PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 54 (68), 1993, 63–70

GENERALIZATION OF THE MERCER THEOREM

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Abstract. It was shown that the bilinear series of the given operator $A = H(I + S)$ $(H = H^*, S = S^*)$ uniformly converges to its kernel.

1. **Introduction**

In this paper we will prove some results related to possibilities of representation of integral operator kernels (close to the selfadjoint ones) by bilinear series. Let A be an integral operator acting on $L^2[a, b]$ with a continuous symetric kernel $K(\cdot, \cdot)$ on $[a, b] \times [a, b]$ $(b - a < \infty)$. Let λ_i be the eigenvalues and ϕ_i the corresponding eigenvectors, forming an orthonormal system.

It is well known [**3**] the following classical Mercer theorem: If all eigenvalues of the operator A are positive, then

$$K(x, y) = \sum_{i=1}^{\infty} \lambda_i \phi_i(x) \cdot \overline{\phi_i(y)}$$

where the convergence is uniform on $[a, b] \times [a, b]$.

In [**1**] a similar statement is proved for operators of the form $A = H(I + S)$, where H and S are integral operators with symetric continuous kernels such that $H > 0$ and the spectar of the operator S lies within $(-1, \|S\|)$. We shall prove that a similar statement is valid for operators S for which $I + S$ is not necessarily positive.

2. **Main result**

We shall consider operators $A = H(I + S)$, where H and S are selfadjoint integral operators with continuous kernels $H(\cdot, \cdot)$ and $S(\cdot, \cdot)$ on $[a, b] \times [a, b]$. It is

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AMS Subject Classification (1990): Primary 47 B 38

obvious that the operator A is compact. Let λ_i and $\overline{\lambda_i}$ be eigenvalues of A and A^*, and ϕ_i and ψ_i appropriate eigenvectors normed such that $(\phi_i, \psi_j) = \delta_{ij}$. (By $(\cdot,\cdot)$ we denote the inner product in $L^2[a,b]$ space, i.e. $(f,g) = \int_a^b f(x)\overline{g(x)}\,dx$, $f,g \in L^2[a,b]$.) Let $\mathcal{A}\ (\cdot,\cdot)$ be the kernel of the integral operator $A = H(I+S)$. The main result of the paper is the following

THEOREM. *If H is a positive operator such that* $\mathrm{Ker}H = \{0\}$ *and* -1 *is a regular point of* S, *then*

$$\mathcal{A}(x,y) = \sum_{i=1}^{\infty} \lambda_i \phi_i(x) \cdot \overline{\psi_i(y)} \tag{1}$$

where convergence is uniform on $[a,b] \times [a,b]$.

Proof. Since the operator S is selfadjoint with continuous kernel on $[a,b] \times [a,b]$, it is also compact, so that the following expansion holds

$$S = \sum_{\nu=1}^{\infty} \mu_\nu (\cdot, e_\nu) e_\nu$$

where $\{e_\nu\}$ is an orthonormal base of the space $L^2[a,b]$, formed by the eigenvectors of S. (Some of μ_ν are equal to zero.) Let $\mu_1, ...\mu_k < -1 < \mu_{k+1}, \mu_{k+2}, ...$ We write

$$I + S = \sum_{\nu=1}^{\infty} (1+\mu_\nu)(\cdot, e_\nu) e_\nu .$$

Let us introduce an operator in the following way

$$|I+S| = \sum_{\nu=1}^{\infty} |1+\mu_\nu| (\cdot, e_\nu) e_\nu .$$

The operator $|I+S|$ is positive and invertible since $-1 \in \rho(S)$. Thus, the operators $|I+S|^{\frac{1}{2}} = \sum_{\nu=1}^{\infty} |1+\mu_\nu|^{\frac{1}{2}} (\cdot, e_\nu) e_\nu$ and $|I+S|^{-\frac{1}{2}} = \sum_{\nu=1}^{\infty} |1+\mu_\nu|^{-\frac{1}{2}} (\cdot, e_\nu) e_\nu$ are positive and invertible. We shall consider the operators $A_1 = |I+S|^{\frac{1}{2}} A |I+S|^{-\frac{1}{2}}$ and $J = (I+S)|I+S|^{-1}$, where $A = H(I+S)$. It is clear that

$$J = -\sum_{\nu=1}^{k} (\cdot, e_\nu) e_\nu + \sum_{\nu=k+1}^{\infty} (\cdot, e_\nu) e_\nu .$$

Let us introduce projectors P_+ and P_- in the following way: $P_- = \sum_{\nu=1}^{k} (\cdot, e_\nu) e_\nu$, $P_+ = \sum_{\nu=k+1}^{\infty} (\cdot, e_\nu) e_\nu$; then we obtain $J = P_+ - P_-$. It is obvious that $P_+ + P_- = I$, $J^* = J, J^2 = J$. Let us now introduce an indefinite product $[x,y] = (Jx,y)$, where $(\cdot,\cdot)$ is the inner product in $L^2(a,b)$, $x,y \in L^2(a,b)$. Using the commutativity of $|I+S|^{\pm\frac{1}{2}}$ and $I+S$ and the fact that $\mathrm{Ker}H = \{0\}$, we obtain

$$[A_1 x, y] = [x, A_1 y] \qquad \forall x,y \in L^2(a,b), \tag{2}$$

$$[A_1 x, x] > 0, \qquad \forall x \in L^2(a,b), x \neq 0. \tag{3}$$

Since the operator A_1 is compact and by (3) J is positive, there exists (see [**2**]) J orthonormal system of the eigenvectors of A_1, so that the spectral theorem is valid, i.e.:

$$A_1 \phi'_n = \lambda_n \phi'_n, \qquad [\phi'_n, \phi'_m] = \delta_{nm} \tag{4}$$

$$A_1 f = \sum_{n=1}^{\infty} \lambda_n \cdot [f, \phi'_n] \phi'_n \tag{5}$$

where the convergence in (5) is the convergence in the L^2 norm. Moreover, the system ϕ'_n is the base of the space $L^2(a,b)$. (see [**2**, p. 271]). By (4) it is clear that $\lambda_n > 0$. Really, by (4) we have $[A_1\phi'_n, \phi'_n] = \lambda_n \cdot [\phi'_n, \phi'_n] = \lambda_n$, so that, by (3), we have that $\lambda_n > 0$. Bearing in mind the definitions of the operators A_1 and J, we can write the relation (5) in the following way

$$|I+S|^{\frac{1}{2}} A |I+S|^{-\frac{1}{2}} f = \sum_{n=1}^{\infty} \lambda_n \cdot (Jf, \phi'_n) \cdot \phi'_n. \tag{6}$$

Certainly, the convergence in (6) is the convergence in the L^2 norm. Applying the bounded operator $|I+S|^{-\frac{1}{2}}$ to the equality (6) and introducing the vectors $\phi_n = |I+S|^{-\frac{1}{2}} \phi'_n$ we obtain $A\phi_n = \lambda_n \phi_n$ and

$$A|I+S|^{-\frac{1}{2}} f = \sum_{n=1}^{\infty} \lambda_n \cdot ((I+S) \cdot |I+S|^{-\frac{1}{2}} f, \phi_n) \cdot \phi_n. \tag{7}$$

The operator $|I+S|^{-\frac{1}{2}}$ has bounded inverse, so that

$$Af = \sum_{n=1}^{\infty} \lambda_n \cdot (f, (I+S)\phi_n) \cdot \phi_n. \tag{8}$$

where convergence in (8) is taken as the convergence in the L^2 norm. From $[\phi'_n, \phi'_n] = \delta_{nm}$, the definitions of the operator J and vectors ϕ_n it follows that $\delta_{mn} = [\phi'_n, \phi'_m] = ((I+S)\phi_n, \phi_n)$. If we introduce vectors $\psi_n = (I+S)\phi_n$, then we have that $(\phi_n, \psi_m) = \delta_{nm}$, which shows that the expansion (8) can be written as

$$Af = \sum_{n=1}^{\infty} \lambda_n (f, \psi_n) \phi_n. \tag{9}$$

The expansion (9) converges in the L^2 norm and is valid for all $f \in L^2(a,b)$. Since the system ϕ'_n is the base of L^2, (because of invertibility of $|I+S|^{-\frac{1}{2}}$ and $I+S$) the systems ϕ_n and ψ_n are also bases of $L^2(a,b)$. Continuity of kernels of the operators A and $A\phi_n = \lambda_n \phi_n$ imply that the functions ϕ_n are continuous on $[a,b]$. That implies that the functions $\psi_n = (I+S)\phi_n$ are also continuous on $[a,b]$ (because the integral operator S has continuous kernel). The expansion (9) implies

$$A^*f = \sum_{n=1}^{\infty} \lambda_n (f,\phi_n)\psi_n \tag{10}$$

so that $A^*\psi_n = \lambda_n \cdot \psi_n$ holds. Since $A = H(I+S)$, we have $A^* = (I+S)H$ so, (because $\psi_n = (I+S)\phi_n$) the expansion (10) can be written as

$$(I+S)Hf = \sum_{n=1}^{\infty} \lambda_n (f,\phi_n)(I+S)\phi_n. \tag{11}$$

The operator $(I+S)^{-1}$ is bounded, so that (11) implies

$$Hf = \sum_{n=1}^{\infty} \lambda_n (f,\phi_n)\phi_n. \tag{12}$$

The convergence in (12) is the convergence in the L^2 norm. The expansion (12) will play the main role in proving the main result. Notice that (12) is not spectral expansion of the selfadjoint operator H because the functions ϕ_n do not form orthonormal system. In the case when $S = 0$ the operator A becomes H, the system ϕ_n becomes orthonormal and the expansion (12) represents the spectral theorem for H. Let us prove that

$$H(x,y) = \sum_{n=1}^{\infty} \lambda_n \phi_n(x)\overline{\phi_n(y)} \tag{13}$$

where convergence in (13) is uniform on $[a,b]\times[a,b]$. Let us introduce a continuous function $H_n(x,y) = H(x,y) - \sum_{k=1}^{n} \lambda_k \phi_k(x)\overline{\phi_k(y)}$. Because of (12) we have that, for $f \in L^2(a,b)$

$$\int_a^b H_n(x,y)f(y)\,dy = \sum_{k=n+1}^{\infty} (f,\phi_k)\phi_k. \tag{14}$$

Let $\mathcal{H}_n$ denotes an integral operator whose kernel is $H_n(\cdot,\cdot)$. From (14) we obtain

$$(\mathcal{H}_n f, f) \geq 0. \tag{15}$$

Let us prove that $H_n(x,x) \geq 0$ on $[a,b]$. Suppose contrary, i.e. that there is $x_0 \in [a,b]$ so that $H_n(x_0,x_0) < 0$. The definition of the function H_n implies

$$\overline{H_n(y,x)} = H_n(x,y) \tag{16}$$

Let us write the function H_n as $H_n(x,y) = P_n(x,y) + iQ_n(x,y)$, where the functions $P_n(\cdot,\cdot)$ and $Q_n(\cdot,\cdot)$ are real and continuous. Now, by (16) we have

$$P_n(x,y) = P_n(y,x), Q_n(x,y) = -Q_n(y,x), \quad \forall x,y \in [a,b]. \tag{17}$$

That implies $Q_n(x_0, x_0) = 0$ so that the condition $H_n(x_0, x_0) < 0$ means nothing but $P_n(x_0, x_0) < 0$. Since $P_n(\cdot, \cdot)$ is a real and continuous function, there is a neighbourhood $U \times U$, where $U = (x_0 - \delta, x_0 + \delta)$, of the point (x_0, x_0) so that any $(x, y) \in U \times U$ satisfies $P_n(x, y) < 0$. We shall introduce a function

$$f_\delta(x) = \begin{cases} 0, & x \notin U \\ 1, & x \in U. \end{cases}$$

The function f_δ is from $L^2(a, b)$ and

$$(\mathcal{H}_n f_\delta, f_\delta) = \iint\limits_{U \times U} H_n(x, y)\, dxdy =$$

$$= \iint\limits_{U \times U} P_n(x, y)\, dxdy + i \iint\limits_{U \times U} Q_n(x, y)\, dxdy.$$

Since (by (17)) the integral $\iint\limits_{U \times U} Q_n(x, y)\, dxdy$ is equal to zero, we have

$$(\mathcal{H}_n f_\delta, f_\delta) = \iint\limits_{U \times U} P_n(x, y)\, dxdy.$$

The function $P(\cdot, \cdot)$ is negative on the square $U \times U$, so that $(\mathcal{H}_n f_\delta, f_\delta) < 0$ what contradics (15). Thus, $H_n(x, x) \geq 0$ for all $x \in [a, b]$ is proved. Now, we can obtain, because of the definition of the function H_n, that

$$\sum_{k=1}^{n} \lambda_k |\phi_k(x)|^2 \leq H(x, x) \tag{18}$$

is true. Since H is continuous, we have $H(x, x) \leq M < \infty$ for $x \in [a, b]$, and $\sum_{k=1}^{\infty} \lambda_k |\phi_k(x)|^2 \leq M$.

Choose $x_0 \in [a, b]$. Then from

$$|\sum_{n=p}^{q} \lambda_n \phi_n(x_0)\overline{\phi_n(y)}|^2 \leq \sum_{n=p}^{q} \lambda_n |\phi_n(x_0)|^2 \cdot \sum_{n=p}^{q} \lambda_n |\phi_n(y)|^2 \leq$$

$$M \sum_{n=p}^{q} \lambda_n |\phi_n(x_0)|^2 \longrightarrow 0 \ \ (p, q \longrightarrow \infty)$$

it follows that the series

$$\sum_{n=1}^{\infty} \lambda_n \phi_n(x_0)\overline{\phi_n(y)} \tag{19}$$

uniformly converges to $B(x_0, y)$, say, which is continuous function of y. Let $f \in C[a, b]$ be an arbitrary function. The uniform convergence of the series (19) shows that

$$\begin{aligned}\int_a^b B(x_0, y) f(y)\, dy &= \sum_{n=1}^{\infty} \lambda_n \phi_n(x_0) \int_a^b f(y) \overline{\phi_n(y)}\, dy \\ &= \sum_{n=1}^{\infty} \lambda_n \phi_n(x_0)(f, \phi_n).\end{aligned} \tag{20}$$

Let us prove that

$$\int_a^b H(x_0, y) f(y)\, dy = \sum_{n=1}^{\infty} \lambda_n \phi_n(x_0)(f, \phi_n). \tag{21}$$

The systems ϕ_n and ψ_n are bases of $L^2[a, b]$ and $(\phi_n, \psi_m) = \delta_{mn}$. Then we obtain $f = \sum_{n=1}^{\infty} (f, \phi_n)\psi_n$. Because of that, we can write

$$\| f - f_n \|_{L^2(a,b)} \longrightarrow 0, \qquad (n \longrightarrow \infty) \tag{22}$$

where $f_n = \sum_{k=1}^{n} (f, \phi_k)\psi_k$. Notice that, by (12), we have $H\psi_k = \lambda_k \phi_k$ and we obtain

$$(Hf_n)(x_0) = \int_a^b H(x_0, y) \sum_{k=1}^{n} (f, \phi_k)\psi_k(y)\, dy = \sum_{k=1}^{n} \lambda_k (f, \phi_k)\phi_k(x_0).$$

Since $(Hf)(x_0) = \int_a^b H(x_0, y) f(y)\, dy$, we have the following estimation

$$\begin{aligned}\left|(Hf)(x_0) - \sum_{\nu=1}^{n} \lambda_\nu (f, \phi_\nu)\phi_\nu(x_0)\right|^2 &= \left|\int_a^b H(x_0, y)(f(y) - f_n(y))\, dy\right|^2 \\ &\le \int_a^b |H(x_0, y)|^2\, dy \cdot \| f - f_n \|^2 \longrightarrow 0 \quad (n \longrightarrow \infty)\end{aligned}$$

(because of (22)) what proves the equality (21). Now, if we combine (20) and (21), we obtain that for any $f \in L^2(a, b)$

$$\int_a^b B(x_0, y) f(y)\, dy = \int_a^b H(x_0, y) f(y)\, dy$$

i.e.

$$\int_a^b (B(x_0, y) - H(x_0, y)) f(y)\, dy = 0.$$

If we have chosen the continuous function $f(y) = \overline{B(x_0, y) - H(x_0, y)}$ instead of f, we would have obtained

$$\int_a^b |B(x_0, y) - H(x_0, y)|^2\, dy = 0$$

implying that $B(x_0, y) = H(x_0, y)$ for all $y \in [a, b]$. If we put $y = x_0$ in the preceding equality, then we will have

$$\sum_{n=1}^{\infty} \lambda_n |\phi_n(x_0)|^2 = H(x_0, x_0).$$

Since x_0, is arbitrary in the interval $[a, b]$ and the function $H(\cdot, \cdot)$ is continuous, we have, by the Dini's theorem, that the series $\sum_{n=1}^{\infty} \lambda_n |\phi_n(x)|^2$ uniformly converges on $[a, b]$ and $\sum_{n=1}^{\infty} \lambda_n |\phi_n(x)|^2 = H(x, x)$

The uniform convergence of the series $\sum_{n=1}^{\infty} \lambda_n |\phi_n(x)|^2 = H(x, x)$ on $[a, b]$ and

$$\psi_n(x) = (I + S)\phi_n(x) = \phi_n(x) + \int_a^b S(x, y)\phi_n(y)\, dy$$

show that series $\sum_{n=1}^{\infty} \lambda_n |\psi_n(x)|^2$ also uniformly converges on $[a, b]$. Let us now prove that the series $\sum_{n=1}^{\infty} \phi_n(x)\overline{\psi_n(y)}$ uniformly converges on $[a, b] \times [a, b]$. It follows directly from the estimation

$$|\sum_{n=p}^{q} \lambda_n \phi_n(x)\overline{\psi_n(y)}|^2 \leq \sum_{n=p}^{q} \lambda_n |\phi_n(x)|^2 \cdot \sum_{n=p}^{q} \lambda_n |\psi_n(y)|^2$$

and from the uniform convergence of the series

$$\sum_{n=1}^{\infty} \lambda_n |\psi_n(x)|^2 \quad \text{and} \quad \sum_{n=1}^{\infty} \lambda_n |\phi_n(x)|^2$$

on the interval $[a, b]$. Since the systems ϕ_n and ψ_n are bases of $L^2(a, b)$ and $(\phi_n, \psi_m) = \delta_{nm}$, the systems Φ_{np} and G_{mq}, where $\Phi_{np}(x, y) = \phi_n(x)\overline{\psi_p(y)}$ and $G_{mq}(x, y) = \psi_m(x)\overline{\phi_q(y)}$, are the bases of $L^2((a, b) \times (a, b))$ and

$$(\Phi_{np}, G_{mq})_{L^2((a,b)\times(a,b))} = \delta_{nm}\delta_{qp}$$

is valid. Because of that we have

$$\mathcal{A}(x, y) = \sum_{n,p=1}^{\infty} (\mathcal{A}, G_{n,p})_{L^2((a,b)\times(a,b))}\Phi_{np}(x, y) \tag{23}$$

where the convergence in (23) is taken as the convergence in $L^2((a, b) \times (a, b))$ norm.

Since $(\mathcal{A}, G_{np})_{L^2((a,b)\times(a,b))} = (A\phi_n, \psi_n)_{L^2(a,b)} = \lambda_n$, the expansion (23) can be written as

$$\mathcal{A}(x, y) = \sum_{n=1}^{\infty} \lambda_n \phi_n(x)\overline{\psi_n(y)}$$

where the convergence in $L^2((a,b)\times(a,b))$ norm. As it is already shown that the series $\sum_{n=1}^{\infty}\lambda_n\phi_n(x)\overline{\psi_n(y)}$ uniformly converges on $[a,b]\times[a,b]$, we obtain

$$\mathcal{A}(x,y)=\sum_{n=1}^{\infty}\lambda_n\phi_n(x)\overline{\psi_n(y)}$$

where the convergence is uniform on $[a,b]\times[a,b]$. That proves the theorem.

Remark 1 The theorem also holds for operators whose kernels are functions $H(\cdot,\cdot),S(\cdot,\cdot):R^{2k}\longrightarrow R$ if those kernels satisfy condition of the theorem.

Remark 2 The theorem can be used to establish the corresponding expansion theorems with respect to system of eigenfunctions for some nonselfadjoint differential operators whoose inverse operators have the form $H(I+S)$, where $H>0$, $\mathrm{Ker}H=\{0\}$, $S=S^*$ and $-1\in\rho(S)$.

Example Let L_1 and L_2 be selfadjoint differential operators generated by differential expresions

$$l_1(y)=\sum_{i=0}^{n}\alpha_i y^{(i)},\quad \alpha_i=\alpha_i(x)\qquad l_2(y)=\sum_{i=0}^{m}\beta_i y^{(i)},\quad \beta_i=\beta_i(x)$$

and by selfadjoint boundary condition

$$U_i^{(1)}(y)=0,\ i=1,2,...,n\qquad U_j^{(2)}(y)=0,\ j=1,2,...,m.$$

Suppose that L_1 is a positive differential operator. Let $\mathrm{Ker}L_2=\mathrm{Ker}L_1=\{0\}$ and -1 be regular point of the operatorL_2. Denote by $G(\cdot,\cdot)$ Green function of the operator L_2+I. We consider the operator

$$Ly=\int_0^1 G(x,\xi)l_2(l_1(y(\xi)))d\xi$$

with boundary condition

$$\begin{cases}U_i^{(1)}(y)=0,\ i=1,2,...,n\\ U_j^{(2)}(l_1(y))=0,\ j=1,2,...,m.\end{cases}\tag{*}$$

Then every function $f\in C^{m+n}[a,b]$ satisfying boundary conditions $(*)$ can be expanded in a uniform convergent series with respect to a system of eigenfunctions of the operator L.

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(Received 28 06 1993)