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COVERING PROPERTIES OF CONTINUA AND MAPPINGS

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Abstract. It is known that monotone mappings preserve the covering property for continua. Similar result is proved for having the covering property hereditarily. An example is constructed which shows that the two results cannot be extended to almost monotone mappings.

Given a (metric) continuum X, a family $\mathcal{F}$ of nonempty closed subsets of X is said to *cover* X provided that $\bigcup\mathcal{F} = X$. We denote by $C(X)$ the *hyperspace of* (nonempty) *subcontinua* of X equipped with the Hausdorff metric (see [13, 0.1, p. 1]; compare [9, 2, p. 9]). By a *Whitney map* for $C(X)$ we mean a mapping $\mu : C(X) \to [0, \infty)$ such that

(a) $\mu(\{x\}) = 0$ for each $x \in X$,
(b) if $A \subsetneq B$, then $\mu(A) < \mu(B)$.

For the existence of a Whitney map for $C(X)$ see e.g., [13, 0.50.1–0.50.3, p. 25–27] and [9, Chapter 4, Section 13, p. 106–108 and Chapter 7, Section 23, p. 205–207].

For each $t \in [0, \mu(X)]$ the preimage $\mu^{-1}(t)$ is called a *Whitney level* for $C(X)$. It is known that Whitney levels are subcontinua of $C(X)$ (i.e., that any Whitney map for $C(X)$ is monotone, see [13, Theorem 14.2, p. 400] and [9, Theorem 19.9, p. 160]).

The reader is referred to [13] and [9] for these and other concepts used in this paper.

A continuum X is said to have the *covering property* (written $X \in CP$) provided that for each Whitney map $\mu : C(X) \to [0, \infty)$ and for each $t \in [0, \mu(X)]$ no proper subcontinuum of $\mu^{-1}(t)$ covers X. A continuum X is said to have the *covering property hereditarily* (written $X \in CPH$) provided that each nondegenerate

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subcontinuum of X has the covering property (see [16, property (2), p. 159]; compare also [13, p. 486]). Obviously $X \in CPH$ implies $X \in CP$ but not conversely, as an example shows of ray spiraling down on a circle, see [16, p. 160].

Bruce Hughes has shown (see [13, Theorem 14.73.3, p. 482]) that a continuum X has the covering property if and only if for each Whitney map each Whitney level is an irreducible continuum. Consequently, we have the following result (compare [13, Theorem 14.73.1, p. 478]).

Theorem 1. *If a continuum X has the covering property, then X is irreducible.*

Another attribute of continua with the covering property has been formulated in [10, Section 6, 2., p. 179] (see [13, Theorem 14.14.1, p. 418]).

Theorem 2. *If a continuum X has the covering property, then X is unicoherent.*

However, a continuum with the covering property, being unicoherent, need not be hereditarily unicoherent, as it can be seen from a continuum which is the union of a circle and a ray spiraling down on it, see [15, Example 28] and [16, p. 160].

The next statement is a consequence of the previous two.

Theorem 3. *If a continuum X has the covering property hereditarily, then X is hereditarily irreducible and hereditarily unicoherent.*

A further progress has been made by B. Hughes [13, Theorem 14.73.21, p. 497], J. Grispolakis and E. D. Tymchatyn [8, Theorem 3.2, p. 178], J. Grispolakis, S. B. Nadler, Jr. and E. D. Tymchatyn [7, Theorem 2.2, p. 199], C. W. Proctor [17, Theorem, p. 294]. The mentioned results are of a special importance because they tie several conditions defined in very different ways (as the structure of some compactifications of the real half-line, being in the Class(W), C^*-smoothness, and CP. The results can be summarized as follows (see [9, conditions (a)–(d), p. 254, and Theorem 67.1, p. 320], where a proof of the equivalences is presented).

Theorem 4. *For a continuum X the following statements are equivalent:*

(4.1) *X is absolutely C^*-smooth;*
(4.2) *$X \in \mathrm{Class}(W)$;*
(4.3) *each compactification Y of $[0,1)$ with X as the remainder has the property that $C(Y)$ is a compactification of $C([0,1))$;*
(4.4) *$X \in CP$.*

The following results are known.

Theorem 5. *Each of the continua below has the covering property hereditarily:*

(5.1) *arc-like continua* (see [10, Theorem 4.2, p. 171 and Section 6, p. 179], [16, Proposition 18, p. 162] and [13, Lemma 14.13.1, p. 415, and Theorem 14.73.12, p. 490]);
(5.2) *hereditarily indecomposable continua* (see [10, Section 6, 2, p. 179], [16, Proposition 18, p. 162] and [13, Theorem 14.14.1, p. 418, and Theorem 14.73.12, p. 490]);

(5.3) *nonplanar circle-like continua* (see [15, Proposition 24], [16, Proposition 19, p. 162], and compare [13, Theorem 14.73.17, p. 493])*; in particular nonplanar solenoids (i.e., solenoids different from a circle)* (see [16, Proposition 20, p. 163], and compare [13, p. 493]).

THEOREM 6. *Let a continuum X be hereditarily decomposable. Then $X \in CPH$ if and only if X is arc-like* (see [13, Theorem 14.73.19, p. 496]).

THEOREM 7. *Let X be a (metric) compactification of the half-open interval $[0,1)$. Then $X \in CP$ if and only if $C(X) = \mathrm{cl}_{C(X)} C([0,1))$ (i.e., if and only if $[0,1)$ approximates each subcontinuum of the remainder)* (see [6, Proposition 4, p. 387]; compare [9, Exercise 67.17, p. 325]).

B. Hughes asked the following question (see [13, Question 14.73.26, p. 501] and [9, Chapter 15, p. 458]).

QUESTION 8. What classes of mappings preserve the covering property?

In connection with this, recall the following. It is shown in [7, Theorem 3.9, p. 204] that *monotone* mappings (i.e., having connected point-inverses) preserve being in the Class(W). By the equivalence of (4.2) and (4.4) in Theorem 4, the next theorem follows (compare [9, Chapter 15, Comment to Question 14.73.26 of [13], p. 458]; see also [13, Chapter 14, Note 3, p. 511]).

THEOREM 9. *Monotone mappings preserve the covering property.*

It is very natural to consider the same for CPH.

QUESTION 10. What classes of mappings preserve the covering property hereditarily?

A partial answer is similar to the previous one.

THEOREM 11. *Monotone mappings preserve the covering property hereditarily.*

Proof. Let X be a continuum with $X \in CPH$, and let $f : X \to Y$ be a monotone surjection. Since X is hereditarily unicoherent according to Theorem 3, and since a continuum is hereditarily unicoherent if and only if each monotone mapping defined on X is hereditarily monotone, see [12, 6.10, p. 53], it follows that f is hereditarily monotone. Take a subcontinuum Q of Y, and let $K = f^{-1}(Q)$. Then K is a subcontinuum of X, thus $K \in CP$. Furthermore, the partial mapping $f|K : K \to f(K) = Q \subset Y$ is monotone, so $Q \in CP$ by Theorem 8. □

It is interesting to know whether or not the above results, namely Theorems 9 and 11, can be extended to other classes of mappings, especially such ones which contain the class of monotone mappings as a proper subclass.

One of such mappings is a confluent mapping. A mapping $f : X \to Y$ between continua is said to be *confluent* provided that for each continuum $Q \subset Y$ and for each component K of $f^{-1}(Q)$ we have $f(K) = Q$ (see e.g., [1, p. 213], [13, 0.45.3, p. 21] or [9, Definition 24.1, p. 207]). Each monotone mapping is obviously a confluent one; and confluent mappings contain open mappings as a subclass, see

[18, 7.5, p. 148]. However, a dyadic solenoid X which has the covering property hereditarily according to (5.3) of Theorem 5 can be mapped onto a circle under an open mapping, see [13, 14.73.26, p. 501] and compare [1, p. 218] for details. The circle does not have the covering property according to either Theorem 1 or Theorem 2. Thus we have the following proposition.

PROPOSITION 12. *Open mappings, and thus confluent ones, need not preserve covering property and covering property hereditarily for continua.*

Other classes of mappings which comprises monotone ones are the following. A mapping $f : X \to Y$ between continua X and Y is said to be:
— *almost monotone* provided that for each subcontinuum Q in Y with nonempty interior the inverse image $f^{-1}(Q)$ is connected;
— *quasi-monotone* provided that for each subcontinuum Q in Y with nonempty interior the inverse image $f^{-1}(Q)$ has a finite number of components and f maps each of them onto Q;
— *weakly monotone* provided that for each subcontinuum Q in Y with nonempty interior each component of the inverse image $f^{-1}(Q)$ is mapped under f onto Q;
— *feebly monotone* provided that if A and B are proper subcontinua of Y such that $Y = A \cup B$, then their inverse images $f^{-1}(A)$ and $f^{-1}(B)$ are connected.

Properties of quasi-monotone mappings are well-known (see e.g., Whyburn's book [18]). Also the concept of a weakly monotone mapping between continua is known for years and was studied by a number of authors. The reader is referred to Maćkowiak dissertation [12] for interrelations between these classes of mappings and their basic properties. Some properties related to almost monotone and feebly monotone mappings (without using these names) were considered by Maćkowiak in [12, proof of Theorem 4.44, p. 25]). Feebly monotone mappings were introduced by the author in [2, p. 210] and studied in [3].

The following diagram (see [3, Proposition 2.1, p. 16]) illustrates the relations between these mappings.

$$\begin{array}{ccccccc} \text{monotone} & \Rightarrow & \text{almost monotone} & \Rightarrow & \text{quasi-monotone} & \Rightarrow & \text{weakly monotone} \\ & & \Downarrow & & & & \\ & & \text{feebly monotone} & & & & \end{array}$$

To see that Theorems 9 and 11 cannot be extended from monotone to almost monotone mappings consider the following example (for other properties of the same example see [12, Example 7.4, p. 59]).

EXAMPLE 13. *There exist a continuum $X \in CPH$, a continuum $Y \notin CP$ and an almost monotone surjective mapping $f : X \to Y$.*

Proof. Let X be the $\sin(1/x)$-curve and let J denotes the limit segment of X. Then X is hereditarily decomposable and arc-like, and thus $X \in CPH$ by Theorem 6. Let a mapping $f : X \to Y$ identify the end points of J to a point $y_0 \in Y$. Note that f is almost monotone.

To see that Y does not have the covering property take an arc $A \subset f(J) \subset Y$ such that $y_0 \in A$ and y_0 is not an end point of A, and observe that the ray $Y \setminus f(J)$

does not contain any sequence of continua converging to the arc A. Therefore the ray does not approximate each subcontinuum of the remainder $f(J)$, and the conclusion $Y \notin CP$ is a consequence of Theorem 7. □

So, we have the next proposition.

PROPOSITION 14. *Almost monotone mappings (and hence quasi-monotone, weakly monotone and feebly monotone ones) need not preserve covering property and covering property hereditarily for continua.*

A mapping $f : X \to Y$ between continua is said to be:
— *simple* provided that $\operatorname{card} f^{-1}(y) \leqslant 2$ for each $y \in Y$;
— *light* provided that $f^{-1}(y)$ is totally disconnected for each $y \in Y$.

Thus each simple mapping is light, and a mapping is light if and only if it is 0-dimensional, i.e., $\dim f^{-1}(y) = 0$ for each $y \in Y$ (see e.g., [18, p. 130]).

Observe that the mapping f of Example 13 is simple, and thus light. Hence the next proposition follows.

PROPOSITION 15. *Simple mappings (and thus light ones) need not preserve covering property and covering property hereditarily for continua.*

Let $\mathcal{M}$ be a class of mappings between continua. A mapping $f : X \to Y$ between continua is said to be *hereditarily* $\mathcal{M}$ provided that for each continuum $K \subset X$ the partial mapping $f|K : K \to f(K) \subset Y$ belongs to $\mathcal{M}$ (see [11, p. 124]; compare also [12, Chapter 4, Part B, p. 16]). Hereditary classes of mappings were studied in [11], [12], [4], [14] and in many other papers quoted therein.

In a discussion on the subject of this paper W. J. Charatonik asked the following questions.

QUESTIONS 16. What *hereditary* classes of mappings preserve a) the covering property, b) the covering property hereditarily?

Since each hereditarily weakly confluent mapping preserves C^*-smoothness of continua, (see [5, Corollary 3.6]) it is especially interesting to know whether the same holds for absolutely C^*-smoothness (which is equivalent to the covering property according to Theorem 4). This can more precisely be formulated as follows.

QUESTION 17. Let a continuum X be absolutely C^*-smooth, and let a surjection $f : X \to Y$ be hereditarily weakly confluent. Is then the continuum Y absolutely C^*-smooth?

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