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ON SEMIGROUPS DEFINED BY THE IDENTITY $xxy = y$

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Abstract. The groupoid identity $x(xy) = y$ appears in definitions of several classes of groupoids, such as Steiner loops (which are closely related to Steiner triple systems) [**9, 10**], orthogonality in quasigroups [**4**] and others [**2, 12**]. We have considered in [**8**] several varieties of groupoids that include this identity among their defining identities, and here we consider the variety $\mathcal{V}$ of semigroups defined by the same identity. The main results are: the decomposition of a $\mathcal{V}$ semigroup as a direct product of a Boolean group and a left unit semigroup; decomposition of the variety $\mathcal{V}$ as a direct product of the variety of Boolean groups and the variety of left unit semigroups; constructions of free objects in $\mathcal{V}$ and the solution of the word problem in $\mathcal{V}$.

1. The structure of $\mathcal{V}$ semigroups

The variety of semigroups with $xxy = y$ will be denoted by $\mathcal{V}$. We will pay attention to some structural properties first.

Proposition 1. *Each $\mathcal{V}$ semigroup is right quasigroup (i.e., it is left cancellative and right solvable).*

Proof. We have $xy = xz \ \Rightarrow \ xxy = xxz \ \Rightarrow \ y = z$ and $z = xy$ is the solution of the equation $xz = y$. □

Proposition 2. *The identity $xyz = yxz$ holds in each $\mathcal{V}$ semigroup.*

Proof. We first prove the special case $xyx = yxx$. We have, $xyx = yxyxxyx = yxyyx = yxx$. Then $xyz = xyxxz = yxxxz = yxz$. □

Corollary 1. *In any $\mathcal{V}$ semigroup the following hold:*

(i)	$xyxz = yz$;	(iii)	$x^2 = y^2 \Longleftrightarrow xy = yx$;
(ii)	$(xy)^2 = y^2$;	(iv)	$x = vy \vee ux = vy \ \Rightarrow \ x^2 = y^2$.

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COROLLARY 2. *For any $\mathcal{V}$ semigroup $(S, \cdot)$ and any $H \subseteq S$, the subsemigroup of S generated by H is $\{h_1 h_2 \ldots h_s h \mid h_i, h \in H,\ i \neq j \ \Rightarrow\ h_i \neq h_j, s \geq 0\}$ where also $h_{\sigma(1)} \ldots h_{\sigma(s)} h = h_1 \ldots h_s h$ for each $h_i \in H$, $s > 0$ and any permutation σ on $\{1, \ldots, s\}$.*

By [**3**, Theorem 1.27] and Proposition 1, we have that every $\mathcal{V}$ semigroup is isomorphic to a direct product of a group and a left unit semigroup (i.e., a right zero semigroup, and from what follows it will be clear why we prefer the term 'left unit' instead of 'right zero'). Though it is known result, we clarify it in this particular case, as follows.

Let $(S, \cdot)$ be a $\mathcal{V}$ semigroup. The set of left units in S and the set of idempotents in S coincide, and x^2 is a left unit (i.e., an idempotent) in S for each $x \in S$. Denote by E the set $E = \{x^2 \mid x \in S\}$. We have that E is nonempty and it is a left unit subsemigroup of S.

PROPOSITION 3. *E contains all left units of S and $ES = S = SE$.*

PROOF. $S = \{x \cdot x^2 \mid x \in S\} \subseteq SE \subseteq S$, hence $ES = S = SE$. □

Let $S/E = \{xE \mid x \in S\}$ and define an operation in S/E by $xE \cdot yE = xyE$. The operation is well defined.

PROPOSITION 4. *S/E is a Boolean group.*

PROOF. Note that $E \in S/E$ since $E = x^2E$, for $x \in S$. We have $E \cdot xE = xE = xE \cdot E$ and $(xE)^2 = xE \cdot xE = x^2E = E$. □

(We note that what we call here Boolean group is also known as "elementary 2-abelian group".)

PROPOSITION 5. *Let S be a $\mathcal{V}$ semigroup generated by a set X. Then*
(1) $E = \{x^2 \mid x \in X\}$,
(2) $S/E = \langle \{xE \mid x \in X\} \rangle$,
(3) $xE = yE \ \Rightarrow\ xy \in E$,
(4) $xE = yE \wedge x^2 = y^2 \ \Rightarrow\ x = y$.

PROOF. (1) For $x \in S$ either $x \in X$ or $x = ay$, $y \in X$, implying $x^2 = y^2$.

(2) It is obvious since $xyE = xEyE$.

(3) Note that $xE = yE$ implies $(\forall z \in X)(\exists t \in X)\ xz^2 = yt^2$ which by Corollary 1(ii) means that $(\forall z \in X)\ xz^2 = yz^2$ and, equivalently, $(\forall z \in X)\ z = xyz$. Hence xy is a left unit i.e., $xy \in E$.

(4) Let $xE = yE$ and $x^2 = y^2$. By (3), $xy = u^2 \in E$ so $x^2 = y^2 = u^2$ and hence $xy = x^2$ i.e., $x = y$. □

If G is a Boolean group and E is a left unit semigroup, then their direct product $G \times E$ is a $\mathcal{V}$ semigroup. The opposite statement is also true.

THEOREM 1. *A semigroup is a $\mathcal{V}$ semigroup iff it is isomorphic to a direct product of a Boolean group and a left unit semigroup.*

PROOF. Let S be a $\mathcal{V}$ semigroup, and define a mapping $\varphi : S \to S/E \times E$ by $\varphi(x) = (xE, x^2)$, where S/E and E are defined as before. Then by Proposition 5(4) we have $\varphi(x) = \varphi(y) \Rightarrow (xE, x^2) = (yE, y^2) \Rightarrow xE = yE \wedge x^2 = y^2 \Rightarrow x = y$, i.e., φ is injective, and it is a bijection too since for given $(xE, y^2) \in S/E \times E$ we have $(xE, y^2) = (yxyE, (yxy)^2) = \varphi(yxy)$. Finally, $\varphi(xy) = (xyE, (xy)^2) = (xEyE, x^2y^2) = \varphi(x)\varphi(y)$, i.e., φ is an isomorphism as well. □

We will consider the Boolean groups for a moment. They are commutative and if X is a base (i.e., a minimal generating set) of a Boolean group G then $G \cong (\mathcal{B}_{\text{fin}}(X), +)$, where $\mathcal{B}_{\text{fin}}(X)$ is the set of all finite subsets of X and $+$ denotes the set theoretical operation of symmetric difference. Namely, given $g \in G$, there exist uniquely determined $x_i \in X$ such that $g = x_1x_2 \dots x_n$ (up to commutativity), and then $g \mapsto \{x_1, x_2, \dots, x_n\}$ is the required isomorphism. As a consequence we have that $|G| = 2^{|X|}$ when X is finite and $|G| = |X|$ when X is infinite, and if Y is another base of G then $|X| = |Y|$. Thus, up to isomorphism, there is a unique Boolean group with given base. Each Boolean group is also isomorphic to the abelian group $(V, +)$ of a vector space over $Z/2Z$ which in n-dimensional case is isomorphic to $(Z/2Z)^n$.

Also, let us state a few facts about left unit semigroups: they are defined by the identity $xy = y$, and there is a unique left unit semigroup on each set.

Now, by Theorem 1, we have the following property.

PROPOSITION 6. *If S is a finite $\mathcal{V}$ semigroup then $|S| = 2^r k$ for some integers $r \geq 0$, $k > 0$.*

It is clear that for any integers $r \geq 0$ and $k > 0$ there is a $\mathcal{V}$ semigroup with cardinality $2^r k$, and S is unique (up to isomorphism) with the property $|S/E| = 2^r$. It follows that the pair (r, k) uniquely determines the semigroup S with $|S/E| = 2^r$, $|E| = k$. Thus we have:

PROPOSITION 7. *The number of all nonisomorphic $\mathcal{V}$ semigroups with m elements is equal to $s + 1$, where $m = 2^s q$, q is odd.*

Let S be a finite $\mathcal{V}$ semigroup with cardinality $2^r k$ generated by n generators, i.e., $S = \langle X \rangle$, $|X| = n$, such that $|S/E| = 2^r$, $|E| = k$. We investigate a bit the connections among n, k and r. Note that $r \leq n$, $k \leq n$. Namely, by Proposition 5 (1), $|E| \leq |X|$ i.e., $k \leq n$ and, by Proposition 5 (2), $|S/E| \leq 2^{|\{xE | x \in X\}|} \leq 2^{|X|}$ i.e., $r \leq n$. Let α denote the equivalence on X defined by $x\alpha y \Leftrightarrow x^2 = y^2$. Let $t = max(|x/\alpha|, x \in X)$. Then $r \geq t - 1$ and k is equal to the number of equivalence classes of α. Indeed, by Proposition 5(4) we have $y \in x/\alpha$, $x \neq y \Rightarrow xE \neq yE$. So, $r = |\{xE \mid x \in X\}| \geq t$ and $r \geq t - 1$ in case when $E \in \{xE \mid x \in X\}$ i.e., $x^2 = x$ for some $x \in X$. Also note that for at most one $y \in x/\alpha$ it is true that $y^2 = y$.

Theorem 1 gives rise to the question of the structure of the variety $\mathcal{V}$. We will show that $\mathcal{V} = \mathcal{BG} \otimes \mathcal{LE}$, where $\mathcal{BG}$ denotes the variety of Boolean groups, and $\mathcal{LE}$ denotes the variety of left unit semigroups. For that aim we will use the results on unindexed products of algebras and varieties stated in [**13**]. Namely, given two algebras $\mathbf{A}$ and $\mathbf{B}$ of arbitrary types an unindexed product $\mathbf{C} = \mathbf{A} \otimes \mathbf{B}$ is an algebra with universe $C = A \times B$ that has a basic n-ary operation t^C for each pair (t_1^A, t_2^B) of n-ary term operations t_1^A in $\mathbf{A}$ and t_2^B in $\mathbf{B}$, defined by $t^C((x_1', x_1''), \dots, (x_n', x_n'')) = (t_1^A(x_1', \dots, x_n'), (t_2^B(x_1'', \dots, x_n''))$. These are all basic operations of $\mathbf{C}$. If $\mathcal{V}_1$ and $\mathcal{V}_2$ are varieties, then the variety generated by all unindexed products $\mathbf{A}_1 \otimes \mathbf{A}_2$, $\mathbf{A}_i \in \mathcal{V}_i$, is called product of $\mathcal{V}_1$ and $\mathcal{V}_2$, and is denoted by $\mathcal{V}_1 \otimes \mathcal{V}_2$.

For unindexed products we have the following properties [**13**]:

$\mathbf{C} \leq \mathbf{A} \otimes \mathbf{B}$ iff there exist $\mathbf{A}_1 \leq \mathbf{A}$, $\mathbf{B}_1 \leq \mathbf{B}$ such that $\mathbf{C} = \mathbf{A}_1 \otimes \mathbf{B}_1$.
$\Theta(\mathbf{A} \otimes \mathbf{B}) = \Theta(\mathbf{A}) \otimes \Theta(\mathbf{B})$, where $\Theta(\mathbf{A})$ denotes the congruence lattice of the algebra $\mathbf{A}$.
There exists an epimorphism $\phi : \mathbf{A} \otimes \mathbf{B} \longrightarrow \mathbf{C}$ iff there exist epimorphisms $\phi_1 : \mathbf{A} \longrightarrow \mathbf{C}_1$ and $\phi_2 : \mathbf{B} \longrightarrow \mathbf{C}_2$ and $\mathbf{C} \cong \mathbf{C}_1 \otimes \mathbf{C}_2$.
An algebra belongs to $\mathcal{V}_1 \otimes \mathcal{V}_2$ if and only if it is isomorphic to an unindexed product of an algebra in $\mathcal{V}_1$ and an algebra in $\mathcal{V}_2$.

LEMMA 1. *Let $\mathbf{G} \in \mathcal{BG}$ and $\mathbf{E} \in \mathcal{LE}$. Then $\mathbf{G} \times \mathbf{E}$ is the same algebra as $\mathbf{G} \otimes \mathbf{E}$.*

PROOF. Denote by $*$, $+$, $\cdot$ the binary operations of $\mathbf{G} \times \mathbf{E}$, $\mathbf{G}$, $\mathbf{E}$ respectively. Let $t = (t_1, t_2)$ be an n-ary operation in $\mathbf{G} \otimes \mathbf{E}$, where t_1, t_2 are n-ary term operations in $\mathbf{G}$, $\mathbf{E}$ respectively. We have $t_1(x_1, \dots, x_n) = x_{i_1} + \dots + x_{i_k}$ where $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ and $t_2(x_1, \dots, x_n) = x_j$ for some $j \in \{1, \dots, n\}$. Then, $t((x_1', x_1''), \dots, (x_n', x_n'')) = (t_1(x_1', \dots, x_n'), t_2(x_1'', \dots, x_n'')) = (x_{i_1}' + \dots + x_{i_k}', x_j'') = (x_{i_1}', x_{i_1}'') * \dots * (x_{i_k}', x_{i_k}'') * (x_j', x_j'') * (x_j', x_j'')$, i.e., t is derived operation from $*$. $\square$

From the previous lemma, Theorem 1 and the results mentioned above we obtain the following theorem.

THEOREM 2. *$\mathcal{V} = \mathcal{BG} \otimes \mathcal{LE}$ and if $\mathbf{G} \in \mathcal{BG}$, $\mathbf{E} \in \mathcal{LE}$ then:*

(i) $\mathbf{S} \leq \mathbf{G} \times \mathbf{E}$ *iff there exist* $\mathbf{G}_1 \leq \mathbf{G}$, $\mathbf{E}_1 \leq \mathbf{E}$ *such that* $\mathbf{S} = \mathbf{G}_1 \times \mathbf{E}_1$.
(ii) $\Theta(\mathbf{G} \times \mathbf{E}) = \Theta(\mathbf{G}) \times \Theta(\mathbf{E})$.
(iii) *There exists an epimorphism* $\phi : \mathbf{G} \times \mathbf{E} \longrightarrow \mathbf{S}$ *iff there exist epimorphisms* $\phi_1 : \mathbf{G} \longrightarrow \mathbf{S}_1$ *and* $\phi_2 : \mathbf{E} \longrightarrow \mathbf{S}_2$ *and* $\mathbf{S} \cong \mathbf{S}_1 \times \mathbf{S}_2$.

Knowing that the congruence lattice of a group is modular, and the lattice of equivalences on a set A is modular iff $|A| \leq 3$, we get the following result.

COROLLARY 3. *The congruence lattice of a $\mathcal{V}$ semigroup $\mathbf{S}$ is modular iff $|E| \leq 3$, where E is the left unit subsemigroup of $\mathbf{S}$.*

REMARK 1. The variety $\mathcal{V}$ is also interesting since it is defined by a single identity that holds in both of the varieties $\mathcal{BG}$, $\mathcal{LE}$. One could obtain the fact $\mathcal{V} = \mathcal{BG} \otimes \mathcal{LE}$ from another (syntactical) point of view. Namely, the varieties $\mathcal{BG}$, $\mathcal{LE}$ are independent, in the sense of [**5**], i.e., there exists a polynomial $p(x, y)$ of two variables in the signature $\{\cdot\}$ such that $p(x, y) = x$ in each Boolean group and $p(x, y) = y$ in each left unit semigroup (e.g. $p(x, y) = xy^2$). The variety $\mathcal{BG}$ has defining (semigroup) identities $xxy = y$, $xy = yx$ and the variety $\mathcal{LE}$ has a single defining identity $xy = y$. Hence, the procedure proposed in [**5**] can be used for finding the defining set of (semigroup) identities for $\mathcal{BG} \otimes \mathcal{LE}$, giving the following set of (semigroup) identities: $xxy = y$, $xyt^2 = yxt^2$, $t(xy)^2 = ty^2$, $xt^2(tx^2)^2 = x$,

$xt^2yt^2t^2 = xyt^2$, $t(tx^2ty^2)^2 = t(xy)^2$. All of them are of course consequences of $xxy = y$ (Proposition 2, Corollary 2). Hence, $\mathcal{BG} \otimes \mathcal{LE}$ is the variety of semigroups satisfying $xxy = y$. Some other similar examples of product varieties are considered in [**6**] and [**7**].

2. Free semigroups in $\mathcal{V}$

Here we focus on free objects in the variety of semigroups $\mathcal{V}$. As an introduction to what follows we give an example of a special class of $\mathcal{V}$ semigroups.

EXAMPLE 1. Let A be a set, and denote by $\mathcal{B}(A)$ its Boolean set, i.e., the set of all subsets of A. Define an operation $*$ on $\mathcal{B}(A) \times \mathcal{B}(A)$ by $(R, S) * (T, U) = (R + S + T, U)$. Then $(\mathcal{B}(A) \times \mathcal{B}(A), *)$ is a $\mathcal{V}$ semigroup which is commutative if and only if $A = \emptyset$. For the semigroup $(\mathcal{B}(A) \times \mathcal{B}(A), *)$ we have that (S, S) is left unit where $S \subseteq A$ and $\{(R, S) \mid R, S \in \mathcal{B}_{\text{fin}}(A)\}$, $\{(R, \{a\}) \mid R \subseteq A,\ a \in A\}$, $\{(R, \{a\}) \mid R \in \mathcal{B}_{\text{fin}}(A),\ a \in A\}$ are its subsemigroups.

We will present three rather simple constructions of free objects and some properties of the free objects will be given too.

The first, direct construction is related to Example 1. Let $B \neq \emptyset$ be a given set and let $F_B = \{(C, b) \mid C \in \mathcal{B}_{\text{fin}}(B),\ b \in B\}$. The set F_B will serve as a universe of a free $\mathcal{V}$ semigroup with free base B. C can be considered as "the content set of an element" and b as "the tail". Further on the singleton set $\{t\}$ will be written simply as t (depending of the context), the set theoretical operation difference of two sets X and Y will be denoted by $X - Y$ and we will identify the element $b \in B$ with the element $(\emptyset, b) \in F_B$, in such a way obtaining $B \subseteq F_B$. Define an operation $*$ on F_B by

$$(C_1, b_1) * (C_2, b_2) = (C_1 + b_1 + C_2, b_2)$$

for all $(C_1, b_1), (C_2, b_2) \in F_B$.

THEOREM 3. *$(F_B, *)$ is a free semigroup in $\mathcal{V}$ with free base B.*

PROOF. Let $u = (C_1, b_1)$, $v = (C_2, b_2)$, $w = (C_3, b_3) \in F_B$. Then we have

$$\begin{aligned}(u * v) * w &= (C_1 + b_1 + C_2, b_2) * (C_3, b_3) = (C_1 + b_1 + C_2 + b_2 + C_3, b_3) = \\ &= (C_1, b_1) * (C_2 + b_2 + C_3, b_3) = u * (v * w)\end{aligned}$$

as well as

$$u * u * v = u * (C_1 + b_1 + C_2, b_2) = (C_1 + b_1 + C_1 + b_1 + C_2, b_2) = (C_2, b_2) = v.$$

Hence, $F_B \in \mathcal{V}$.

The rest of the proof is derived by induction on the cardinality of the content set.

The set B generates F_B since $(C, b) = (\emptyset, c) * (C - c, b) = c * (C - c, b)$ if $c \in C$.

Let $(G, \circ)$ be any $\mathcal{V}$ semigroup and $f : B \to G$ a mapping. We extend f to a mapping $\hat{f} : F_B \to G$ inductively in the following way.

$$\hat{f}((C, b)) = \begin{cases} f(b), & C = \emptyset \\ f(c) \circ \hat{f}((C - c, b)), & c \in C \end{cases}$$

The fact that $\hat{f}$ is well defined follows by induction and Proposition 2. Clearly, it is well defined for the base elements, and for elements with $|C| = 1$. If $c_1 \neq c_2 \in C$ and if $\hat{f}$ is well defined for elements with smaller content set and $g = \hat{f}((C - \{c_1, c_2\}, b))$, then

$$f(c_1) \circ \hat{f}((C - c_1, b)) = f(c_1) \circ f(c_2) \circ g = f(c_2) \circ f(c_1) \circ g = f(c_2) \circ \hat{f}((C - c_2, b)).$$

To complete the proof we check that $\hat{f}$ is a homomorphism. Let $u = (C_1, b_1)$, $v = (C_2, b_2)$. At first we consider several base cases:

(1) $C_1 = \emptyset$, $C_2 = \emptyset$: Then $u * v = (b_1, b_2)$ and $\hat{f}(u) \circ \hat{f}(v) = f(b_1) \circ f(b_2) = \hat{f}((b_1, b_2)) = \hat{f}(u * v)$.

(2) $C_1 = \emptyset$, $C_2 \neq \emptyset$, $b_1 \in C_2$: Then $u * v = (b_1 + C_2, b_2) = (C_2 - b_1, b_2)$ and $\hat{f}(u) \circ \hat{f}(v) = f(b_1) \circ \hat{f}((C_2, b_2)) = f(b_1) \circ f(b_1) \circ \hat{f}((C_2 - b_1, b_2)) = \hat{f}((C_2 - b_1, b_2)) = \hat{f}(u * v)$.

(3) $C_1 = \emptyset$, $C_2 \neq \emptyset$, $b_1 \notin C_2$: Then $u * v = (b_1 + C_2, b_2)$ and $\hat{f}(u) \circ \hat{f}(v) = f(b_1) \circ \hat{f}((C_2, b_2)) = \hat{f}((C_2 + b_1, b_2)) = \hat{f}(u * v)$.

Finally we consider the inductive case:

(4) $C_1 \neq \emptyset$, $c \in C_1$: $\hat{f}(u) \circ \hat{f}(v) = f(c) \circ \hat{f}((C_1 - c, b_1)) \circ \hat{f}((C_2, b_2)) =$
$f(c) \circ \hat{f}((C_1 - c, b_1) * (C_2, b_2)) = f(c) \circ \hat{f}((((C_1 - c) + b_1 + C_2, b_2)) = R$

There are two subcases here.

(4.1) $c \in b_1 + C_2$: $u * v = (C_1 + b_1 + C_2, b_2) = ((C_1 - c) + ((b_1 + C_2) - c), b_2)$ and $R = f(c) \circ f(c) \circ \hat{f}(((C_1 - c) + ((b_1 + C_2) - c), b_2)) = \hat{f}(u * v)$,

(4.2) $c \notin b_1 + C_2$: $R = \hat{f}(((C_1 - c) + b_1 + C_2 + c), b_2)) =$
$= \hat{f}((C_1 + b_1 + C_2, b_2)) = \hat{f}(u * v)$. □

A free $\mathcal{V}$ semigroup with free base B can also be constructed using the free semigroup B^+ consisting of all nonempty words over the alphabet B with concatenation of words as operation. For that purpose we will use the construction of $(F_B, *)$, although it can be done independently. Let $<$ be a well ordering of the set B and let F'_B consists of all words $b_1 b_2 \dots b_n b$ ($b, b_i \in B$, $n \geq 0$) in B^+ such that $i < j \Rightarrow b_i < b_j$. If we define an operation $*'$ by $b_1 b_2 \dots b_n b *' c_1 c_2 \dots c_k c = d_1 d_2 \dots d_m c$ iff $b_1 * b_2 * \cdots * b_n * b * c_1 * c_2 * \cdots * c_k * c = d_1 * d_2 * \cdots * d_m * c$, then $(F'_B, *')$ is a free $\mathcal{V}$ semigroup with free base B.

Theorem 2 enables us to give one more construction of the free $\mathcal{V}$ semigroup. If B is a nonempty set then $(\mathcal{B}_{\text{fin}}(B), +)$ is a free Boolean group with free base B and the left unit semigroup $(B, \cdot)$ on B is also free. Now, consider the direct product $(F''_B, *'') = (\mathcal{B}_{\text{fin}}(B), +) \times (B, \cdot)$ where $(C_1, b_1) *'' (C_2, b_2) = (C_1 + C_2, b_2)$. $(F''_B, *'')$ is certainly free in $\mathcal{V}$ because of the properties of product varieties, with free base $\{(b, b) \mid b \in B\}$. Namely, given $(C, b) \in F''_B$, where $C = \{b_1, \dots, b_n\} \subseteq B$, $b \in B$, we have $(C, b) = (b_1, b_1) *'' \cdots (b_n, b_n) *'' (b, b) *'' (b, b)$. The mapping $\varphi : F_B \to F''_B$ defined by $\varphi((C, b)) = (C + b, b)$ is an isomorphism from $(F_B, *)$ onto $(F''_B, *'')$.

Knowing how the free $\mathcal{V}$ semigroups look like, it is clear that any free $\mathcal{V}$ semigroup with finite free base is also finite, in fact we have the following theorem.

THEOREM 4. *Let S be a finitely generated $\mathcal{V}$ semigroup with n element base. Then S is free if and only if $|S| = 2^n n$.*

PROOF. If S is free with n element free base then, according to previous constructions, $|S| = 2^n n$. On the other hand, if $|S| = 2^n n$ and S has a base B with n generators then, by Proposition 5(i),(ii), we have $|S/E| \leq 2^n$ and $|E| \leq n$ which means that $|S/E| = 2^n$, $|E| = n$, i.e., S is free by the third construction of free objects in $\mathcal{V}$. □

In most of the varieties we considered in [**8**] the class of free objects was inheritable. But this is not a case in the variety $\mathcal{V}$, since any nontrivial free $\mathcal{V}$ semigroup contains a left unit subsemigroup. Furthermore, having the 'right' number of elements in a subsemigroup of a free $\mathcal{V}$ semigroup is still not sufficient for freeness, as shown by the next example.

EXAMPLE 2. Let $(F'_B, *')$ be the free $\mathcal{V}$ semigroup with a four element base $B = \{a, b, c, d\}$, $a < b < c < d$. Then $G = \{aa,\ bb,\ cc,\ dd,\ bcda,\ acdb,\ abdc,\ abcd\}$ is its subsemigroup which is not free, even though it has $2^2 \cdot 2$ elements.

3. The word problem for $\mathcal{V}$

Let S be a $\mathcal{V}$ semigroup with a finite base B. Let S be a factor semigroup of F''_B by some congruence $\sim$ generated by a finite set of defining pairs $t_i \sim s_i$, where $t_i, s_i \in F''_B$ for $i = 1, \ldots, n$. The word problem is solvable for S if there is an algorithm such that for any $u, v \in F''_B$ it is possible to determine whether $u \sim v$. The word problem is globally solvable for $\mathcal{V}$ if there exists an algorithm that solves the word problem for any such $\mathcal{V}$ semigroup.

Since F''_B is finite when B is finite, the word problem for the variety $\mathcal{V}$ is trivially solvable. Here we show that solving the word problem for $\mathcal{V}$ semigroups reduces to solving the word problem for Boolean groups and left unit semigroups.

Let $S = F''_B/\sim$ where B is finite set and the congruence $\sim$ is generated by a finite set of defining pairs $t_i \sim s_i$, where $t_i, s_i \in F''_B$ for $i = 1, \ldots, n$, and let $u, v \in F''_B$. Then $t_i = (t'_i, t''_i)$, $s_i = (s'_i, s''_i)$, $u = (u', u'')$, $v = (v', v'')$. Denote by $\sim_1$ the congruence on $(\mathcal{B}_{\text{fin}}(B), +)$ generated by the pairs $t'_i \sim_1 s'_i$, and by $\sim_2$ the congruence on the left unit semigroup $(B, \cdot)$ generated by the pairs $t''_i \sim_2 s''_i$.

LEMMA 2. *$(u', u'') \sim (v', v'')$ if and only if $u' \sim_1 v'$ and $u'' \sim_2 v''$, for any $(u', u''), (v', v'') \in F''_B$.*

PROOF. One direction of this statement is easy i.e., $(u', u'') \sim (v', v'') \Rightarrow u' \sim_1 v' \wedge u'' \sim_2 v''$. For the other direction note the following: $(t'_i, t''_i) \sim (s'_i, s''_i)$ implies $(\emptyset, t''_i) \sim (\emptyset, s''_i)$ (since $(\emptyset, t''_i) = (t'_i, t''_i) *'' (t'_i, t''_i)$) and also implies $(t'_i, b) \sim (s'_i, b)$ for all $b \in B$ (since $(t'_i, b) = (t'_i, t''_i) *'' (\emptyset, b)$). Hence, $t'_i \sim_1 s'_i \Rightarrow (t'_i, b) \sim (s'_i, b)$ for all $b \in B$ and $t''_i \sim_2 s''_i \Rightarrow (\emptyset, t''_i) \sim (\emptyset, s''_i)$. Moreover, for any $x', y' \in \mathcal{B}_{\text{fin}}(B)$, $x' \sim_1 y' \Rightarrow (x', b) \sim (y', b)$ for all $b \in B$ and for any $x'', y'' \in B$, $x'' \sim_2 y'' \Rightarrow (\emptyset, x'') \sim (\emptyset, y'')$. Hence, if $u' \sim_1 v'$ and $u'' \sim_2 v''$ then $(u', u'') = (u', b) *'' (\emptyset, u'') \sim (v', b) *'' (\emptyset, v'') = (v', v'')$. □

Note that if $\sim_1$ is defined as previously, then $u' \sim_1 v'$ if and only if there exist $i_1, \dots, i_k \in \{1, 2, \dots, n\}$ such that $u' + v' = \sum_{j=1}^{k} t'_{i_j} + s'_{i_j}$. Also the word problem is trivially solvable for left unit semigroups, since any equivalence is a congruence in them.

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