PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 71(85) (2002), 55–62

REGULAR VARIATION AND THE FUNCTIONAL CENTRAL LIMIT THEOREM FOR HEAVY TAILED RANDOM VECTORS

Mark M. Meerschaert* and Steven J. Sepanski

ABSTRACT. Multivariable regular variation is used, along with the martingale central limit theorem, to give a very simple proof that the partial sum process for a sequence of independent, identically distributed random vectors converges to a Brownian motion whenever the summands belong to the generalized domain of attraction of a normal law. This includes the heavy tailed case, where the covariance matrix is undefined because some of the marginals have infinite variance.

1. Introduction

Regular variation is a powerful tool in probability theory, and in many other areas of mathematics. Bingham, Goldie, and Teugels [**2**] catalog numerous applications of the one variable theory. Feller [**4**] gives an elegant proof of the central limit theorem using regular variation, including the infinite variance case. In fact, the centered and normalized partial sums of independent and identically distributed random variables are asymptotically normal if and only if the truncated variance function is slowly varying, so that even the statement of the central limit theorem properly includes the idea of regular variation. Meerschaert and Scheffler [**16**, Theorem 8.1.7] provides an extension of this result to random vectors, using multivariable regular variation. Multivariable regular variation can be traced back to the works of Stam [**25**], Stadtmüller and Trautner [**24**], Jakimiv [**10**], de Haan et al. [**5, 6**], and Ostrogorski [**17, 18, 19**]. The theory was extended in Meerschaert [**13**] and Meerschaert and Scheffler [**15**] to the generality needed for operator-normed limit theorems in probability. The book [**16**] contains a detailed introduction to the general theory of multivariable regular variation, independent of probabilistic considerations.

In this paper, multivariable regular variation is used to prove the functional central limit theorem for independent and identically distributed random vectors

2000 *Mathematics Subject Classification.* Primary 60F17; Secondary 62E20.
Key words and phrases. martingale; invariance principle; Donsker's Theorem; partial sum process; generalized domain of attraction; operator normalization.
* Partially supported by NSF grants DES-9980484 and DMS-0139927

on a finite dimensional real vector space. It is assumed that the sequence of partial sums can be centered (by subtracting the mean) and normalized (by applying an appropriate linear operator) so that the resulting sequence converges in distribution to a Gaussian limit. In this case, the distribution of these random vectors is said to belong to the generalized domain of attraction of a multivariate normal law. This includes the classical situation, in which the covariance matrix of the summands exists, but also allows that the covariance matrix is undefined. This can occur when some of the marginals have an infinite variance, as long as the multivariable regular variation condition specified in [**16**] holds. The main idea of the proof is to verify the conditions for the martingale central limit theorem, using multivariable regular variation.

2. Result

Let $\xi, \xi_1, \xi_2, \ldots$ be independent and identically distributed random vectors in $\mathbb{R}^d$. We say that the probability distribution $\mathcal{L}(\xi)$ belongs to the generalized domain of attraction (GDOA) of a d-variate normal law if there exist a sequence of invertible linear operators B_n on $\mathbb{R}^d$ and a sequence of nonrandom vectors b_n such that

$$\mathcal{L}\left(B_n \sum_{i=1}^{n} \xi_i - b_n\right) \Rightarrow N(0, I). \tag{2.1}$$

Here $\Rightarrow$ denotes weak convergence, and I is the identity matrix. Then $\mathcal{L}(\xi)$ is full (e.g., see Lemma 3.3.3 in [**16**]). That is, $\mathcal{L}(\xi)$ is not supported on any $d-1$ dimensional hyperplane. Equivalently, $\langle \xi, \theta \rangle$ is nondegenerate $\forall\theta$. Here $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $\mathbb{R}^d$. Throughout this article all distributions will be assumed to be full. Moreover, if $\mathcal{L}(\xi)$ is in the GDOA of the multivariate normal law then Theorem 3 in Hudson, Veeh, and Weiner [**9**] shows that $E\|\xi\| < \infty$. Therefore, without loss of generality, we will assume that $E\xi = 0$, and in this case one may take $b_n = 0$, see Hahn and Klass [**8**]. According to Meerschaert [**14**], we may assume that $B_n = n^{-1/2} L_n$ where L_n is slowly varying, which means that $L_{[\lambda n]} L_n^{-1} \to I$ for all $\lambda > 0$. Consequently, $\|B_n\| \to 0$. Define the partial sum process $M_n(t) = 0$ for $t \in [0, 1/n)$, and for $t \geqslant 1/n$, let

$$M_n = B_n \sum_{i=1}^{[nt]} \xi_i, \tag{2.2}$$

where $[\,\cdot\,]$ is the greatest integer function. Then M_n has sample paths in the space $D([0,\infty), \mathbb{R}^d) = \{f : [0,\infty) \to \mathbb{R}^d : \forall t\ f(t+) = f(t) \text{ and } f(t-) \text{ exists}\}$. We endow this space with the usual topology, and we let W be a realization of standard Brownian motion on this space.

THEOREM 2.1. *Suppose that $\xi_1, \xi_2, \ldots$ are independent and identically distributed mean zero random vectors in $\mathbb{R}^d$ whose distribution belongs to the generalized domain of attraction of a d-variate normal law, so that* (2.1) *holds. Then $M_n \Rightarrow W$ in $D([0,\infty), \mathbb{R}^d)$ where M_n is the partial sum process defined in* (2.2) *above.*

PROOF. The proof of Theorem 2.1 is an application of the martingale central limit theorem that appears in Ethier and Kurtz [**3**, Theorem 7.1.4a, p. 339]. In the statement below, which is included for convenience, M_n^i is the i^{th} coordinate of M_n, and $[M_n^i, M_n^j]$ is the cross variation process. (See [**3**, p. 79].) □

THEOREM 2.2. (Martingale Central Limit Theorem) *For* $n = 1, 2 \ldots$, *let* $\{\mathcal{F}_t^n\}$ *be a filtration and let* M_n *be an* $\{\mathcal{F}_t^n\}$ *local martingale with* $M_n(0) = 0$ *and sample paths in* $D([0,\infty), \mathbb{R}^d)$. *Let* $A_n = (A_n^{ij})$ *be symmetric* $d \times d$ *matrix valued processes such that* $A_n^{ij} = [M_n^i, M_n^j]$ *has sample paths in* $D([0,\infty), \mathbb{R})$ *and* $A_n(t) - A_n(s)$ *is nonnegative definite for* $t > s \geqslant 0$. *If:*

(i) *for each* $T > 0$, $\lim_{n\to\infty} E \sup_{t\leqslant T} \|M_n(t) - M_n(t-)\| = 0$; *and*
(ii) $A_n(t) \to tI$ *in probability;*

then $M_n \Rightarrow W$, *a standard Brownian motion on* $D([0,\infty), \mathbb{R}^d)$.

Define the empirical covariance operator

$$C_n = \sum_{i=1}^{n} \xi_i \xi_i^T \tag{2.3}$$

where ^{T} denotes the transpose, and let $A_n(t) = 0$ for $t < 1/n$,

$$A_n(t) = B_n C_{[nt]} B_n^T = \sum_{i=1}^{[nt]} B_n \xi_i \xi_i^T B_n^T$$

for $t \geqslant 1/n$. It is obvious that M_n is a martingale adapted to the natural filtration, $\mathcal{F}_t^n = \sigma(M_n(s) : s \leqslant t) = \sigma(\xi_i : i \leqslant nt)$ with sample paths in $D([0,\infty), \mathbb{R}^d)$.

Clearly, A_n^{ij} has sample paths in $D([0,\infty), \mathbb{R})$. Also, if $t > s \geqslant 0$, and $\theta \in \mathbb{R}^d$, then

$$\begin{aligned}
\langle A_n(t)\theta, \theta\rangle &= \sum_{i=1}^{[nt]} \langle B_n \xi_i \xi_i^T B_n^T \theta, \theta\rangle = \sum_{i=1}^{[nt]} \langle \xi_i \xi_i^T B_n^T \theta, B_n^T \theta\rangle \\
&= \sum_{i=1}^{[nt]} (B_n^T\theta)^T \xi_i \xi_i^T (B_n^T \theta) = \sum_{i=1}^{[nt]} \langle \xi_i, B_n^T \theta\rangle^2 \\
&\geqslant \sum_{i=1}^{[ns]} \langle \xi_i, B_n^T \theta\rangle^2 = \langle A_n(s)\theta, \theta\rangle
\end{aligned}$$

so that $A_n(t) - A_n(s)$ is nonnegative definite. For $\theta, \phi \in \mathbb{R}^d$ arbitrary

$$\begin{aligned}
\langle \theta, A_n(t)\phi\rangle &= \theta^T A_n(t)\phi = \theta^T B_n C_{[nt]} B_n^T \phi = (B_n^T\theta)^T C_{[nt]} (B_n^T \phi) \in \mathbb{R} \\
&= \left((B_n^T\theta)^T C_{[nt]} (B_n^T\phi)\right)^T = (B_n^T\phi)^T C_{[nt]}^T (B_n^T\theta) = (B_n^T\phi)^T C_{[nt]} (B_n^T\theta) \\
&= \phi^T B_n C_{[nt]} B_n^T \theta = \phi^T A_n(t)\theta = \langle A_n(t)\theta, \phi\rangle
\end{aligned}$$

so $A_n(t)$ is also symmetric.

We now verify condition (i). First, fix $T > 0$. Without loss of generality, we may assume that T is an integer. Furthermore, assume that $n > T$. Now $M_n(t) =$

$M_n(t-)$ except possibly at $t = j/n$, $j = 1, 2, \ldots$. In which case, $M_n(t) = \sum_{i=1}^{j} B_n\xi_i$ and $M_n(t-) = \sum_{i=1}^{j-1} B_n\xi_i$. Therefore, $M_n(t) - M_n(t-) = B_n\xi_j$. Now,

$$\begin{aligned} E\left[\sup_{t\leqslant T}\|M_n(t) - M_n(t-)\|\right] &= E\left[\max_{j\leqslant nT}\|B_n\xi_j\|\right] \\ &= E\left[\max\left\{\max_{(k-1)n<j\leqslant kn}\|B_n\xi_j\| : k = 1,\ldots,T\right\}\right] \\ &\leqslant E\left[\sum_{k=1}^{T}\max_{(k-1)n<j\leqslant kn}\|B_n\xi_j\|\right] \\ &= TE\left[\max_{j\leqslant n}\|B_n\xi_j\|\right] \end{aligned}$$

Therefore, it suffices to show that $E\max_{j\leqslant n}\|B_n\xi_j\| \to 0$. The following lemma is a significant strengthening of results in Sepanski [**20**] and Maller [**12**], where the convergence was shown in probability. The proof is an application of multivariable regular variation.

LEMMA 2.1. *If $\xi, \xi_1, \xi_2, \ldots$ are independent and identically distributed random vectors in $\mathbb{R}^d$ with distribution in the generalized domain of attraction of a d-variate normal law, then $E\max_{j\leqslant n}\|B_n\xi_j\| \to 0$.*

PROOF. Define $T_n = \max_{j\leqslant n}\|B_n\xi_j\|$. Write

$$\begin{aligned} ET_n &= \int_0^\infty \Pr[T_n > t]dt \\ &= \int_0^\delta \Pr[T_n > t]dt + \int_\delta^\infty \Pr[T_n > t]dt \\ &\leqslant \delta + \int_\delta^\infty (1 - \Pr[T_n \leqslant t])dt \\ &= \delta + \int_\delta^\infty (1 - \Pr[\|B_n\xi\| \leqslant t]^n)dt \\ &= \delta + \int_\delta^\infty (1 - (1 - \varepsilon_n(t))^n)dt \\ &= \delta + I_n \end{aligned}$$

where $\delta > 0$ is small and $\varepsilon_n(t) = \Pr[\|B_n\xi\| > t] \leqslant \varepsilon_n(\delta) \to 0$ uniformly in $t > \delta$ as $n \to \infty$ since $n\Pr[\|B_n\xi\| > \delta] \to 0$ by the standard convergence criteria for triangular arrays (Theorem 3.2.2 in [**16**]). Note that $1 - (1-x)^n \leqslant nx$ for all $x > 0$ small, and so for all n large

$$I_n \leqslant n\int_\delta^\infty \varepsilon_n(t)dt = nE\|B_n\xi\|I(\|B_n\xi\| > \delta) \to 0$$

by the following regular variation argument. Recall that we are assuming that $E\xi = 0$. Note that

$$\|B_n\xi\| \leqslant \sum_{j=1}^{d} |\langle B_n\xi, e_j\rangle|$$

where $e_1 \cdots e_k$ is the standard basis for $\mathbb{R}^d$. Then

$$I_n \leqslant nE \sum_{j=1}^{d} |\langle B_n\xi, e_j\rangle| I(\|B_n\xi\| > \delta)$$

where

$$nE|\langle B_n\xi, e_j\rangle| I(\|B_n\xi\| > \delta) = J_1 + J_2$$

with

$$\begin{aligned} J_1 &= nE|\langle B_n\xi, e_j\rangle| I(|\langle B_n\xi, e_j\rangle| > \delta) \\ J_2 &= nE|\langle B_n\xi, e_j\rangle| I(\|B_n\xi\| > \delta \text{ and } |\langle B_n\xi, e_j\rangle| \leqslant \delta) \end{aligned}$$

Define the truncated moments U_b, V_c as in Feller [**4**] i.e., let

$$\begin{aligned} U_b(r,x) &= E|\langle \xi, x\rangle|^b I[|\langle \xi, x\rangle| \leqslant r], \\ V_c(r,x) &= E|\langle \xi, x\rangle|^c I[|\langle \xi, x\rangle| > r]. \end{aligned}$$

Since every full multivariable normal law on $\mathbb{R}^d$ is of the same type, we can assume without loss of generality that the limit is standard normal, with characteristic function $e^{-\frac{1}{2}Q(t)}$ where $Q(t) = \|t\|^2$. Corollary 8.1.4 in [**16**] shows that ξ belongs to the generalized domain of attraction of this multivariate normal law if and only if the truncated second moment function $M(t) = E\big(\langle \xi, t\rangle^2 I(|\langle \xi, t\rangle| \leqslant 1)\big)$ satisfies the regular variation condition

(2.4) $$nM(B_n^T t_n) \to Q(t) \text{ whenever } t_n \to t \neq 0 \text{ in } \mathbb{R}^d.$$

This condition is equivalent to regular variation of $M(t)$ at zero with exponent $-\frac{1}{2}I$, and then Theorem 5.3.4 in [**16**] shows that $U_2(r,x)$ is slowly varying uniformly in $\|x\| = 1$ as $r \to \infty$. Write $B_n^T e_j = r_n^{-1}\theta_n$ where $\|\theta_n\| = 1$ and $r_n \to \infty$ so that $|\langle B_n\xi, e_j\rangle| = |\langle \xi, B_n^T e_j\rangle| = r_n^{-1}|\langle \xi, \theta_n\rangle|$. Then

$$\begin{aligned} J_1 &= nE|\langle B_n\xi, e_j\rangle| I(|\langle B_n\xi, e_j\rangle| > \delta) \\ &= nr_n^{-1} E|\langle \xi, \theta_n\rangle| I(|\langle \xi, \theta_n\rangle| > \delta r_n) \\ &= nr_n^{-1} V_1(\delta r_n, \theta_n) \\ &= \frac{r_n V_1(\delta r_n, \theta_n)}{U_2(\delta r_n, \theta_n)} \cdot \frac{U_2(\delta r_n, \theta_n)}{U_2(r_n, \theta_n)} \cdot nr_n^{-2} U_2(r_n, \theta_n) \to 0 \end{aligned}$$

since the first factor tends to zero by Theorem 5.3.11 in [**16**] (a uniform version of Feller [**4**, XVII.9, Theorem 2]), the second factor tends to 1 since U_2 is uniformly slowly varying, and the third factor

$$\begin{aligned} nr_n^{-2}U_2(r_n,\theta_n) &= nr_n^{-2}E|\langle\xi,\theta_n\rangle|^2 I(|\langle\xi,\theta_n\rangle| \leqslant r_n) \\ &= nE|\langle\xi,r_n^{-1}\theta_n\rangle|^2 I(|\langle\xi,r_n^{-1}\theta_n\rangle| \leqslant 1) \\ &= nE|\langle\xi,B_n^T e_j\rangle|^2 I(|\langle\xi,B_n^T e_j\rangle| \leqslant 1) \\ &= M(B_n^T e_j) \to Q(e_j) = 1 \end{aligned}$$

as $n \to \infty$ in view of (2.4).

Next write

$$\begin{aligned} J_2 &= nE|\langle B_n\xi, e_j\rangle| I(\|B_n\xi\| > \delta \text{ and } |\langle B_n\xi, e_j\rangle| \leqslant \delta) \\ &\leqslant \delta \cdot n \Pr[\|B_n\xi\| > \delta] \to 0 \end{aligned}$$

by the standard convergence criteria for triangular arrays (Theorem 3.2.2 in [**16**]). This concludes the proof of the lemma, and so condition (i) is established.

Finally we verify condition (ii). Recall that C_n is the empirical covariance operator defined by equation (2.3). Let H_n be the normalizing sequence constructed in Hahn and Klass [**7, 8**], and note that $H_n = H_n^T$. Let B_n be any other norming sequence for which the basic convergence in distribution (2.1) holds. As shown in Sepanski [**21**], $H_n C_n H_n \to I$ in probability. In fact, Sepanski [**22**] shows that $H_n C_n^{1/2} \to I$ in probability. However, since both sequences $B_n \sum_{i=1}^n \xi_i$ and $H_n \sum_{i=1}^n \xi_i$ converge in law to $N(0,I)$, Billingsley's convergence of types theorem [**1**] says that, $B_n = \delta_n R_n H_n$, where $\delta_n \to I$, and R_n are orthogonal transformations. But then we also have $B_n C_n B_n^T \to I$ in probability, since

$$\begin{aligned} &\|B_n C_n B_n^T - I\| \\ &\quad = \|\delta_n R_n H_n C_n H_n R_n^T \delta_n^T - I\| \\ &\quad = \|(\delta_n R_n H_n C_n H_n R_n^T \delta_n^T - \delta_n) + (\delta_n - I)\| \\ &\quad \leqslant \|\delta_n R_n H_n C_n H_n R_n^T \delta_n^T - \delta_n\| + \|\delta_n - I\| \\ &\quad \leqslant \|\delta_n\| \; \|R_n H_n C_n H_n R_n^T \delta_n^T - I\| + \|\delta_n - I\| \\ &\quad = \|\delta_n\| \; \|(R_n H_n C_n H_n R_n^T \delta_n^T - \delta_n^T) + (\delta_n^T - I)\| + \|\delta_n - I\| \\ &\quad \leqslant \|\delta_n\| \left\{ \|R_n H_n C_n H_n R_n^T \delta_n^T - \delta_n^T\| + \|\delta_n^T - I\| \right\} + \|\delta_n - I\| \\ &\quad \leqslant \|\delta_n\| \left\{ \|R_n H_n C_n H_n R_n^T - I\| \; \|\delta_n^T\| + \|\delta_n^T - I\| \right\} + \|\delta_n - I\| \\ &\quad = \|\delta_n\| \; \|\delta_n^T\| \; \|R_n H_n C_n H_n R_n^T - R_n R_n^T\| + \|\delta_n\| \; \|\delta_n^T - I\| + \|\delta_n - I\| \\ &\quad \leqslant \|\delta_n\| \; \|\delta_n^T\| \; \|R_n\| \; \|R_n^T\| \; \|H_n C_n H_n - I\| + \|\delta_n\| \; \|\delta_n^T - I\| + \|\delta_n - I\| \end{aligned} \tag{2.5}$$

tends to zero in probability. Next recall that $B_n = n^{-1/2} L_n$ where L_n is slowly varying, so that $L_{[nt]} L_n^{-1} \to I$. Then we have

$$\begin{aligned} B_n B_{[nt]}^{-1} &= n^{-1/2} L_n \, L_{[nt]}^{-1} [nt]^{1/2} = n^{-1/2} [nt]^{1/2} L_n L_{[nt]}^{-1} \\ &= n^{-1/2} [nt]^{1/2} (L_{[nt]} L_n^{-1})^{-1} \to t^{1/2} I^{-1} = t^{1/2} I \end{aligned} \tag{2.6}$$

while

$$(2.7)\qquad \begin{aligned}(B_{[nt]}^{-1})^T B_n^T &= [nt]^{1/2}(L_{[nt]}^{-1})^T L_n^T n^{-1/2}\\ &= [nt]^{1/2}n^{-1/2}(L_{[nt]}^{-1})^T L_n^T\\ &= [nt]^{1/2}n^{-1/2}(L_n L_{[nt]}^{-1})^T\\ &= [nt]^{1/2}n^{-1/2}((L_{[nt]}L_n^{-1})^{-1})^T\\ &\to t^{1/2}(I^{-1})^T = t^{1/2}I.\end{aligned}$$

Combining (2.5), (2.6), and (2.7) we see that

$$\begin{aligned}A_n(t) &= B_n C_{[nt]} B_n^T\\ &= B_n B_{[nt]}^{-1}\cdot B_{[nt]}C_{[nt]}B_{[nt]}^T\cdot(B_{[nt]}^{-1})^T B_n^T\\ &\to t^{1/2}I\cdot I\cdot t^{1/2}I = tI\end{aligned}$$

in probability, and this concludes the proof. □

3. Remarks

Theorem 2.1 was proven by Sepanski [**23**] using specialized Fourier analytic methods. The proof given in this paper is considerably simpler. It may also be possible to prove Theorem 2.1 using the more general result of Kuelbs [**11**], which shows that a functional limit theorem for Banach space valued random variables follows from the central limit theorem, under suitable conditions.

References

[1] P. Billingsley (1966), *Convergence of types in k-space*, Z. Wahrsh. Verw. Gebiete **5** 175–179.

[2] N. Bingham, C. Goldie, and J. Teugels (1987), *Regular Variation.* Encyclopedia of Mathematics and its Applications **27**, Cambridge University Press, London.

[3] S.N Ethier and T.G. Kurtz (1986), *Markov Processes: Characterization and Convergence*, Wiley, New York.

[4] W. Feller (1971), *An Introduction to Probability Theory and Its Applications*, Vol. II, 2nd Ed., Wiley, New York.

[5] L. de Haan and S. Resnick (1979), *Derivatives of regularly varying functions in* $\mathbb{R}^d$ *and domains of attraction of stable distributions*, Stoch. Proc. Appl. **8**, 349–355.

[6] L. de Haan, E. Omey, and S. Resnick (1984), *Domains of attraction and regular variation in* $\mathbb{R}^d$, J. Multivariate Anal. **14**, 17–33.

[7] M.G. Hahn and M.J. Klass (1980), *Matrix normalization of sums of random vectors in the domain of attraction of the multivariate normal*, Ann. Probab. **8**, 262–280.

[8] M.G. Hahn and M.J. Klass (1980), *The generalized domain of attraction of spherically symmetric stable laws on* $\mathbb{R}^d$, Proc. Conf. Probability in Vector Spaces II, Poland. Lecture Notes in Math. **828**, 52–81.

[9] W. Hudson, J. Veeh, and D. Weiner (1988), *Moments of distributions attracted to operator stable laws*, J. Multivariate Anal. **24**, 1–10.

[10] A. Jakimiv (1981), *Many-dimensional Tauberian theorems and their application to Bellman-Harris branching processes*, Math. Sbornik **115**, 463–477. (Russian)

[11] J Kuelbs (1973), *The invariance principle for Banach space valued random variables*, J. Multivariate Anal. **3**, 161–172.

[12] R.A. Maller (1993), *Quadratic negligibility and the asymptotic normality of operator normed sums*, J. Multivariate Anal. **44**, 191–219.

[13] M. Meerschaert (1988), *Regular variation in* $\mathbb{R}^k$, Proc. Amer. Math. Soc. **102**, 341–348.
[14] M. Meerschaert (1994), *Norming operators for generalized domains of attraction*, J. Theoret. Probab., **7**, 739–798.
[15] M. Meerschaert and H. P. Scheffler (1998), *Multivariable regular variation of functions and measures*, J. Appl. Anal. **5**, 77–98.
[16] M. Meerschaert, M. and H. P. Scheffler (2001), *Limit Theorems for Sums of Independent Random Vectors*. Wiley, New York.
[17] T. Ostrogorski (1995), *Regular variation on homogeneous cones*, Publ. Inst. Math. (Beograd) (N.S.) **58 (72)**, 51–70.
[18] T. Ostrogorski (1997), *Regular variation in* $\mathbb{R}^n_+$, Mathematica **39(62)**, no. 2, 265–276.
[19] T. Ostrogorski (1998), *Regular variation and the Banach-Steinhaus theorem*, Nieuw Arch. Wisk. **(4) 16**, no. 1–2, 27–36.
[20] S. Sepanski (1994), *Probabilistic characterizations of the generalized domain of attraction of the multivariate normal law*, J. Theoret. Probab. **7**, 857–866.
[21] S. Sepanski (1994), *Asymptotics for multivariate t-statistic and Hotelling's* t^2 *statistic under infinite second moments via bootstrapping*, J. Multivariate Anal. **49**, 41–54.
[22] S. Sepanski (1996), *Asymptotics for multivariate t-statistic for random vectors in the generalized domain of attraction of the multivariate normal law*, Stat. Probab. Lett. **30**, 179–188.
[23] S. Sepanski (1997), *Some invariance principles for random vectors in the generalized domain of attraction of the multivariate normal law*, J. Theoret. Probab. **10**, 1053–1063.
[24] U. Stadtmüller and R. Trautner (1981), *Tauberian theorems for Laplace transforms in dimension* $D > 1$, J. Reine Angew. Math. **323**, 127–138.
[25] A. Stam (1977), *Regular variation in* $\mathbb{R}^d_+$ *and the Abel-Tauber theorem*, Technical Report T. W. 189, Mathematisch Instituut Rijksuniversiteit Groningen, The Netherlands.

Department of Mathematics
University of Nevada
Reno, NV 89557-0045

Department of Mathematical Sciences
Saginaw Valley State University
University Center, MI 48710

sepanski@svsu.edu