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A FUNCTION FROM DIOPHANTINE APPROXIMATIONS

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Abstract. Some remarkable properties of a function defined from consideration of Diophantine approximations are established. For example, the function is continuous only at irrational points whose images are transcendental numbers, and the range of the function has Hausdorff dimension 0.

1. A remarkable function. The number of integer points in an interval of length θ is either $[\theta]$ or $[\theta]+1$, with the precise answer depending on the positions of the end points of the interval. For a family of intervals associated with a parameter the required analysis amounts to the study of the parameter for the condition under which we should have $\{\theta_1\}+\{\theta_2\}>1$; here $\{\theta\}=\theta-[\theta]$ is the fractional part of θ. We consider such a problem by introducing a real function defined in an interval.

For $0<\alpha<1$ and $m=1,2,\dots$, we set

$$d_m = d_m(\alpha) = \begin{cases} 1 & \text{if } \{m/\alpha\}+\{1/\alpha\}\geq 1, \\ 0 & \text{otherwise}, \end{cases} \tag{1.1}$$

and define a function $f:(0,1)\to[0,1)$ by letting (d_m) to be the sequence of binary digits for $f(\alpha)$, that is

$$f(\alpha)=\sum_{m=1}^{\infty}\frac{d_m(\alpha)}{2^m}. \tag{1.2}$$

It follows almost immediately from (1.1) that (d_m) is periodic if and only if α is rational, so that $f(\alpha)$ is rational only for such α. As we shall see in Section 3, we need only deal with the subdomain

$$I=\{\alpha:\tfrac{1}{2}<\alpha<1\}, \tag{1.3}$$

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wherein f is strictly decreasing. It turns out that f has some remarkable properties similar to those possessed by what physicists call the *devil's staircase* associated with the classical Cantor's ternary set; see, for example, [**3**, Chapter 13]. We shall prove the following two theorems.

THEOREM 1. *The function f is strictly decreasing in I, and is continuous at each irrational point. Furthermore, it has a jump discontinuity of $1/(2^p - 1)$ at a rational point p/q.*

THEOREM 2. *The range of f has Hausdorff dimension 0. Furthermore, if α is irrational, then $f(\alpha)$ is transcendental.*

2. Notation and preliminary remarks. Except for functions and sets of numbers, we use Roman and Greek letters to denote integers and real numbers, respectively. Continued fractions play a dominating role in the analysis of f, and we shall use the notation in [**2**], where proofs for the results used on such fractions can be found. Thus, for an irrational $\alpha \in I$, we shall write

$$\alpha = [1, a_2, a_3, \ldots], \tag{2.1}$$

where a_n is the n-th partial quotient for α. When α is rational, there are only finitely many such quotients, and we may choose to have either an odd or an even number of them in the representation of α by making an appropriate adjustment to the last partial quotient. We also write

$$\alpha_n = [1, a_2, \ldots a_n] = \frac{p_n}{q_n}, \qquad d(m,n) = d_m(\alpha_n), \tag{2.2}$$

so that α_n is the n-th convergent for α in (2.1), and $d(m,n)$ is the m-th digit for $f(\alpha_n)$.

For a fixed positive integer p, we let

$$r(x) = x - \left[\frac{x}{p}\right]p, \tag{2.3}$$

the remainder of x when divided by p. When $\alpha = p/q$, we let $r = r(q)$, $s = r(-q)$ and $\rho = r(q)/r(-q)$, so that for $1 < p < q < 2p$ we have

$$r = q - p, \qquad s = 2p - q, \qquad \rho = \frac{r}{s}. \tag{2.4}$$

In particular, when $\alpha_n = p_n/q_n$ is the n-th convergent to α in (2.1), we let r_n, s_n and ρ_n be given by (2.4), so that the usual iterative formula $u_n = a_n u_{n-1} + u_{n-2}$ holds when u_n is any of the terms p_n, q_n, r_n, s_n with their own initial values. In fact $1/\rho_n = [a_3, a_4, \ldots, a_n]$ when $\frac{1}{2} < \alpha < \frac{2}{3}$, and we shall require

$$\rho_n - \rho_{n-1} = \frac{(-1)^n}{s_n s_{n-1}}. \tag{2.5}$$

A rational number with denominator 2^p has the terminating binary representation $\sum_{m\le p} d_m/2^m$, to which we may attach the finite string $D = (d_1 d_2 \dots d_p)$. Extra brackets may be inserted in between the digits d_m, but all brackets are meant to be ignored, so that we can concatenate such strings to form new strings. Thus, if $D' = d'_1 d'_2 \dots d'_q$ is another string, then we may write $DD' = (d_1 d_2 \dots d_p)(d'_1 d'_2 \dots d'_q)$ $= d_1 d_2 \dots d_p d'_1 d'_2 \dots d'_q$, and we also let $D^a = DD^{a-1}$ for $a \ge 2$. We call 0^a a null-string of length a, and all other strings, which must have at least one digit $d_m = 1$, are said to be positive.

The range of f in I will be denoted by R, and we let $\phi(n)$ be Euler's totient function, which counts the integers $a \le n$ that are coprime with n. The proof of the second part of Theorem 2 requires the following theorem, the proof of which can be found in [**1**].

THUE-SIEGEL-ROTH THEOREM. *Let α be a real irrational algebraic number, and $\epsilon > 0$. Then there are only finitely many fractions u/v such that*

$$\left|\alpha - \frac{u}{v}\right| < \frac{1}{v^{2+\epsilon}}.$$

3. The image of f at a rational point. Let $\alpha = p/q$, where p, q are coprime integers satisfying $1 \le p < q$. With $r(x)$ defined by (2.3), the condition in (1.1) for $d_m = 1$ becomes $r(mq) + r(q) \ge p$. Since $r(mq) = r(m'q)$ whenever $m \equiv m' \pmod p$, it follows that the sequence (d_m) is periodic with period p. Note also that (d_m) depends on q only to the extend that it actually depends on $r(q)$, so that we need only study the case $kp < q \le (k+1)p$ for one particular value of k, which we now choose to be 1. It is clear that $f(\frac{1}{2}) = 0$, so that we may restrict ourselves to $1 < p < q < 2p$, that is the study of $f(\alpha)$ with $\alpha \in I$, the interval given by (1.3). With the notation in (2.3) and the introduction of r and s in (2.4), the condition (1.1) for $d_m = 1$ can now be rewritten as

$$r(mq) \ge s. \tag{3.1}$$

We set

$$\delta = \delta(\alpha) = \sum_{m\le p} \frac{d_m}{2^m}, \qquad Q = Q(\alpha) = \sum_{m\le p} d_m 2^{p-m}, \tag{3.2}$$

so that $Q = 2^p\delta$. From the periodicity of (d_m) and (1.2) we find that

$$f(\alpha) = \delta\Big(1 + \frac{1}{2^p} + \frac{1}{2^{2p}} + \cdots\Big) = \frac{\delta 2^p}{2^p - 1} = \frac{Q}{P}, \qquad P = 2^p - 1, \tag{3.3}$$

so that $f(\alpha)$ is completely specified by the string attached to δ, namely

$$D = d_1 d_2 \dots d_p. \tag{3.4}$$

We now call D the string associated with α, and our immediate task is to find an explicit formula for the digits d_m. Such a string D always ends with the digits $(10) = (1)(0)$, that is $d_{p-1} = 1$, $d_p = 0$, and it turns out that r and s in (2.4) are precisely the numbers of digits in D taking the values 1 and 0, respectively.

LEMMA 1. *Let $D = d_1 d_2 \dots d_p$ be the string associated with $\alpha = p/q$. Then $d_{p-1} = 1$, and*

$$\begin{cases} d_m = 1 \quad \textit{when} \quad m = \ell + \left[\dfrac{s\ell}{r}\right], \quad 1 \le \ell < r, \\ d_m = 0 \quad \textit{when} \quad m = \ell + \left[\dfrac{r\ell}{s}\right], \quad 1 \le \ell \le s. \end{cases} \tag{3.5}$$

Proof. If $m = p - 1$, then $mq \equiv -q \equiv s \pmod{p}$ by (2.4), so that $r(mq) = s$ and hence $d_{p-1} = 1$ by (3.1). Now let m be one of the values given by the first formula in (3.5), with $1 \le \ell < r$. Since $p = r + s$, we have $m = [p\ell/r]$, and we may now write $p\ell = mr + b$, where $1 \le b < r$. From $r(mq) \equiv mq \equiv mr \equiv -b \pmod{p}$, we deduce that $r(mq) = p - b > p - r = s$, so that $d_m = 1$ by (3.1).

Similarly, let m be one of the values given by the second formula in (3.5), with $1 \le \ell \le s$. On writing $p\ell = ms + b$, where $0 \le b < s$, we find that $r(mq) = b$ and hence $d_m = 0$ by (3.1). Lemma 1 is proved.

It follows from Lemma 1 and (3.2) that $Q \equiv 2 \pmod{4}$, and we shall see later that P, Q are coprime, so that the representation of $f(\alpha)$ in (3.3) is already in lowest fraction. We also remark that if $\alpha' = p/q'$, where $q' + q = 3p$, then $d_m(\alpha) + d_m(\alpha') = 1$ for $1 \le m < p - 1$, and $Q(\alpha) + Q(\alpha') = 2^p$, giving

$$f(\alpha) + f(\alpha') = 1 + \frac{1}{P}, \qquad P = 2^p - 1.$$

This follows from (3.2) and (3.3) together with the observation that $s = r(q')$ by (2.4).

4. Iterative formulae for D_n. In order to study the local behaviour for f at a point α, we need to consider the rational approximations to α. From the reciprocal relationship giving the values for d_m in (3.5), we see that the natural approach is to use continued fractions. For our purpose, we need to find the relationship between D_n and D_{n+1}, the strings associated with α_n and α_{n+1}, the successive convergents for α. By (2.2) and (3.4),

$$D_n = d_1 d_2 \dots d_p, \qquad d_m = d(m, n), \qquad p = p_n. \tag{4.1}$$

Since $a_1 = 1$, the first convergent is $\alpha_1 = 1$, so that $p_1 = q_1 = 1$, and hence D_1 is the null-string which corresponds to $f(1) = 0$. Next, if $a_2 = 1$ also, then the second convergent is $\alpha_2 = \frac{1}{2}$, so that D_2 is also the null-string corresponding to $f(\frac{1}{2}) = 0$, but the third convergent α_3 will have an associated string D_3 which is positive.

On the other hand, if $a_2 > 1$ then D_2 is already a positive string. We therefore distinguish between the two cases depending on whether $a_2 = 1$ or $a_2 > 1$, which corresponds to $\alpha < \frac{2}{3}$ or $\alpha > \frac{2}{3}$, and we remark that $f(\frac{2}{3}) = \frac{2}{3}$. It turns out that, for either case, there is a rather simple relationship between D_n and D_{n+1}.

LEMMA 2. *Let* $\frac{1}{2} < \alpha < \frac{2}{3}$. *Then* $D_1 = D_2 = (0)$ *and*

$$D_3 = (1^{a_3}0), \qquad D_4 = D_3^{a_4-1}(1)D_3, \tag{4.2}$$

and, for $n > 1$,

$$D_{2n+1} = D_{2n-1}D_{2n}^{a_{2n+1}}, \qquad D_{2n+2} = D_{2n+1}^{a_{2n+2}}D_{2n}. \tag{4.3}$$

If $\frac{2}{3} < \alpha < 1$, *then* $D_1 = (0)$, $D_2 = (0^{a_2-2}10)$, $D_3 = (0)D_2^{a_3}$, $D_4 = D_3^{a_4}D_2$ *and* (4.3) *also holds.*

Proof. Let $\frac{1}{2} < \alpha < \frac{2}{3}$, so that $\alpha = [1, 1, a_3, a_4, \ldots]$. From (2.2) and (2.4) we find that $r_1 = r_2 = 0$ and

$$\begin{cases} r_3 = a_3, \\ s_3 = 1, \end{cases} \qquad \begin{cases} r_4 = a_4a_3 + 1, \\ s_4 = a_4, \end{cases} \tag{4.4}$$

so that $D_1 = D_2 = (0)$. By Lemma 1, we have $d(m, n) = 0$ when

$$m = \ell + [\rho_n \ell], \qquad 1 \le \ell \le s_n. \tag{4.5}$$

From (4.4) we see that $d(m, 3) = 0$ only when $m = p_3$, so that $D_3 = 1^{a_3}0$. It also follows from (4.4) that $d(m, 4) = 0$ when $m = p_4$ and $m = (a_3 + 1)\ell$ for $1 \le \ell < a_4$, which then gives $D_4 = D_3^{a_4-1}1D_3$. Therefore (4.2) is proved.

Now let $n \ge 5$ be an odd number. We first show that

$$[\rho_n \ell] = [\rho_{n-2}\ell], \qquad 1 \le \ell \le s_{n-2}. \tag{4.6}$$

From (2.5) we have

$$\rho_n = \rho_{n-1} - \frac{1}{s_n s_{n-1}} = \rho_{n-2} + \frac{1}{s_{n-1}s_{n-2}} - \frac{1}{s_n s_{n-1}}, \tag{4.7}$$

which implies $0 < (\rho_n - \rho_{n-2})\ell < 1/s_{n-1}$. Since the fractional part $\{\rho_{n-2}\ell\} \le 1 - 1/s_{n-2}$, the required result (4.6) is established. It now follows from (4.5) and (4.6) that

$$d(m, n) = d(m, n-2), \qquad 1 \le m \le p_{n-2}, \tag{4.8}$$

which means that the first p_{n-2} digits for D_n are the same as those for D_{n-2}. Next, we let $0 \le a < a_n$ and consider the digits $d(m, n)$ in the block $p_{n-2} + ap_{n-1} < m \le p_{n-2} + (a + 1)p_{n-1}$. Such digits take the value 0 when m has the form (4.5) with

$s_{n-2}+as_{n-1}<\ell\le s_{n-2}+(a+1)s_{n-1}$, and we now put $\ell=s_{n-2}+as_{n-1}+\ell'$, so that $1\le\ell'\le s_{n-1}$. From (4.7) we find that

$$\rho_n s_{n-2}=r_{n-2}+\frac{1}{s_{n-1}}-\frac{s_{n-2}}{s_n s_{n-1}},\quad a\rho_n s_{n-1}=ar_{n-1}-\frac{1}{s_n},\quad \rho_n\ell'=\rho_{n-1}\ell'-\frac{\ell'}{s_n s_{n-1}},$$

so that the relevant values for m are given by

$$m=p_{n-2}+ap_{n-1}+\ell'+\Big[\rho_{n-1}\ell'+\frac{s_n-\ell'-s_{n-2}}{s_n s_{n-1}}\Big].$$

Observe that the first term inside the square bracket is short of an integer by at least $1/s_{n-1}$, which exceeds the value for the remaining term. Therefore the value of the square bracket is simply $[\rho_{n-1}\ell']$, and hence

$$d(p_{n-2}+ap_{n-2}+m,n)=d(m,n-1),\qquad 1\le m\le p_{n-1},\quad 0\le a<a_n.$$

Together with (4.8), we have established the formula $D_n=D_{n-2}D_{n-1}^{a_n}$.

We omit the similar proof of the remaining formula in (4.3) corresponding to n being even. The case when $\frac{2}{3}<\alpha<1$ can be dealt with in the same way. Lemma 2 is proved.

5. The images of f at successive convergents. For the convergent $\alpha_n=p_n/q_n$ for α, we now rewrite (3.2) as

$$\delta_n=\sum_{m\le p_m}\frac{d_m}{2^m},\qquad Q_n=2^{p_n}\delta_n, \tag{5.1}$$

so that (3.3) becomes

$$f(\alpha_n)=\frac{Q_n}{P_n},\qquad P_n=2^{p_n}-1. \tag{5.2}$$

Most of the properties for f can be derived from the following lemma.

LEMMA 3. *For odd values of $n>1$, we have*

$$f(\alpha_{n+1})=f(\alpha_n)+\frac{P_n+1}{P_nP_{n+1}}=f(\alpha_{n+2})+\frac{P_{n+2}+1}{P_{n+1}P_{n+2}}. \tag{5.3}$$

Proof. By (5.2) the first equation in (5.3) amounts to

$$P_nQ_{n+1}-P_{n+1}Q_n=P_n+1,\qquad n\text{ odd}. \tag{5.4}$$

Let T_n denote the left hand side of this equation, which we proceed to establish by finding an iterative formula for Q_n. For a fixed integer n write

$$a=a_{n+1},\qquad \epsilon=\frac{1}{2^{p_n}}.$$

From (4.3) we have $D_{n+1} = D_n^a D_{n-1}$ for odd n, and (5.1) now gives

$$\delta_{n+1} = \delta_n(1 + \epsilon + \cdots + \epsilon^{a-1}) + \delta_{n-1}\epsilon^a = \delta_n \frac{1-\epsilon^a}{1-\epsilon} + \delta_{n-1}\epsilon^a.$$

Since $p_{n+1} = ap_n + p_{n-1}$, we have $2^{p_{n+1}} = 2^{ap_n}2^{p_{n-1}} = \epsilon^{-a}2^{p_{n-1}}$, and hence

$$2^{p_{n+1}}\delta_{n+1} = 2^{p_{n-1}}\epsilon^{-1}\delta_n \frac{\epsilon^{-a}-1}{\epsilon^{-1}-1} + 2^{p_{n-1}}\delta_{n-1}.$$

This then gives

$$Q_{n+1} = 2^{p_{n-1}}E_nQ_n + Q_{n-1}, \qquad n \text{ odd}, \tag{5.5}$$

where

$$E_n = \frac{\epsilon^{-a}-1}{\epsilon^{-1}-1} = \frac{2^{a_{n+1}p_n}-1}{2^{p_n}-1} = \frac{(P_n+1)^{a_{n+1}}-1}{P_n}.$$

From $2^{p_{n-1}}E_nP_n = P_{n+1} - P_{n-1}$, we obtain

$$(P_n+1)2^{p_{n-1}}EP_n = P_{n+1} - P_{n-1} + \frac{P_{n+1}-P_{n-1}}{P_n},$$

so that

$$(P_n+1)Q_{n+1} = \Big(P_{n+1} - P_{n-1} + \frac{P_{n+1}-P_{n-1}}{P_n}\Big)Q_n + 2^{p_n}Q_{n-1}.$$

Therefore, for odd values of n, we have

$$\begin{aligned} P_nQ_{n+1} - P_{n+1}Q_n &= -Q_{n+1} - \Big(P_{n-1} - \frac{P_{n+1}-P_{n-1}}{P_n}\Big)Q_n + 2^{p_n}Q_{n-1} \\ &= -\Big(2^{p_{n-1}}E_n + P_{n-1} - \frac{P_{n+1}-P_{n-1}}{P_n}\Big)Q_n + \big(2^{p_n}-1\big)Q_{n-1} \\ &= -P_{n-1}Q_n + P_nQ_{n-1}. \end{aligned}$$

Similarly, we find that

$$Q_{n+1} = 2^{ap_n}Q_{n-1} + E_nQ_n, \qquad n \text{ even},$$

and from

$$(P_n+1)Q_{n+1} = 2^{(a+1)p_n}Q_{n-1} + 2^{p_n}E_nQ_n$$

we obtain

$$\begin{aligned} P_nQ_{n+1} - P_{n+1}Q_n &= 2^{(a+1)p_n}Q_{n-1} + 2^{p_n}E_nQ_n - Q_{n+1} - P_{n+1}Q_n \\ &= 2^{(a+1)p_n}Q_{n-1} + \big(2^{p_n}E_nQ_n - P_{n+1}\big)Q_n - 2^{ap_n}Q_{n-1} - E_nQ_n \\ &= 2^{ap_n}\big(2^{p_n}-1\big)Q_{n-1} + (P_nE_n - P_{n+1})Q_n \\ &= 2^{ap_n}\big(P_nQ_{n-1} - P_{n-1}Q_n\big), \end{aligned}$$

because $-2^{ap_n}P_{n-1} = 2^{ap_n} - 2^{p_{n+1}} = (P_n+1)^a - P_{n+1} - 1 = P_nE_n - P_{n+1}$.

We have therefore proved that T_n is $-T_{n-1}$ or $-2^{a_{n+1}p_n}T_{n-1}$ depending on whether n is odd or even, and hence

$$T_n = 2^{a_np_{n-1}}T_{n-2}, \qquad \text{odd } n \geq 5.$$

Repeated application of this formula now gives $T_n = 2^{A_n}T_3$ for odd $n \geq 3$, where

$$\begin{aligned} A_n &= a_np_{n-1} + a_{n-2}p_{n-3} + \cdots + a_5p_4 \\ &= (p_n - p_{n-2}) + (p_{n-2} - p_{n-4}) + \cdots + (p_5 - p_3) \\ &= p_n - p_3. \end{aligned}$$

It remains to evaluate T_3, and we now let $\epsilon = 1/2^{p_3}$ and $a = a_4$. From $D_3 = 1^{a_3}0$, we have $\delta_3 = 1 - 2\epsilon$, and hence $Q_3 = \epsilon^{-1}\delta_3 = 2^{p_3} - 2 = P_3 - 1$. Again, from $D_4 = D_3^{a-1}1D_3$, the argument leading to (5.5) now yields $\delta_4 = \delta_3(1-\epsilon^a)/(1-\epsilon) + \epsilon^a$ and $P_3Q_4 = Q_4(P_3 - 1) + 2P_3$. Thus $T_3 = 2P_3 - Q_3 = P_3 + 1 = 2^{p_3}$.

Therefore $T_n = 2^{p_n} = P_n + 1$ for every odd $n \geq 3$, so that (5.4) is established. The second equation in (5.3) can be proved in the same way. Lemma 3 is proved.

6. Proof of Theorem 1. Let α be an irrational number given by (2.1). From Lemma 2, we see that $f(\alpha_{n+1})$ and $f(\alpha_{n+2})$ must have the same first p_n binary digits, and that these same digits are also the initial digits for $f(\alpha)$ itself. It follows from (1.2) that $f(\alpha_n) \to f(\alpha)$ as $n \to \infty$. Although this does not immediately imply that f is continuous at α, we can deduce that f is monotonic first. We certainly have $f(\alpha_{2n-1}) < f(\alpha_{2n+1}) < f(\alpha_{2n+2}) < f(\alpha_{2n})$ by (5.3) so that $f(\alpha_{2n-1}) < f(\alpha) < f(\alpha_{2n})$. Now let $\frac{1}{2} < \lambda < \mu < 1$. By considering the continued fraction expansions for λ and μ, we can find $\alpha_n = [1, a_2, a_3, \ldots, a_n]$ such that $\lambda < \alpha_{n-1} < \alpha_n < \alpha_{n-2} < \mu$, and hence, by what we have just established, $f(\lambda) > f(\alpha_n) > f(\mu)$. Therefore f is strictly decreasing in I.

Again, let α be an irrational point given by (2.1). Since f is monotonic, in the evaluation of the limit of $f(\beta)$ as $\beta \to \alpha$, we may let $\beta = \alpha_n = [1, a_2, a_3, \ldots, a_n]$ and $n \to \infty$. In fact, $\beta \to \alpha_+$ and $\beta \to \alpha_-$ now correspond to the restrictions that n be odd and even respectively. For odd n, we have $f(\alpha_{n+2}) < f(\alpha) < f(\alpha_{n+1})$ and, by (5.3),

$$f(\alpha_{n+1}) - f(\alpha_{n+2}) = \frac{P_{n+2}+1}{P_{n+1}P_{n+2}} \to 0 \quad \text{as} \quad n \to \infty,$$

and the same conclusion also holds when n is even. Therefore $f(\beta) \to f(\alpha)$ as $\beta \to \alpha$ when α is irrational, so that f is continuous at each irrational point.

Now let $\alpha = [1, a_2, a_3, \ldots, a_n]$ be rational, so that n is fixed, except that we may choose to have it being odd or even. Since f is monotonic, in considering the limit of $f(\beta)$ as $\beta \to \alpha$, we may assume that $\beta = [1, a_2, a_3, \ldots, a_n, a_{n+1}]$, where $a_{n+1} \to \infty$. In fact we then have $\beta \to \alpha_-$ or $\beta \to \alpha_+$ depending on whether we choose n to be odd or even in the representation of α. Note that P_n stays fixed with n, but $P_{n+1} \to \infty$ as $a_{n+1} \to \infty$. For odd n, we see from (5.3) that

$$f(\beta) = f(\alpha) + \frac{P_n + 1}{P_n P_{n+1}} \to f(\alpha) \quad \text{as} \quad \beta \to \alpha_-,$$

so that f is continuous from the left at each rational point α. On the other hand, when n is even, we find from (5.3) that

$$f(\beta) = f(\alpha) - \frac{P_{n+1} + 1}{P_n P_{n+1}} \to f(\alpha) - \frac{1}{P_n} \quad \text{as} \quad \beta \to \alpha_+.$$

Therefore, as $\beta \to \alpha = p/q$ from the right, there is a jump discontinuity of $1/P = 1/(2^p - 1)$ at $f(\alpha)$. Theorem 1 is proved.

7. Proof of Theorem 2. For $1 < p < q < 2p$, $(p, q) = 1$, let $J(p, q)$ denote the interval $(Q-1)/P \leq \theta < Q/P$, where Q and P are defined by (3.2) and (3.3). By Theorem 1, the range R of f does not contain any point in $J(p, q)$, and in fact $R = (0, 1) \setminus J$, where J is given by the following union:

$$J = \bigcup_{N \geq 2} J_N, \qquad J_N = \bigcup_{1 < p \leq N} J(p), \qquad J(p) = \bigcup_{\substack{p<q<2p \\ (q,p)=1}} J(p, q).$$

The montonicity of f implies that the intervals $J(p, q)$ are disjoint, so that the Lebesgue measure of J is given by

$$\mu(J) = \sum_{p=2}^{\infty} \frac{1}{P} \sum_{\substack{p<q<2p \\ (q,p)=1}} 1 = \sum_{p=2}^{\infty} \frac{\phi(p)}{2^p - 1}.$$

Thus

$$1 + \mu(J) = \sum_{p=1}^{\infty} \phi(p) \sum_{k=1}^{\infty} \frac{1}{2^{kp}} = \sum_{m=1}^{\infty} \frac{1}{2^m} \sum_{d|m} \phi(d) = \sum_{m=1}^{\infty} \frac{m}{2^m} = 2,$$

so that $\mu(J) = 1$ and hence $\mu(R) = 0$.

Essentially the same argument shows that R has Hausdorff dimension 0. Let

$$\Phi_N = \sum_{1 \leq p \leq N} \phi(p),$$

and we remark that $\Phi_N \sim 6N^2/\pi^2$ as $N \to \infty$, although the trivial bound $\Phi_N \leq N^2$ suffices for our purpose. We consider $R = \cap R_N$, where R_N is the complement of J_N. Thus $R_N \cup \{0\}$ is made up of Φ_N intervals of the form $c/U_m < \theta \leq d/U_n$, where $U_m = 2^m - 1$, and $cU_n - dU_m = 1$, with $m, n \leq N$, $m + n > N$, $(m, n) = 1$. These intervals therefore have lengths $1/U_m U_n$ which are bounded by $1/2^{N-1}$, and there is an apparent relationship between them and the Farey sequence of order N, namely by having U_m as denominator instead of $m \le N$. We can obtain R_N from R_{N-1} by the removal of the $\phi(N)$ intervals that make up $J(N)$. For example, R_5 is the union of the 10 intervals $(0, \frac{1}{31})$, $[\frac{2}{31}, \frac{1}{15})$, $[\frac{2}{15}, \frac{1}{7})$, $[\frac{2}{7}, \frac{9}{31})$, $[\frac{10}{31}, \frac{1}{3})$, $[\frac{2}{3}, \frac{21}{31})$, $[\frac{22}{31}, \frac{5}{7})$, $[\frac{6}{7}, \frac{13}{15})$, $[\frac{14}{15}, \frac{29}{31})$, $[\frac{30}{31}, 1)$, and R_6 is obtained from R_5 by replacing the first and last intervals by the four intervals $(0, \frac{1}{63})$, $[\frac{2}{63}, \frac{1}{31})$, $[\frac{30}{31}, \frac{61}{63})$, $[\frac{62}{63}, 1)$. For the Hausdorff dimension of R, we consider the τ-cover of R_N (see, for example, [**3**, Chapter 13]) for some parameter $\tau > 0$, which is given by the sum

$$\sum_{\substack{m,n \le N \\ m+n > N \\ (m,n)=1}} \left(\frac{1}{U_m U_n}\right)^{\tau} \le \frac{\Phi_N}{2^{\tau(N-1)}} \le \frac{N^2}{2^{\tau(N-1)}} \to 0 \quad \text{as} \quad N \to \infty.$$

Since this holds for every $\tau > 0$, the required Hausdorff dimension has the value 0.

Now let $\alpha = [1, a_2, a_3, \ldots]$ be irrational, and write

$$\beta = f(\alpha) = [1, b_2, b_3, \ldots], \qquad \beta_n = [1, b_2, b_3, \ldots, b_n] = \frac{U_n}{V_n}, \tag{7.1}$$

so that β_n is the n-th convergent for β. We proceed to prove that β is transcendental by showing that, for infinitely many n, the convergents β_n are unusually good rational approximations to β.

There is a close relationship between β_n and $f(\alpha_n)$, and in fact $\beta_{n-1} = f(\alpha_n) - c_n/P_n$, where $c_n = 0$ or 1 corresponding to $n \equiv 1$ or 0 (mod 2). In other words,

$$V_{n-1} = P_n, \qquad U_{n-1} = Q_n - c_n. \tag{7.2}$$

This can be proved as follows. By Lemma 2, if n is odd then at least the first $M = 2p_n + 1$ digits in the string D_{n+2} are the same as those in D_n^3. Consequently, we have

$$0 < f(\alpha) - f(\alpha_n) < \frac{1}{2^M} < \frac{1}{2P_n^2},$$

and since P_n is the denominator of $f(\alpha_n)$ it follows from Legendre's theorem (see [**2**, Theorem 184]) that $f(\alpha_n)$ is a convergent from below to $\beta = f(\alpha)$. Thus $\beta_{n-1} = f(\alpha_n)$ and (7.2) holds when n is odd. The case when n is even can be deduced from (5.3) together with $U_n V_{n+1} - U_{n+1} V_n = (-1)^{n+1}$. It also follows from (7.1) and (5.3) that

$$b_n = \frac{P_{n+1} - P_{n-1}}{P_n} = \frac{2^{p_{n-1}}\left((P_n + 1)^{a_n} - 1\right)}{P_n}. \tag{7.3}$$

Thus $b_n = 2^{p_{n-1}}$ if $a_n = 1$, and $b_n > P_n$ if $a_n > 1$. Suppose now that there are infinitely many n such that

$$a_{n+2} \ge 2 \tag{7.4}$$

in $\alpha = [1, a_2, a_3, \ldots]$. Then $V_{n+1} = P_{n+2} = 2^{p_{n+2}} - 1 \geq 2^{2p_{n+1}} - 1 \geq (2^{p_{n+1}} - 1)^2 = P_{n+1}^2 = V_n^2$ and hence (see [**2**, Theorem 164])

$$\left|\beta - \frac{U_n}{V_n}\right| < \frac{1}{V_n V_{n+1}} \leq \frac{1}{V_n^3}. \tag{7.5}$$

On the other hand, if (7.4) fails to hold for infinitely many n, then $a_n = 1$ for all large n, so that $p_{n+1} = p_n + p_{n-1}$, and hence $p_n \sim \lambda p_{n+1}$ as $n \to \infty$, where

$$\lambda = \lim_{n\to\infty} \frac{p_n}{p_{n+1}} = \frac{\sqrt{5}-1}{2} = [1, 1, 1, \ldots]. \tag{7.6}$$

Now fix any τ in $1 < \tau < 1/\lambda$, so that $p_{n+2} > \tau p_{n+1}$, and hence $V_{n+1} > 2^{\tau p_{n+1}} - 1 > (2^{p_{n+1}} - 1)^\tau = V_n^\tau$. In place of (7.5) we now have, for all large n,

$$\left|\beta - \frac{U_n}{V_n}\right| < \frac{1}{V_n V_{n+1}} < \frac{1}{V_n^{1+\tau}}. \tag{7.7}$$

Thus, either (7.5) holds for infinitely many n, or else (7.7) holds for all large n. Since $1 + \tau > 2$, the transcendence of β follows from the Thue-Siegel-Roth theorem, and Theorem 2 is proved.

We conclude with the following remarks. If $\alpha = \lambda$, the *golden ratio* given in (7.6), then $(p_n) = (1, 1, 2, 3, 5, 8, 13, \ldots)$ is the Fibonacci sequence, and $f(\lambda) = [1, 2, 2, 4, 8, 32, 256, \ldots]$ is transcendental. Here the partial quotients are given by $b_n = 2^{p_{n-1}}$ for $n > 1$ according to (7.3), so that $\log\log b_n > n/10$ for all large n. Moreover, for any $\alpha \in I$, the sequence of partial quotients for the image $f(\alpha)$ must increase at least at fast as this particular sequence (b_n) corresponding to $\alpha = \lambda$. Indeed, it is clear from the derivation of (7.5) from (7.4) that the measure of transcendence for $f(\alpha)$ can be estimated or computed from the upper limit of the sequence of partial quotients (a_n) for α.

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