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DERIVATIONS OF SKEW POLYNOMIAL RINGS

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Dedicated to Professor Yukio Tsushima on his 60th birthday

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ABSTRACT. Let R be a commutative ring of characteristic zero. Under certain conditions we determine the type of derivations of a skew polynomial ring $A_n = R[X_1, X_2, \dots, X_n; d_1, d_2, \dots, d_n]$ over R, where $d_1, d_2, \dots, d_n$ are derivations of R commuting to each other, and we examine properties of the ideals of A_n.

0. Introduction

Let R be a commutative ring with identity 1 and d a derivation of R. A skew polynomial ring $R[X; d]$ is defined as the set of all polynomials $\sum_{i=0}^{n} r_i X^i$ with usual addition and the following multiplication:

$$Xr = rX + d(r) \quad \text{for all } r \in R.$$

For derivations $d_1,\ d_2, \dots, d_n$ of R, we can also construct a skew polynomial ring $A_n = R[X_1, X_2, \dots, X_n; d_1, d_2, \dots, d_n]$ such that

$$X_i r = rX_i + d_i(r) \quad \text{and} \quad X_i X_j = X_j X_i$$

for any $r \in R$. The properties of these skew polynomial rings have been discussed by many authors (see for example [C-F], [J1] and [V2]). In [V2], Voskoglou has given the properties of the skew polynomial ring over a ring R of prime characteristic which are connected with the $\mathcal{D}$-simplicity of R with respect to a set of derivations $\mathcal{D}$ of R.

In this paper, we determine the type of derivations D of the skew polynomial ring A_n and we examine properties of its ideals.

In the following, R will denote a commutative ring with identity 1, and $n \cdot 1 = n$ not a zero divisor in R for any integer $n > 0$.

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1. Preliminaries

Let d be a derivation of R and let $R[X;d]$ be the skew polynomial ring over R defined with respect to d. Firstly, we treat derivations of $R[X;d]$. The following relation is easily obtained by $Xr = rX + d(r)$ and by applying induction on n.

$$X^n r = rX^n + nd(r)X^{n-1} + \frac{n(n-1)}{2}d^2(r)X^{n-2} + \cdots + d^n(r). \tag{1.1}$$

In this paper, for R-algebras A and B, an additive map $D : A \to B$ is called a *derivation* if for any x, $y \in A$,

$$D(xy) = D(x)y + xD(y) \quad \text{and} \quad D(R) \subseteq R.$$

The following lemma is elementary in our computation.

LEMMA 1.1. *Let $f \in R[X;d]$ and let $s \in R$. Assume that there exists an element $\alpha \in R$ such that $d(\alpha) \neq 0$ is not a zero divisor. Then, if $f\alpha = \alpha f + s$, $f = r_1X + r_0$ for some $r_0, r_1 \in R$ and $s = r_1d(\alpha)$. In particular, if $s = 0$, then $f = r_0 \in R$.*

PROOF. We set $f = r_nX^n + r_{n-1}X^{n-1} + \cdots + r_1X + r_0$ $(r_i \in R)$. Then, by (1.1) and $f\alpha - \alpha f - s = 0$, the coefficient $nr_nd(\alpha)$ of X^{n-1} is zero, which means $r_n = 0$. Therefore, by induction, we have $f = r_1X + r_0$ and $s = r_1d(\alpha)$. □

LEMMA 1.2. *Assume that there exists an element $\alpha \in R$ such that $d(\alpha) \neq 0$ is not a zero divisor. If $D : R[X;d] \to R[X;d]$ is a derivation, then $D(X) = r_1X + r_0$ for some r_0, $r_1 \in R$. In particular if there exists an element $\xi \in R$ such that $(Dd - dD)(\xi) = 0$ and $d(\xi)$ is not a zero divisor, then $D(X) = r_0 \in R$.*

PROOF. We set $D(X) = f = r_nX^n + r_{n-1}X^{n-1} + \cdots + r_1X + r_0$ $(r_i \in R)$. Since $Xr = rX + d(r)$ for any $r \in R$, we have

$$D(X)r = rD(X) + (Dd - dD)(r) \quad \text{for any } r \in R, \tag{1.2}$$

which shows that $fr = rf + (Dd - dD)(r)$. Thus, the result is obtained by Lemma 1.1. □

COROLLARY 1.1. *Assume that there exists an element $\alpha \in R$ such that $d(\alpha) \neq 0$ is not a zero divisor. If I_g is an inner derivation by $g \in R[X;d]$, then $g = r_1X + r_0$. In particular if $I_gd = dI_g$, then $d(r_1)d(r) = 0$ for any $r \in R$.*

PROOF. If I_g is an inner derivation by $g \in R[X;d]$, then, by the definition of derivation, $I_g(R) \subset R$ and so $I_g(r) = gr - rg \in R$. Then, by Lemma 1.1, $g = r_1X + r_0$. Moreover if $I_gd = dI_g$, then, by $I_gd(r) = dI_g(r)$, we get the result. □

2. Ideals of $\mathbf{R[X;d]}$

In this section, we treat some properties of ideals of $R[X;d]$. Let D be a derivation of a ring A and I an ideal of A. I is called a *D-ideal* if $D(I) \subseteq I$. If A has no D-ideal except 0 and A, then A is said to be *D-simple.* For a subset S of R, we set $S[X;d] = \left\{ \sum_{i=0}^{n} s_iX^i \mid s_i \in S,\ n = 0, 1, 2, \cdots \right\}$. Then we have the following:

LEMMA 2.1. (1) *Let d be a derivation of R. If I is a d-ideal of R, then $I[X;d]$ is an ideal of $R[X;d]$.*

(2) *If $\Im$ is an ideal of $R[X;d]$, then $\Im \cap R$ is a d-ideal of R.*

PROOF. See Lemma 1.3 (cases i and ii) of [J1]. □

LEMMA 2.2. *Assume that $n = n \cdot 1$ is invertible in R for any integer $n > 0$. Assume also that there exists an element $\alpha \in R$ such that $d(\alpha)$ is invertible. If $\Im$ is an ideal of $R[X;d]$, then $\Im \cap R$ is equal to the set of all coefficients of all polynomials f in $\Im$.*

PROOF. Let $\Im$ be an ideal of $R[X;d]$ and let $f = \sum_{i=0}^{n} r_i X^i \in \Im$. Since

$$f\alpha - \alpha f = r_n n d(\alpha) X^{n-1} + \text{terms of lower degree}$$

is contained in $\Im$ and $nd(\alpha)$ is invertible, we see

$$r_n X^{n-1} + s_{n-2} X^{n-2} + \cdots + s_1 X + s_0 \in \Im$$

for some $s_i \in R$. Repeating this method, we have $r_n \in \Im$. Thus, $f - r_n X^n = r_{n-1}X^{n-1} + \cdots + r_1 X + r_0 \in \Im$. Using this process, we see that if $f = \sum_{i=0}^{n} r_i X^i \in \Im$, then all the coefficients of f are contained in $\Im$ and thus $\Im \cap R$ is equal to the set of all coefficients of all polynomials in $\Im$. □

COROLLARY 2.1. *Let D be a derivation of $R[X;d]$ and let $\Im$ be an ideal of $R[X;d]$. Then under the assumptions of Lemma 2.2, $\Im$ is a D-ideal if and only if $\Im \cap R$ is a D-ideal.*

PROOF. Let $f = \sum_{i=0}^{n} r_i X^i \in \Im$. Then, by $D(f) = \sum_{i=0}^{n} D(r_i)X^i + \sum_{i=0}^{n} r_i D(X^i)$ and $r_i \in \Im$ for any i, $D(f) \in \Im$ if and only if $\sum_{i=0}^{n} D(r_i)X^i \in \Im$. By Lemma 2.2, this is equivalent to $D(r_i) \in \Im$ for any i. □

Let Γ be the set of all d-ideals of R and let Λ be the set of all ideals of $R[X;d]$. Then we have a correspondence

$$\Phi : \Gamma \ni I \mapsto I[X;d] \in \Lambda \quad \text{and} \quad \Psi : \Lambda \ni \Im \mapsto \Im \cap R \in \Gamma.$$

Under these notations, we see the following:

THEOREM 2.1. *Assume that $n = n \cdot 1$ is invertible in R for any integer $n > 0$ and there exists an element $\alpha \in R$ such that $d(\alpha)$ is invertible. Then there exists an order preserving lattice isomorphism of Γ and Λ.*

PROOF. If I_1 and I_2 are d-ideals of R such that $I_1 \subset I_2$, then, by Lemma 2.1, $I_1[X;d] \subset I_2[X;d]$ is clear. Conversely, if $\Im_1$ and $\Im_2$ are ideals of $R[X;d]$ such that $\Im_1 \subset \Im_2$, then, by Lemma 2.2, $\Im_1 \cap R \subset \Im_2 \cap R$. Since $\Im_1 \cap R$ is the set of all coefficients of polynomials in $\Im_1$, $\Psi\Phi = I_\Gamma$ and $\Phi\Psi = I_\Lambda$ are clear. □

3. Derivations and ideals of $\mathbf{R[X_1, X_2, \cdots, X_n; d_1, d_2, \cdots, d_n]}$

It is well-known (e.g., [C-F, p. 42]) that if R is a commutative Noetherian $\mathbb{Q}$-algebra with nonzero derivation d such that R is d-simple, then $R[X; d]$ is a simple ring. And Jordan [J2] has shown that if k is a field of characteristic zero and $2 \leqslant n$, then the commutative polynomial ring $A = k[X_1, X_2, \cdots, X_n]$ admits a k-derivation d such that A is d-simple and $d(A)$ contains no units.

For derivations $d_1, d_2, \cdots, d_n$ of R commuting to each other, we consider the skew polynomial ring $A_n = R[X_1, X_2, \ldots, X_n; d_1, d_2, \cdots, d_n]$ such that

$$X_i r = rX_i + d_i(r) \quad \text{and} \quad X_i X_j = X_j X_i$$

for any $r \in R$. In this section, we characterize derivations of A_n under the following conditions:

(3.1) $n = n \cdot 1$ is not a zero divisor for any integer $n > 0$.

(3.2) There exist elements $\alpha_i \in R$ such that $d_i(\alpha_i) \neq 0$ is not a zero divisor and $d_i(\alpha_j) = 0$ $(i \neq j)$ for any $i,\ j = 1, 2, \ldots, n$.

There exists a ring which satisfies the conditions (3.1) and (3.2) as follows:

EXAMPLE 3.1. Let k be an integral domain with characteristic 0 and $R = k[Y_1, Y_2, \ldots, Y_n]$ a commutative polynomial ring of n-variables. Then $d_i = \dfrac{\partial}{\partial Y_i}$ is a derivation such that $d_i(Y_i) = 1$ and $d_i(Y_j) = 0$ $(i \neq j)$.

Thus, the conditions (3.1) and (3.2) occur naturally.

LEMMA 3.1. *Let* $f = \sum_{i=0}^{k} a_i X_n^i$ *be in* A_n, *where* $a_i \in A_{n-1}$. *If* $fr - rf \in A_{n-1}$ *for any* $r \in R$, *then* $f = r_1 X_n + a_0$ *for some* $r_1 \in R$ *and* $a_0 \in A_{n-1}$. *In particular, if* $fr = rf$ *for any* $r \in R$, *then* $f \in R$.

PROOF. We note the following: if $d_i(r) = 0$, then, by $X_i r = rX_i$, $fr = rf$ for any $f \in R[X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n]$. Let $f = \sum_{i=0}^{k} a_i X_n^i$ be in A_n such that $a_i \in A_{n-1}$. Firstly, we show that $f = a_1 X_n + a_0$ for some $a_i \in A_{n-1}$. By $X_n \alpha_n = \alpha_n X_n + d_n(\alpha_n)$ and $X_i \alpha_n = \alpha_n X_i$ for any $1 \leqslant i \leqslant n-1$, we have

$$f\alpha_n - \alpha_n f = a_k k d_n(\alpha_n) X_n^{k-1} + \text{terms of degree} < k-1 \text{ in } X_n. \tag{3.3}$$

Since the coefficients of the lower term of X_n^{k-2} are in A_{n-1} and $kd_n(\alpha_n)$ is invertible, we get $a_k = 0$. Repeating this argument, we obtain that $f = a_1 X_n + a_0$ for some $a_i \in A_{n-1}$. Secondly, using α_i for $1 \leqslant i \leqslant n-1$, we have $a_1 \alpha_i = \alpha_i a_1$. Since a_1 is in A_{n-1}, we denote $a_1 = \sum_{i=0}^{\ell} b_i X_{n-1}^i$ for some $b_i \in A_{n-2}$. Then, by $a_1 \alpha_{n-1} = \alpha_{n-1} a_1$, we have $a_1 = b_0 \in A_{n-2}$. Repeating this argument, we obtain that $f = r_1 X_n + a_0$ for some $r_1 \in R$ and $a_0 \in A_{n-1}$. In particular, if $fr = rf$, then $r_1 = 0$ and so $f = a_0 \in A_{n-1}$. Thus, by induction, we have $f \in R$. □

THEOREM 3.1. *Let* D *be a derivation of* A_n *such that* $Dd_i = d_i D$ *for any* $1 \leqslant i \leqslant n$. *Then* $D(X_i) = r_i$ *for some* $r_i \in R$.

PROOF. By $D(X_ir) = D(rX_i + d_i(r))$ for any $r \in R$, we see $D(X_i)r = rD(X_i)$. Since $D(X_i)$ is a polynomial in A_n, then, by Lemma 3.1, $D(X_i) = r_i$ for some $r_i \in R$. □

LEMMA 3.2. *Assume that $n = n \cdot 1$ is invertible in R for any integer $n > 0$ and there exists an element $\alpha_i \in R$ such that $d_i(\alpha_i)$ is invertible and $d_i(\alpha_j) = 0$ for any $j \neq i$ $(1 \leqslant i, j \leqslant n)$. If $\Im$ is an ideal of A_n, then $\Im \cap R$ is equal to the set of all coefficients of all polynomials f in $\Im$.*

PROOF. Let $f = \sum_{i=0}^{k} a_i X_n^i$ be in $\Im$. By using (3.3), we have $a_k X_n^{k-1} + \cdots \in \Im$ and inductively we get $a_i \in \Im$ for any $0 \leqslant i \leqslant k$. Since $a_i = \sum_{j=0}^{\ell} b_j X_{n-1}^j$ and $b_j \in A_{n-2}$, by the similar computations, we can show that all coefficients of f are contained in $\Im$. This completes the proof of the Lemma. □

Let $\mathcal{D}$ be a set of derivations of a ring A. An ideal I of A is called a *$\mathcal{D}$-ideal* if I is a d-ideal for all $d \in \mathcal{D}$.

LEMMA 3.3. (1) *Let $\mathcal{D} = \{d_1, d_2, \cdots, d_n\}$ be a set of derivations of R. Then, if I is a $\mathcal{D}$-ideal of R, $I[X_1, X_2, \ldots, X_n; d_1, d_2, \ldots, d_n]$ is an ideal of A_n.*

(2) *If $\Im$ is an ideal of A_n, then $\Im \cap R$ is a $\mathcal{D}$-ideal of R.*

PROOF. (1) See Lemma 3.1 of [V1].

(2) For any $r \in \Im \cap R$, we have $X_i r - rX_i = d_i(r) \in \Im$ for all $1 \leqslant i \leqslant n$. Hence, $\Im \cap R$ is a $\mathcal{D}$-ideal. □

Let Γ_n be the set of all $\mathcal{D}$-ideals of R and let Λ_n be the set of all ideals of $A_n = R[X_1, X_2, \ldots, X_n; d_1, d_2, \ldots, d_n]$. Then we have a correspondence

$$\Phi : \Gamma_n \ni I \mapsto I[X_1, X_2, \ldots, X_n; d_1, d_2, \ldots, d_n] \in \Lambda_n,$$
$$\Psi : \Lambda_n \ni \Im \mapsto \Im \cap R \in \Gamma_n.$$

Under these notations, we see the following:

THEOREM 3.2. *Assume that $n = n \cdot 1$ is invertible in R for any integer $n > 0$ and there exists an element $\alpha_i \in R$ such that $d_i(\alpha_i)$ is invertible and $d_i(\alpha_j) = 0$ for any $j \neq i$ $(1 \leqslant i, j \leqslant n)$. Then Φ and Ψ are order preserving lattice isomorphism of Γ_n and Λ_n such that $\Psi\Phi = I_{\Gamma_n}$ and $\Phi\Psi = I_{\Lambda_n}$.*

PROOF. If I_1 and I_2 are $\mathcal{D}$-ideals of R such that $I_1 \subset I_2$, then, by Lemma 3.4(1), we have

$$I_1[X_1, X_2, \cdots, X_n; d_1, d_2, \cdots, d_n] \subset I_2[X_1, X_2, \cdots, X_n; d_1, d_2, \cdots, d_n].$$

Conversely, if $\Im_1$ and $\Im_2$ are ideals of A_n such that $\Im_1 \subset \Im_2$, then, by Lemma 3.3, we have $\Im_1 \cap R \subset \Im_2 \cap R$. Moreover, by Lemma 3.3, we easily see that $\Psi\Phi = I_{\Gamma_n}$ and $\Phi\Psi = I_{\Lambda_n}$. □

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References

[C-F] J. Cozzen, C. Faith *Simple Noetherian Rings*, Cambridge Tracts in Mathematics 69, Cambridge University Press, 1975.

[J1] D. A. Jordan, *Noetherian Ore extensions and Jacobson rings*, J. London Math. Soc. **10** (1975), 281–291.

[J2] D. A. Jordan, *Differentially simple rings with no invertible derivatives*, Quart. J. Math. **32** (1981), 417–424.

[V1] M. G. Voskoglou, *Simple skew polynomial rings*, Publ. Inst. Math. (Beograd) **37(51)** (1985), 37–41.

[V2] M. G. Voskoglou, *A note on skew polynomial rings*, Publ. Inst. Math. (Beograd) **55(69)** (1994), 23–28.

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