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THIRD ORDER EXTENDED REGULAR VARIATION

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Abstract. A theory of third order extended regular variation is developed, somewhat similar to the corresponding theories of first and second order.

1. Introduction

The theory of regularly varying functions initiated by J. Karamata (1930, 1933) has turned out to be very useful and in fact indispensable in the probabilistic and statistical theory of extreme values.

An extended form of regular variation serves as necessary and sufficient condition for the domain of attraction of extreme value distributions. The same conditions are sufficient for the consistency of the estimators of the extreme value index γ.

Second order extended regular variation has proven very useful (almost indispensable) for establishing asymptotic normality of estimators of γ, in particular for calculating the asymptotic variance and bias. In second order extended regular variation an extra parameter shows up that we call ρ. Then asymptotic bias depends on this parameter.

One way of reducing the (asymptotic) bias of an estimator is subtracting the estimated bias from the estimator. In order to do so one needs an estimator for the second order parameter ρ. Such estimators have been developed. They are generally consistent under second order extended regular variation. But for proving asymptotic normality of a ρ-estimator third order extended regular variation comes into play.

It has been proved (Fraga Alves, de Haan, and Lin, 2003) that indeed third order extended regular variation is sufficient for the asymptotic normality of ρ-estimators. The cited paper also gives a sketch of the third order theory but this sketch had to be so short that it is extremely difficult to understand.

This paper offers full proofs of the main results for third order extended regular variation.

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2. The results

A measurable function f is said to satisfy the extended regular variation property if there is a positive function a such that for $x > 0$,

$$\lim_{t\to\infty} \frac{f(tx)-f(t)}{a(t)} = \frac{x^\gamma - 1}{\gamma}, \tag{2.1}$$

where γ is a real parameter (Notation $f \in ERV$ or $f \in ERV_\gamma$).

The speed of convergence in this limit relation can be captured by a relation of second order. The measurable function f is said to satisfy the second extended regular variation property if there exist functions $a > 0$ and A, positive or negative, with $\lim_{t\to\infty} A(t) = 0$, such that for all $x > 0$

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{A(t)} \left(\frac{f(tx)-f(t)}{a(t)} - \frac{x^\gamma - 1}{\gamma} \right) &= \int_1^x y^{\gamma-1} \int_1^y u^{\rho-1} du\, dy \\ &= \frac{1}{\rho}\left(\frac{x^{\gamma+\rho}-1}{\gamma+\rho} - \frac{x^\gamma - 1}{\gamma} \right), \end{aligned} \tag{2.2}$$

where $\rho \leqslant 0$ is the second order parameter (Notation $f \in 2ERV$ or $f \in 2ERV_{\gamma,\rho}$).

Now the third order relation becomes obvious. The function f satisfies the third order extended regular variation property if it satisfies (2.2) and there exists a positive or negative function B, with $\lim_{t\to\infty} B(t) = 0$, such that for all $x > 0$

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{B(t)} \left[\frac{1}{A(t)} \left(\frac{f(tx)-f(t)}{a(t)} - \frac{x^\gamma - 1}{\gamma} \right) - \int_1^x y^{\gamma-1} \int_1^y u^{\rho-1} du\, dy \right] \\ = \int_1^x y^{\gamma-1} \int_1^y u^{\rho-1} \int_1^u s^{\eta-1} ds\, du\, dy, \end{aligned} \tag{2.3}$$

where $\eta \leqslant 0$ is the third order parameter (Notation $f \in 3ERV$ or $f \in 3ERV_{\gamma,\rho,\eta}$).

One of the results established in the Theorem is that only in trivial cases the limit function in (2.3) is not of the stated form.

THEOREM 2.1. *Write*

$$D_\gamma(x) := \int_1^x y^{\gamma-1} dy = \frac{x^\gamma - 1}{\gamma} \qquad \text{and}$$

$$H_{\gamma,\rho}(x) := \int_1^x y^{\gamma-1} \int_1^y u^{\rho-1} du\, dy = \frac{1}{\rho}\left(\frac{x^{\gamma+\rho}-1}{\gamma+\rho} - \frac{x^\gamma - 1}{\gamma} \right).$$

Suppose f is a measurable function and there exist functions a_0 (positive) and a_1 and a_2 (positive or negative) with $\lim_{t\to\infty} a_i(t) = 0$ for $i = 1, 2$ such that for all $x > 0$

$$\lim_{t\to\infty} \left\{ f(tx) - f(t) - a_0(t) D_\gamma(x) - a_1(t) H_{\gamma,\rho}(x) \right\} / a_2(t) =: R(x), \tag{2.4}$$

exists ($t \to \infty$), then for a judicious choice of a_0, a_1 and a_2 and given that D, H and R are not linearly dependent, the limit function R will be of the form

$$R_{\gamma,\rho,\eta}(x) := \int_1^x y^{\gamma-1} \int_1^y u^{\rho-1} \int_1^u s^{\eta-1} ds\, du\, dy, \tag{2.5}$$

with $\rho \leqslant 0$, $\eta \leqslant 0$. *Moreover,*

$$\lim_{t\to\infty} \frac{a_2(tx)}{a_2(t)} = x^{\gamma+\rho+\eta}, \tag{2.6}$$

$$\lim_{t\to\infty} \frac{a_1(tx) - a_1(t)x^{\gamma+\rho}}{a_2(t)} = x^{\gamma+\rho} \cdot \frac{x^\eta - 1}{\eta}, \tag{2.7}$$

$$\lim_{t\to\infty} \frac{a_0(tx) - a_0(t)x^\gamma - a_1(t)x^\gamma \cdot (x^\rho - 1)/\rho}{a_2(t)} = x^\gamma \cdot H_{\rho,\eta}(x) \tag{2.8}$$

and for each $\epsilon > 0$ *there exists* t_0 *such that for* $t \geqslant t_0$, $tx \geqslant t_0$

$$e^{-\epsilon|\log x|} x^{-\gamma-\rho-\eta} \left| \frac{f(tx) - f(t) - a_0(t)D_\gamma(x) - a_1(t)H_{\gamma,\rho}(x)}{a_2(t)} - R_{\gamma,\rho,\eta}(x) \right| \leqslant \epsilon. \tag{2.9}$$

Proof. We write for $t, x, y > 0$

$$f(txy) - f(t) = (f(txy) - f(tx)) + (f(tx) - f(t))$$

and we connect the three parts to (2.4) as follows

$$\begin{aligned} &\{f(txy) - f(t) - a_0(t)D_\gamma(xy) - a_1(t)H_{\gamma,\rho}(xy)\}/a_2(t) \\ &= \left[\{f(txy) - f(tx) - a_0(tx)D_\gamma(y) - a_1(tx)H_{\gamma,\rho}(y)\}/a_2(tx)\right] a_2(tx)/a_2(t) \\ &\quad + \{f(tx) - f(t) - a_0(t)D_\gamma(x) - a_1(t)H_{\gamma,\rho}(x)\}/a_2(t) \\ &\quad + \{-a_0(t)\left(D_\gamma(xy) - D_\gamma(x)\right) + a_0(tx)D_\gamma(y)\}/a_2(t) \\ &\quad + \{-a_1(t)\left(H_{\gamma,\rho}(xy) - H_{\gamma,\rho}(x)\right) + a_1(tx)H_{\gamma,\rho}(y)\}/a_2(t). \end{aligned} \tag{2.10}$$

Note that

$$-a_0(t)\left(D_\gamma(xy) - D_\gamma(x)\right) + a_0(tx)D_\gamma(y) = \left(a_0(tx) - x^\gamma a_0(t)\right) D_\gamma(y)$$

and

$$\begin{aligned} &- a_1(t)\left(H_{\gamma,\rho}(xy) - H_{\gamma,\rho}(x)\right) + a_1(tx)H_{\gamma,\rho}(y) \\ &\qquad = \left(a_1(tx) - x^{\gamma+\rho}a_1(t)\right) H_{\gamma,\rho}(y) - a_1(t)\, x^\gamma \frac{x^\rho - 1}{\rho} \frac{y^\gamma - 1}{\gamma} \end{aligned}$$

so that the sum of the last two terms of (2.10) is

$$\frac{a_1(tx) - a_1(t)x^{\gamma+\rho}}{a_2(t)} H_{\gamma,\rho}(y) + \frac{a_0(tx) - a_0(t)x^\gamma + a_1(t)x^\gamma(x^\rho - 1)/\rho}{a_2(t)} D_\gamma(y). \tag{2.11}$$

Let $t \to \infty$ in both sides of the equation (2.10) with the last two terms substituted by (2.11). We get for $x, y > 0$

$$\begin{aligned} R(xy) - R(x) = \lim_{t\to\infty} \Bigg[& R(y)(1 + o(1)) \frac{a_2(tx)}{a_2(t)} + H_{\gamma,\rho}(y) \frac{a_1(tx) - a_1(t) \cdot x^{\gamma+\rho}}{a_2(t)} \\ & + D_\gamma(y) \frac{a_0(tx) - a_0(t)x^\gamma - a_1(t)x^\gamma \cdot (x^\rho - 1)/\rho}{a_2(t)} \Bigg]. \end{aligned} \tag{2.12}$$

We have required that there do not exist constants c_2 and c_3 such that

$$R(y) \equiv c_2 H_{\gamma,\rho}(y) + c_3 D_\gamma(y).$$

Consequently, the set of vectors $\{(R(y), H(y), D(y))\}_{y>0}$ is not contained in a plane, hence there are y_1, y_2, y_3 such that the matrix

$$\begin{pmatrix} R(y_1) & H(y_1) & D(y_1) \\ R(y_2) & H(y_2) & D(y_2) \\ R(y_3) & H(y_3) & D(y_3) \end{pmatrix}$$

has rank 3. Then also the transposed matrix has rank 3. So there are no z_1, z_2, z_3, not all of them zero, such that $\sum_i z_i R(y_i) = \sum_i z_i H(y_i) = \sum_i z_i D(y_i) = 0$.

Now take z_1, z_2, z_3 such that

$$\sum_i z_i H(y_i) = \sum_i z_i D(y_i) = 0; \tag{2.13}$$

then we must have $\sum_i z_i R(y_i) \neq 0$.

Now multiply the two sides of (2.12) by z_i, $i = 1, 2, 3$, and add the three equations. The resulting equation (use (2.13)) shows that $\lim_{t\to\infty} a_2(tx)/a_2(t)$ exists for $x > 0$. The limit could be zero (but that is not possible here) or else a power of x, say $x^{\gamma+\rho+\eta}$. This is the definition of η. Similarly (repeating the above reasoning for this case)

$$\lim_{t\to\infty} \frac{a_1(tx) - a_1(t) \cdot x^{\gamma+\rho}}{a_2(t)} \tag{2.14}$$

and

$$\lim_{t\to\infty} \frac{a_0(tx) - a_0(t) \cdot x^{\gamma} - a_1(t) \cdot x^{\gamma} \cdot (x^{\rho} - 1)/\rho}{a_2(t)} \tag{2.15}$$

must exist. The limits in (2.14) and (2.15) must be [de Haan and Stadtmüller 1996, Theorem 1]

$$c_1 x^{\gamma+\rho} \cdot (x^{\eta} - 1)/\eta \quad \text{and} \quad x^{\gamma}\left(c_1 H_{\rho,\eta}(x) + c_2 D_{\rho}(x)\right).$$

By replacing a_2 with $c_1 a_2$ and a_1 with $a_1 + c_2 a_2$ in (2.4) the limits become $x^{\gamma+\rho} \cdot (x^{\eta} - 1)/\eta$ and $x^{\gamma} H_{\rho,\eta}(x)$, respectively.

Now (2.12) leads to the functional equation $(x, y > 0)$

$$x^{\gamma+\rho+\eta} R(y) = R(xy) - R(x) - x^{\gamma} \cdot H_{\rho,\eta}(x) \frac{y^{\gamma} - 1}{\gamma} - x^{\gamma+\rho} \cdot H_{\gamma,\rho}(y) \cdot \frac{x^{\eta} - 1}{\eta}. \tag{2.16}$$

The reader may want to check that the function $R_{\gamma,\rho,\eta}$ is a solution of (2.16). Let R' be any other solution and define $V := R_{\gamma,\rho,\eta} - R'$. Then V satisfies the equation

$$x^{\gamma+\rho+\eta} V(y) = V(xy) - V(x),$$

hence (as in de Haan 1970, section 1.4),

$$V(x) = c \cdot \frac{x^{\gamma+\rho+\eta} - 1}{\gamma + \rho + \eta}.$$

Next we get rid of V. Suppose the limit in (2.4) is $R' := R_{\gamma,\rho,\eta} + V$. Note that by changing the functions a_0, a_1 and a_2 in (2.4) into

$$\begin{cases} \tilde{a}_2 := c_2^{-1} a_2 \\ \tilde{a}_1 := a_1 + c_1 \tilde{a}_2 \\ \tilde{a}_0 := a_0 + c_0 \tilde{a}_2 \end{cases}, \tag{2.17}$$

the limit function R changes into $c_2 R - c_0 D_\gamma - c_1 H_{\gamma,\rho}$. Also note that if both $\eta \neq 0$ and $\eta + \rho \neq 0$,

$$R_{\gamma,\rho,\eta}(x) = \frac{1}{\eta(\eta+\rho)} \frac{x^{\gamma+\rho+\eta} - 1}{\gamma+\rho+\eta} - \frac{1}{\eta} H_{\gamma,\rho}(x) - \frac{1}{\eta(\eta+\rho)} D_\gamma(x),$$

i.e.,

$$\frac{x^{\gamma+\rho+\eta} - 1}{\gamma+\rho+\eta} = \eta(\eta+\rho) \left\{ R_{\gamma,\rho,\eta}(x) + \frac{1}{\eta} H_{\gamma,\rho}(x) + \frac{1}{\eta(\eta+\rho)} D_\gamma(x) \right\}.$$

If $\eta = 0$, then

$$\frac{x^{\gamma+\rho+\eta} - 1}{\gamma+\rho+\eta} = \rho H_{\gamma,\rho}(x) + D_\gamma(x)$$

and if $\eta + \rho = 0$, then

$$\frac{x^{\gamma+\rho+\eta} - 1}{\gamma+\rho+\eta} = D_\gamma(x)$$

Hence by the device (2.17) we can make sure that the limit function in (2.4) is exactly $R_{\gamma,\rho,\eta}$.

In order to prove (2.9) we distinguish various cases.

1. $\gamma = \rho = \eta = 0$. The proof is similar to Omey and Willekens (1988). For $x, y > 0$

$$\begin{aligned} f(txy) &- f(ty) - f(tx) - f(t) - a_1(t)(\log x)(\log y) \\ &= f(txy) - f(t) - a_0(t)\log(xy) - a_1(t)\tfrac{1}{2}\left(\log(xy)\right)^2 \\ &\quad - \left\{ f(ty) - f(t) - a_0(t)\log y - a_1(t)\tfrac{1}{2}\left(\log(y)\right)^2 \right\} \\ &\quad - \left\{ f(tx) - f(t) - a_0(t)\log x - a_1(t)\tfrac{1}{2}\left(\log(x)\right)^2 \right\}. \end{aligned}$$

Hence by (2.4) and (2.5) for the function $g_x(t) := f(tx) - f(t)$, we have

$$\begin{aligned} \lim_{t\to\infty} \frac{g_x(ty) - g_x(t) - a_1(t)\log x \, \log y}{a_0(t)} &= \frac{1}{6}\left(\log(xy)\right)^3 - \frac{1}{6}\left(\log y\right)^3 - \frac{1}{6}\left(\log x\right)^3 \\ &= \frac{1}{2}\left(\log x\right)^2 \log y + \frac{1}{2}\left(\log y\right)^2 \log x \end{aligned}$$

for all $x, y > 0$, i.e.,

$$\lim_{t\to\infty} \frac{g_x(ty) - g_x(t) - \left\{ a_1(t)(\log x) - a_0(t)\frac{1}{2}\left(\log x\right)^2 \right\} \log y}{a_0(t)\log x} = \frac{1}{2}\left(\log y\right)^2.$$

Hence g_x is second order Π-varying for each $x > 0$. It follows from Omey and Willekens (1988), that the function $h_x(t) := g_x(t) - \frac{1}{t}\int_0^t g_x(s)\,ds$ is in the class Π with auxiliary function $a_0(t)\log x$, i.e., for $y > 0$

$$\lim_{t\to\infty} \frac{h_x(ty) - h_x(t)}{a_0(t)\log x} = \log y.$$

The theory of Π-variation applies, hence by Geluk and de Haan (1987)

$$\lim_{t\to\infty} \frac{1}{a_0(t)\log x}\left(h_x(t) - \frac{1}{t}\int_0^t h_x(s)ds\right) = 1. \tag{2.18}$$

Now note that

$$h_x(t) = h^{(1)}(tx) - h^{(1)}(t) \quad \text{with} \quad h^{(1)}(t) := f(t) - \frac{1}{t}\int_0^t f(u)\,du.$$

Consequently, it follows from (2.18) that

$$\lim_{t\to\infty} \frac{1}{a_0(t)\log x}\left(h^{(1)}(tx) - h^{(1)}(t) - \frac{1}{tx}\int_0^{tx} h^{(1)}(s)\,ds + \frac{1}{t}\int_0^t h^{(1)}(s)\,ds\right) = 1,$$

i.e.,

$$\lim_{t\to\infty} \frac{h^{(2)}(tx) - h^{(2)}(t)}{a_0(t)\log x} = 1 \tag{2.19}$$

with

$$h^{(2)}(t) := h^{(1)}(t) - \frac{1}{t}\int_0^t h^{(1)}(s)ds.$$

Again by the theory of Π-variation, relation (2.19) implies

$$\lim_{t\to\infty} \frac{h^{(2)}(tx) - h^{(2)}(t)}{h^{(3)}(t)} = \log x$$

with

$$h^{(3)}(t) := h^{(2)}(t) - \frac{1}{t}\int_0^t h^{(2)}(s)ds.$$

Then by Drees' inequalities (1998), for any $\epsilon_1, \epsilon_2 > 0$, there exists $t_0 = t_0(\epsilon_1, \epsilon_2)$ such that for $t \geqslant t_0, tx \geqslant t_0$

$$e^{-\epsilon_2|\log x|}\left|\frac{h^{(2)}(tx) - h^{(2)}(t)}{h^{(3)}(t)} - \log x\right| < \epsilon_1. \tag{2.20}$$

It is easily checked using the definition of $h^{(2)}$ that

$$h^{(1)}(t) = h^{(2)}(t) + \int_0^t h^{(2)}(s)\frac{ds}{s}$$

hence

$$h^{(1)}(tx) - h^{(1)}(t) = h^{(2)}(tx) - h^{(2)}(t) + \int_1^x h^{(2)}(ts)\frac{ds}{s},$$

i.e.,

(2.21)
$$h^{(1)}(tx) - h^{(1)}(t) - h^{(2)}(t)\log x = h^{(2)}(tx) - h^{(2)}(t) + \int_1^x \left(h^{(2)}(ts) - h^{(2)}(t)\right)\frac{ds}{s}$$

and consequently

$$\frac{h^{(1)}(tx) - h^{(1)}(t) - \left(h^{(2)}(t) + h^{(3)}(t)\right)\log x}{h^{(3)}(t)} - \frac{(\log x)^2}{2} = \frac{h^{(2)}(tx) - h^{(2)}(t)}{h^{(3)}(t)} - \log x + \int_1^x \left(\frac{h^{(2)}(ts) - h^{(2)}(t)}{h^{(3)}(t)} - \log s\right)\frac{ds}{s}.$$

Integrating (2.21) with respect to x we get

$$\int_1^x \left(h^{(1)}(ts) - h^{(1)}(t)\right)\frac{ds}{s} - h^{(2)}(t)\int_1^x (\log s)\frac{ds}{s} = \int_1^x \left(h^{(2)}(ts) - h^{(2)}(t)\right)\frac{ds}{s} + \int_1^x \int_1^u \left(h^{(2)}(ts) - h^{(2)}(t)\right)\frac{ds}{s}\frac{du}{u}. \tag{2.22}$$

Also we have the following obvious analogue of (2.21):

(2.23)
$$f(tx) - f(t) - h^{(1)}(t)\log x = \left(h^{(1)}(tx) - h^{(1)}(t)\right) + \int_1^x \left(h^{(1)}(ts) - h^{(1)}(t)\right)\frac{ds}{s}.$$

Combining (2.22) and (2.23) we get

$$\begin{aligned} f(tx) - f(t) &= h^{(1)}(t)\log x + h^{(2)}(t)(\log x)^2/2 \\ &\quad + \int_1^x \int_1^u \left(h^{(2)}(ts) - h^{(2)}(t)\right)\frac{ds}{s}\frac{du}{u} + \int_1^x \left(h^{(2)}(ts) - h^{(2)}(t)\right)\frac{ds}{s} \\ &\quad + \left(h^{(1)}(tx) - h^{(1)}(t)\right), \end{aligned}$$

which by (2.21) equals

$$\begin{aligned} &\left(h^{(1)}(t) + h^{(2)}(t)\right)\log x + h^{(2)}(t)(\log x)^2/2 + \int_1^x \int_1^u \left(h^{(2)}(ts) - h^{(2)}(t)\right)\frac{ds}{s}\frac{du}{u} \\ &\qquad + 2\int_1^x \left(h^{(2)}(ts) - h^{(2)}(t)\right)\frac{ds}{s} + h^{(2)}(tx) - h^{(2)}(t). \end{aligned}$$

Hence

$$\begin{aligned} &\frac{f(tx) - f(t) - \left(h^{(1)}(t) + h^{(2)}(t) + h^{(3)}(t)\right)\log x - \left(h^{(2)}(t) + 2h^{(3)}(t)\right)(\log x)^2/2}{h^{(3)}(t)} \\ &- \frac{(\log x)^3}{6} = \int_1^x \int_1^u \left(\frac{h^{(2)}(ts) - h^{(2)}(t)}{h^{(3)}(t)} - \log s\right)\frac{ds}{s}\frac{du}{u} \\ &\qquad + 2\int_1^x \left(\frac{h^{(2)}(ts) - h^{(2)}(t)}{h^{(3)}(t)} - \log s\right)\frac{ds}{s} + \frac{h^{(2)}(tx) - h^{(2)}(t)}{h^{(3)}(t)} - \log x. \end{aligned}$$

Hence by (2.20) for any $\epsilon_1, \epsilon_2 > 0$, $t \geqslant t_0(\epsilon_1, \epsilon_2)$, $tx \geqslant t_0(\epsilon_1, \epsilon_2)$

$$\left| \frac{f(tx) - f(t) - \left(h^{(1)}(t) + h^{(2)}(t) + h^{(3)}(t)\right) \log x - \left(h^{(2)}(t) + 2h^{(3)}(t)\right) (\log x)^2/2}{h^{(3)}(t)} - \frac{(\log x)^3}{6} \right|$$
$$< \epsilon_1 \int_1^x \int_1^u e^{\epsilon_2 |\log s|} \frac{ds}{s} \frac{du}{u} + 2\epsilon_1 \int_1^x e^{\epsilon_2 |\log s|} \frac{ds}{s} + \epsilon_1 e^{\epsilon_2 |\log x|}$$
$$< \epsilon_1 \left(1 + \frac{1}{\epsilon_2} + \frac{1}{\epsilon_2^2}\right) e^{\epsilon_2 |\log x|}.$$

2. $\eta < 0$: By Theorem 1.10 of Geluk and de Haan (1987) relation (2.7), that is,

$$\lim_{t\to\infty} \frac{(tx)^{-(\gamma+\rho)} a_1(tx) - t^{-(\gamma+\rho)} a_1(t)}{t^{-(\gamma+\rho)} a_2(t)} = \frac{x^\eta - 1}{\eta}$$

with $\eta < 0$, implies that for some constant $c > 0$

$$c - t^{-(\gamma+\rho)} a_1(t) \sim -\eta^{-1} t^{-(\gamma+\rho)} a_2(t),$$

that is, $a_1(t) = ct^{\gamma+\rho} + \eta^{-1} a_2(t)(1 + o(1))$ as $t \to \infty$. Hence we can replace a_1 in (2.8) with a_1^* defined by $a_1^*(t) := ct^{\gamma+\rho} + \eta^{-1} a_2(t)$.

By inserting this choice in (2.8) and rearranging we get for $x > 0$

$$\lim_{t\to\infty} \frac{a_0(tx) - a_0(t)x^\gamma - ct^{\gamma+\rho} x^\gamma \cdot (x^\rho - 1)/\rho}{a_2(t)} = x^\gamma \cdot H_{\rho,\eta}(x) - \eta^{-1} x^\gamma \cdot \frac{x^\rho - 1}{\rho}$$

which is the same as

$$\lim_{t\to\infty} \frac{(tx)^{-\gamma} a_0(tx) - t^{-\gamma} a_0(t) - ct^\rho \cdot (x^\rho - 1)/\rho}{t^{-\gamma} a_2(t)} = \frac{1}{\eta} \cdot \frac{x^{\rho+\eta} - 1}{\rho + \eta}.$$

We can rewrite this as

$$\lim_{t\to\infty} \frac{\left((tx)^{-\gamma} a_0(tx) - c((tx)^\rho - 1)/\rho\right) - \left(t^{-\gamma} a_0(t) - c(t^\rho - 1)/\rho\right)}{t^{-\gamma} a_2(t)} = \frac{1}{\eta} \cdot \frac{x^{\rho+\eta} - 1}{\rho + \eta},$$

where $\rho + \eta < 0$. Again by Theorem 1.10 of Geluk and de Haan (1987) this implies that for some constant c_1,

$$c_1 - t^{-\gamma} a_0(t) - c\frac{t^\rho}{\rho} \sim \frac{1}{\eta(\rho + \eta)} t^{-\gamma} a_2(t), \quad t \to \infty,$$

hence in (2.4) we can replace a_0 with a_0^* defined by

$$a_0^*(t) := c_1 t^\gamma + \frac{c}{\rho} t^{\gamma+\rho} - \frac{1}{\eta(\rho + \eta)} a_2(t).$$

Inserting the special choices a_1^* and a_0^* in (2.4) we get

$$\begin{aligned}(2.24)\qquad &\lim_{t\to\infty}\frac{1}{a_2(t)}\left[f(tx)-c_1\frac{(tx)^\gamma-1}{\gamma}-\frac{c}{\eta}\frac{(tx)^{\gamma+\rho}-1}{\gamma+\rho}\right.\\ &\qquad\left.-\left(f(t)-c_1\frac{t^\gamma-1}{\gamma}-\frac{c}{\eta}\frac{t^{\gamma+\rho}-1}{\gamma+\rho}\right)-\frac{c}{\eta}\frac{t^{\gamma+\rho}}{\eta}\frac{x^\gamma-1}{\gamma}\right]\\ &=R_{\gamma,\rho,\eta}(x)+\frac{1}{\eta}H_{\gamma,\rho}(x)-\frac{1}{\eta(\rho+\eta)}\frac{x^\gamma-1}{\gamma}\\ &=\frac{1}{\eta}\left(H_{\gamma,\rho+\eta}(x)-\frac{1}{\rho+\eta}\frac{x^\gamma-1}{\gamma}\right)=\frac{1}{\eta}\frac{x^{\gamma+\rho+\eta}-1}{\gamma+\rho+\eta}.\end{aligned}$$

Relation (2.24) is a *second order* relation for which uniform bounds are well known (Drees, 1998). Hence for some functions $\hat{a}_1$ and $\hat{a}_2$

$$\begin{aligned}e^{-\epsilon|\log x|}x^{-\gamma-\rho-\eta}&\left|\left[\left(f(tx)-c_1\frac{(tx)^\gamma-1}{\gamma}-\frac{c}{\eta}\frac{(tx)^{\gamma+\rho}-1}{\gamma+\rho}\right)\right.\right.\\ &\left.\left.-\left(f(t)-c_1\frac{t^\gamma-1}{\gamma}-\frac{c}{\eta}\frac{t^{\gamma+\rho}-1}{\gamma+\rho}\right)-\hat{a}_1(t)\frac{x^\gamma-1}{\gamma}\right]\frac{1}{\hat{a}_2(t)}\right|\leqslant\epsilon\end{aligned}$$

provided $t\geqslant t_0(\epsilon)$ and $tx\geqslant t_0(\epsilon)$. This gives (2.9) for $\eta<0$.

3. $\rho<0$, $\eta=0$. We start from (2.8), i.e.,

$$\lim_{t\to\infty}\frac{(tx)^{-\gamma}a_0(tx)-t^{-\gamma}a_0(t)-t^{-\gamma}a_1(t)\cdot(x^\rho-1)/\rho}{t^{-\gamma}a_2(t)}=H_{\rho,0}(x).$$

This relation is discussed in de Haan and Stadtmüller (1996), Theorem 2,(ii). Relation (2.11) of that paper gives

$$\lim_{t\to\infty}\frac{\left\{(tx)^{-\gamma}a_0(tx)-\frac{1}{\rho}(tx)^{-\gamma}a_1(tx)\right\}-\left\{t^{-\gamma}a_0(t)-\frac{1}{\rho}t^{-\gamma}a_1(t)\right\}}{t^{-\gamma}a_2(t)}=-\frac{1}{\rho}\cdot\frac{x^\rho-1}{\rho}.$$

The latter relation is discussed in Proposition 2(ii) of the same paper. It follows that for some $c>0$

$$\lim_{t\to\infty}\frac{t^{-\gamma}a_0(t)-\frac{1}{\rho}t^{-\gamma}a_1(t)-c}{t^{-\gamma}a_2(t)}=-\frac{1}{\rho^2},$$

i.e., relation (2.4) holds with a_1 replaced by a_1^* defined by

$$a_1^*(t):=-c\rho t^\gamma+\rho a_0(t)+\frac{1}{\rho}a_2(t).$$

After some rearranging we get

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{a_2(t)}&\left[\left(f(tx)-c\frac{(tx)^\gamma-1}{\gamma}\right)-\left(f(t)-c\frac{t^\gamma-1}{\gamma}\right)-(ct^\gamma-a_0(t))\frac{x^{\gamma+\rho}-1}{\gamma+\rho}\right]\\ &=R_{\gamma,\rho,0}(x)+\frac{1}{\rho}H_{\gamma,\rho}(x)=\frac{1}{\rho}H_{\gamma+\rho,0}(x).\end{aligned}$$

This is a *second order* relation for which uniform bounds are well known (Drees, 1998). This gives (2.9) for $\eta=0$, $\rho<0$.

4. In the case $\gamma \neq 0$, $\rho = \eta = 0$ we have, as $t \to \infty$:

$$\frac{f(tx)-f(t)-a_0(t)(x^\gamma-1)/\gamma-a_1(t)H_{\gamma,0}(x)}{a_2(t)} \to R_{\gamma,0,0}(x) = \frac{x^\gamma \log^2 x}{2\gamma} - \frac{1}{\gamma}H_{\gamma,0}(x).$$

Hence

$$\frac{f(tx) - f(t) - a_0(t)(x^\gamma - 1)/\gamma - \big(a_1(t) - a_2(t)/\gamma\big)H_{\gamma,0}(x)}{a_2(t)} \to \frac{x^\gamma \log^2 x}{2\gamma}.$$

Note that

$$H_{\gamma,0}(x) = \frac{x^\gamma \log x}{\gamma} - \frac{x^\gamma - 1}{\gamma^2}.$$

We have

$$\frac{1}{a_2(t)}\left[f(tx)-f(t)-\Big(a_0(t)-\frac{1}{\gamma}a_1(t)+\frac{1}{\gamma^2}a_2(t)\Big)\frac{x^\gamma-1}{\gamma}-\Big(a_1(t)-\frac{1}{\gamma}a_2(t)\Big)\frac{x^\gamma \log x}{\gamma}\right] \to \frac{x^\gamma \log^2 x}{2\gamma}.$$

Define

$$\hat{a}_0(t) := a_0(t) - \frac{1}{\gamma}a_1(t) + \frac{1}{\gamma^2}a_2(t), \quad \hat{a}_1(t) := a_1(t) - \frac{1}{\gamma}a_2(t).$$

Then

$$\frac{1}{a_2(t)}\Big[f(tx) - f(t) - \hat{a}_0(t)\frac{x^\gamma - 1}{\gamma} - \hat{a}_1(t)\frac{x^\gamma \log x}{\gamma}\Big] \to \frac{x^\gamma \log^2 x}{2\gamma},$$

which can be rewritten as

$$\frac{\big(f(tx) - \gamma^{-1}\hat{a}_0(tx)\big) - \big(f(t) - \gamma^{-1}\hat{a}_0(t)\big)}{a_2(t)} + \frac{\hat{a}_0(tx) - \hat{a}_0(t)x^\gamma - \hat{a}_1(t)x^\gamma \log x}{\gamma a_2(t)} \to \frac{x^\gamma \log^2 x}{2\gamma}. \tag{2.25}$$

Relations (2.6), (2.7) and (2.8) imply

$$\frac{\hat{a}_0(tx) - \hat{a}_0(t)x^\gamma - \hat{a}_1(t)x^\gamma \log x}{a_2(t)} \to \frac{x^\gamma \log^2 x}{2}. \tag{2.26}$$

From (2.25) and (2.26) we have

$$\frac{(f(tx) - \gamma^{-1}\hat{a}_0(tx)) - (f(t) - \gamma^{-1}\hat{a}_0(t))}{a_2(t)} \to 0. \tag{2.27}$$

• If $\gamma < 0$, by Theorem 3.1.10 of Bingham, Goldie and Teugels (1987, due to Ash, Erdös and Rubel, 1974), the limit

$$c := \lim_{t\to\infty} f(t) - \gamma^{-1}\hat{a}_0(t)$$

exists, finite, and

$$\lim_{t\to\infty} \frac{c - f(t) + \gamma^{-1}\hat{a}_0(t)}{a_2(t)} = 0,$$

i.e.,

$$\hat{a}_0(t) = \gamma\,(f(t) - c) + o(a_2(t)),$$

as $t \to \infty$. Combining this with (2.26) gives

$$\frac{\gamma f(tx) - \gamma f(t) x^\gamma - \hat{a}_1(t) x^\gamma \log x}{a_2(t)} \to \frac{x^\gamma \log^2 x}{2}, \tag{2.28}$$

i.e.,

$$\frac{(tx)^{-\gamma} f(tx) - t^{-\gamma} f(t) - \hat{a}_1(t) x^\gamma (\log x)/\gamma}{a_2(t)/\gamma} \to \frac{\log^2 x}{2}.$$

Note that this is just a second order condition for which uniform inequalities have been derived in Drees (1998). They can be rewritten as follows: one can find functions $\hat{a}_1^*$ and $\hat{a}_2^*$ such that for any $\epsilon > 0$, $t, tx \geqslant t_0(\epsilon)$,

$$(2.29)\quad x^{-\gamma} \exp\{-|\log x|\epsilon\}$$
$$\times \left| \frac{1}{\hat{a}_2^*(t)} \left[\gamma f(tx) - \gamma f(t) - \gamma^2 f(t) \frac{x^\gamma - 1}{\gamma} - \hat{a}_1^*(t) x^\gamma \log x \right] - \frac{x^\gamma \log^2 x}{2} \right| < \epsilon.$$

- For $\gamma > 0$, Theorem 3.1.12 of of Bingham, Goldie and Teugels (1987, due to Bojanic and Karamata, 1963), applied to (2.27) implies

$$\frac{f(t) - \gamma^{-1} \hat{a}_0(t)}{a_2(t)} \to 0,$$

i.e.,

$$\hat{a}_0(t) = \gamma f(t) + o(a_2(t)),$$

as $t \to \infty$. Combining this with (2.26) gives (2.28) once again and hence (2.29) is obtained. □

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