2x+1)e^{-\alpha x}}{nx^2}. \end{aligned}$$

Using (3.5) to (3.8), we have, for  $n$  sufficiently large,

$$|E_3| \leq \frac{3x^{-2}(2x+1)}{n} \sum_{k=1}^n V_x^{x-x/\sqrt{k}}(g_x) + O(1) \frac{x^{-2}(2x+1)e^{-\alpha x}}{n}. \quad (3.9)$$

Using (3.2) to (3.4) and (3.9) we are lead to the proposition (3.1) of the theorem which can easily be checked to be asymptotically the best.

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