PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 89(103) (2011), 49–56 DOI: 10.2298/PIM1103049I

HEREDITARILY INDECOMPOSABLE HAUSDORFF CONTINUA HAVE UNIQUE HYPERSPACES 2^X AND $C_n(X)$

Alejandro Illanes

Communicated by Miloš Kurilić

Abstract. Let X be a Hausdorff continuum (a compact connected Hausdorff space). Let 2^X (respectively, $C_n(X)$) denote the hyperspace of nonempty closed subsets of X (respectively, nonempty closed subsets of X with at most n components), with the Vietoris topology. We prove that if X is hereditarily indecomposable, Y is a Hausdorff continuum and 2^X (respectively $C_n(X)$) is homeomorphic to 2^Y (respectively, $C_n(Y)$), then X is homeomorphic to Y.

1. Introduction

A *Hausdorff continuum* is a compact connected Hausdorff space X with more than one point. A *subcontinuum* A of X is a closed connected subset of X.

For a Hausdorff continuum X and a positive integer n, define the hyperspaces

$$\begin{aligned} 2^X &= \{A \subset X : A \text{ is closed and nonempty}\}, \\ C_n(X) &= \{A \in 2^X : A \text{ has at most } n \text{ components}\}, \\ C(X) &= C_1(X), \\ F_n(X) &= \{A \in 2^X : A \text{ has at most } n \text{ points}\}. \end{aligned}$$

The hyperspace 2^X is endowed with the Vietoris topology. That is, the basis for the topology of 2^X is the family $\mathcal{B} = \{\langle U_1, \dots, U_n\rangle : n$ is a positive integer and $U_1, \dots, U_n$ are open subsets of $X\}$, where $\langle U_1, \dots, U_n\rangle = \{A \in 2^X : A \subset U_1 \cup \dots \cup U_n$ and $A \cap U_i \neq \emptyset$, for each $i \in \{1, \dots, n\}\}$.

The Hausdorff continuum X is *hereditarily indecomposable* provided that given $A, B \in C(X)$, either $A \cap B = \emptyset$ or $A \subset B$ or $B \subset A$ and X is said to *have unique hyperspace* 2^X (resp., $C_n(X)$), provided that, if Y is a Hausdorff continuum and

2010 *Mathematics Subject Classification:* Primary 54B20.
Key words and phrases: generalized arc, Hausdorff continuum, hereditarily indecomposable, hyperspace, unique hyperspace, Vietoris topology.

2^X is homeomorphic to 2^Y (resp., $C_n(X)$ is homeomorphic to $C_n(Y)$), then X is homeomorphic to Y.

Uniqueness of hyperspaces has been widely studied (see [**2**], [**3**, p. 285] and [**4**]). Macías proved in [**7**] and [**8**, Theorem 6.1] that if X is a metric hereditarily indecomposable Hausdorff continuum, then X has unique hyperspaces 2^X and $C_n(X)$ (for each n). Lončar [**6**, Theorem 2.4] proved that rim metrizable hereditarily indecomposable continua have unique hyperspace $C(X)$. Using techniques of inverse limits, Peláez generalized these results by proving that if X is a rim metrizable (X has a basis of neighborhoods with metrizable boundary) hereditarily indecomposable Hausdorff continuum, then X has unique hyperspaces 2^X and $C_n(X)$ (for each n).

In this paper we generalize these results by proving:

THEOREM 1.1. *If X is a hereditarily indecomposable Hausdorff continuum, then X has unique hyperspaces 2^X and $C_n(X)$ (for each n).*

2. Generalized arcs

The proof of Theorem 1.1 is based in an analysis of the generalized arcs (see below) in the hyperspace 2^X.

A *generalized arc joining p and q* in a topological space Z is a subcontinuum α of Z such that $p, q \in \alpha$ and each point $z \in \alpha - \{p, q\}$, separates p and q in α. Given $z \in \alpha - \{p, q\}$, let $\alpha - \{z\} = U_z \cup V_z$, where U_z and V_z are disjoint open subsets of α such that $p \in U_z$ and $p \in V_z$. Let $U_p = \emptyset = V_q$, $V_p = \alpha - \{p\}$ and $U_q = \alpha - \{q\}$. If $z, w \in \alpha$ and $z \neq w$, define $z < w$ if and only if $z \in U_w$.

The following lemma summarizes the basic facts about generalized arcs. For the proof of (b), see Theorem 6.16 of [**10**]. The rest of Lemma 2.1 is easy to prove.

LEMMA 2.1. *Let α be a generalized arc joining points p and q in a topological space Z. Then:*

(a) *the relation $<$ is a well defined linear order,*
(b) *α has the topology induced by $<$,*
(c) *given $z, w \in \alpha$, the intervals $[z, w), (z, w), (z, w]$ and $[z, w]$, defined in the natural way, are connected subsets of α,*
(d) *$(\alpha, <)$ has the property of the supremum (every nonempty subset of α has a supremum in α),*
(e) *the relation $\backsim$ defined in Z by $z \backsim w$ if an only if there exists a generalized arc joining z and w in Z, is an equivalence relation; the equivalence classes are called* g-arcwise components.

Let X be a Hausdorff continuum. Given $A, B \in 2^X$, with $A \subsetneq B$, an *order arc in 2^X, from A to B* is a subcontinuum α of 2^X such that $A \subset C \subset B$ for each $C \in \alpha$ and, for every $C, D \in \alpha$, either $C \subset D$ or $D \subset C$. The fundamental theorem for order arcs in hyperspaces is the following.

THEOREM 2.2. [**5**, Theorem 15.3] *Let X be a Hausdorff continuum and let $A, B \in 2^X$ be such that $A \subsetneq B$. Then there exists an order arc in 2^X, from A to B if and only if each component of B intersects A.*

LEMMA 2.3. *Let X be a Hausdorff continuum, $A, B \in 2^X$ and let α be and order arc in 2^X, from A to B. Then α is a generalized arc joining A and B in 2^X.*

PROOF. It is enough to show that for each $C \in \alpha - \{A, B\}$, C separates A and B in α. Let $\mathcal{A} = \{D \in \alpha : D \subset C\}$ and $\mathcal{B} = \{D \in \alpha : C \subset D\}$. It is easy to show that $\mathcal{A}$ and $\mathcal{B}$ are closed subsets of α. By the definition of order arc, we have $\alpha = \mathcal{A} \cup \mathcal{B}$, $A \in \mathcal{A} - \{C\}$, $B \in \mathcal{B} - \{C\}$ and $\mathcal{A} \cap \mathcal{B} = \{C\}$. This proves that C separates A and B in α. □

3. Property of Kelley

A Hausdorff continuum X is said to have the *Property of Kelley* (see [**1**, p. 115]), provided that for every $p \in X$, $A \in C(X)$ and open subset $\mathcal{U}$ of $C(X)$ such that $p \in A \in \mathcal{U}$, there exists an open subset T of X such that $p \in T$ and, for each $q \in T$, there exists $B \in C(X)$ such that $q \in B \in \mathcal{U}$.

For metric Hausdorff continua, property of Kelley has been widely studied, see ([**5**, pp. 167 and 405] and [**9**, Chapter XIV]). In [**9**, Theorem 16.27] it is proved that hereditarily indecomposable Hausdorff metric continua have property of Kelley. Next, we extend this result to Hausdorff continua.

THEOREM 3.1. *Let X be a hereditarily indecomposable Hausdorff continuum. Then X has property of Kelley.*

PROOF. Suppose that X does not have the property of Kelley. Then there exist $p \in X$, $A \in C(X)$ and $\mathcal{V} = \langle V_1, \dots, V_m \rangle \cap C(X)$, a basic set in $C(X)$, such that $p \in A \in \mathcal{V}$ and for each open subset T of X such that $p \in T$ there exists a point $q \in T$ for which there is no element $B \in C(X)$ with the property that $q \in B \in \mathcal{V}$. Note that $A \neq X$.

Let W be an open subset of X such that $A \subset W \subset \mathrm{cl}_X(W) \subset V_1 \cup \dots \cup V_m$ and $W \neq X$. Let $\mathcal{T} = \{T \subset X : T$ is an open subset of X such that $p \in T \subset W\}$. For each $T \in \mathcal{T}$, choose a point $q_T \in T$ for which there is no element $B \in C(X)$ with the property that $q_T \in B \in \mathcal{V}$ and let D_T be the component of $\mathrm{cl}_X(W)$ such that $q_T \in D_T$. Since $D_T \subset V_1 \cup \dots \cup V_m$ and $D_T \notin \mathcal{V}$, there exists $i \in \{1, \dots, m\}$ such that $D_T \cap V_i = \emptyset$. Let $E = \{q_T \in X : T \in \mathcal{T}\}$. Then $p \in \mathrm{cl}_X(E)$. For each $i \in \{1, \dots, m\}$, let $\mathcal{T}_i = \{T \in \mathcal{T} : D_T \cap V_i = \emptyset\}$ and $E_i = \{q_T \in X : T \in \mathcal{T}_i\}$. Then $\mathcal{T} = \mathcal{T}_1 \cup \dots \cup \mathcal{T}_m$ and $E = E_1 \cup \dots \cup E_m$. Thus, there exists $i \in \{1, \dots, m\}$ such that $p \in \mathrm{cl}_X(E_i)$. We may assume that $p \in \mathrm{cl}_X(E_1)$.

For each $Q \in \mathcal{T}$, let $F(Q) = \mathrm{cl}_X\left(\bigcup\{D_T : T \in \mathcal{T}_1 \text{ and } q_T \in Q\}\right)$. Since $p \in \mathrm{cl}_X(E_1)$, it follows that $p \in F(Q)$. Thus, $F(Q)$ is a nonempty compact subset of X. Let $F = \bigcap\{F(Q) : Q \in \mathcal{T}\}$ and let G be the component of F such that $p \in G$.

We need to show that $\mathrm{Fr}_X(W) \cap G \neq \emptyset$. Suppose to the contrary that $\mathrm{Fr}_X(W) \cap G = \emptyset$. By [**5**, Theorem 12.9], there exist disjoint compact subsets K and L of X such that $F = K \cup L$, $G \subset K$ and $K \cap \mathrm{Fr}_X(W) = \emptyset$. Let R and S be disjoint open subsets of X such that $K \subset R$ and $\mathrm{Fr}_X(W) \cup L \subset S$. Since $F \subset R \cup S$, there exist $k \geqslant 1$ and elements $Q_1, \dots, Q_k \in \mathcal{T}$ of X such that $F \subset F(Q_1) \cap \dots \cap F(Q_k) \subset R \cup S$. Let $Q_0 = Q_1 \cap \dots \cap Q_k$. Then Q_0 is an open subset of X such that $p \in Q_0 \subset W$ and

$F(Q_0) \subset R \cup S$. Since $p \in R \cap Q_0$ and $p \in \mathrm{cl}_X(E_1)$, there exists $T \in \mathcal{T}_1$ such that $q_T \in R \cap Q_0 \cap D_T$. Then $D_T \subset F(Q_0) \subset R \cup S$. Since D_T is connected, $D_T \subset R$, so $\mathrm{Fr}_X(W) \cap D_T = \emptyset$, this contradicts [**5**, Theorem 12.10]. We have shown that $\mathrm{Fr}_X(W) \cap G \neq \emptyset$.

Recall that $G \in C(X)$ and $p \in G \cap A$. Since X is hereditarily indecomposable, $G \subset A$ or $A \subset G$. Since $\mathrm{Fr}_X(W) \cap G \neq \emptyset$ and $\mathrm{Fr}_X(W) \cap A = \emptyset$, we obtain that $A \subset G$. Since $A \cap V_1 \neq \emptyset$, we can choose a point $x \in A \cap V_1 \subset F \cap V_1$. Then $x \in V_1 \cap F(W)$. Thus, there exists $T \in \mathcal{T}_1$ such that $V_1 \cap D_T \neq \emptyset$. This contradicts the definition of $\mathcal{T}_1$ and ends the proof of the theorem. □

4. Main results

LEMMA 4.1. *Let X be a hereditarily indecomposable Hausdorff continuum. Suppose that $A, B \in 2^X$, $C \in C(X)$, $A \subset C$, $B \not\subseteq C$ and α is a generalized arc joining A and B in 2^X. Then $C \in \alpha$.*

PROOF. Let $\mathcal{D} = \{D \in \alpha : D \subset C\}$. Then $\mathcal{D}$ is a nonempty proper closed subset of α. Since α is connected, there exists $D_0 \in \mathrm{Fr}_\alpha(\mathcal{D})$. Since $\mathcal{D}$ is closed in α, $D_0 \subset C$. Let $<$ be the order defined, as in the paragraph previous to Lemma 2.1, for α satisfying $A < B$. We claim that $D_0 = C$. Suppose to the contrary that $D_0 \neq C$. Fix a point $x_0 \in C - D_0$. Since $\langle X - \{x_0\}\rangle \cap \alpha$ is an open subset of α containing D_0, by Lemma 2.1(b), we may assume that there exists an element $D_1 \in \alpha$, such that $D_0 < D_1$ such that $[D_0, D_1] \subset \langle X - \{x_0\}\rangle$ and $[D_0, D_1] \not\subseteq \mathcal{D}$. Let $E = \bigcup\{F : F \in [D_0, D_1]\}$. Notice that $D_0 \subset E$, $x_0 \notin E$ and $E \not\subseteq C$. We check that E is closed in X. Let $x \in X - E$. Then $\{\langle X - \mathrm{cl}_X(U)\rangle : x \in U$ and U is an open subset of $X\}$ is an open cover of the compact set $[D_0, D_1]$. Thus, there exists an open subset U_0 of X such that $x \in U_0$ and $[D_0, D_1] \subset \langle X - \mathrm{cl}_X(U_0)\rangle$. Hence, $U_0 \cap E = \emptyset$. This proves that E is closed in X. Fix a point $y_0 \in E - C$.

We claim that the component G of E that contains y_0 intersects C. Suppose to the contrary that $G \cap C = \emptyset$. By [**5**, Theorem 12.9], there exists an open and closed subset K of E such that $y_0 \in K$ and $K \cap C = \emptyset$. Let $L = E - K$. Then L is a compact subset of X and $D_0 \subset L$. Let $\mathcal{E} = \{F \in [D_0, D_1] : F \subset L\}$ and $\mathcal{F} = \{F \in [D_0, D_1] : F \cap K \neq \emptyset\}$. Clearly, $\mathcal{E}$ and $\mathcal{F}$ are closed disjoint subsets of $[D_0, D_1]$, $D_0 \in \mathcal{E}$, $\mathcal{F} \neq \emptyset$ and $[D_0, D_1] = \mathcal{E} \cup \mathcal{F}$. This contradicts the connectedness of $[D_0, D_1]$ and completes the proof that $G \cap C \neq \emptyset$.

Since X is hereditarily indecomposable, G and C are subcontinua of X and $G \cap C \neq \emptyset$, we have $G \subset C$ or $C \subset G$. Since $y_0 \in G - C$ and $G \subset E \subset X - \{x_0\}$, we obtain a contradiction. This ends the proof that $C = D_0 \in \alpha$. □

The following lemma can be proved imitating the proof of Theorem 11.3 of [**9**] and using Theorem 2.2.

LEMMA 4.2. *Let Y be a Hausdorff continuum, $n \geqslant 1$ and $A \in 2^Y - C(Y)$. Then $2^Y - \{A\}$ and $C_n(Y) - \{A\}$ are arcwise connected.*

LEMMA 4.3. *Let X be a hereditarily indecomposable continuum and $n \geqslant 1$. Let $A \in C(X) - (\{X\} \cup F_1(X))$. Let $\mathcal{E}$ be the g-arcwise component of $C_n(X) - \{A\}$ that contains X. Then $\mathcal{E}$ is the only dense g-arcwise component of $C_n(X) - \{A\}$.*

Proof. Let $\mathcal{F}$ be a g-arcwise component of $C_n(X) - \{A\}$ such that $\mathcal{F} \neq \mathcal{E}$. Given $B \in C_n(X) - \{A\}$ such that $B \not\subseteq A$, taking an order arc from B to X, we obtain that $B \in \mathcal{E}$. This implies that $\mathcal{F} \subset C_n(A)$. Since $C_n(A)$ is compact and $C_n(A) \neq C_n(X)$, we conclude that $\mathcal{F}$ is not dense in $C_n(X) - \{A\}$.

Now we check that $\mathcal{E}$ is dense in $C_n(X) - \{A\}$. Let $B \in C_n(X) - \{A\}$ and $\mathcal{U} = \langle U_1, \dots, U_m\rangle \cap C_n(X)$ be a basic set in $C_n(X)$ such that $B \in \mathcal{U}$, where each U_i is a nonempty open subset of X. We may assume that $B \notin \mathcal{E}$. Then $B \subsetneq A$. Let $B_1, \dots, B_k$ be the components of B. Then $k \leqslant n$. Fix points $p_1 \in B_1, \dots, p_k \in B_k$. For each $i \in \{1, \dots, k\}$, let $F_i = \{j \in \{1, \dots, m\} : B_i \cap U_j \neq \emptyset\}$ and $\mathcal{U}_i = \{C \in C(X) : C \subset \bigcup\{U_j : j \in F_i\}$ and $C \cap U_j \neq \emptyset$ for each $j \in F_i\}$. Then $\mathcal{U}_i$ is open in $C(X)$ and $B_i \in \mathcal{U}_i$. By Theorem 3.1, there exists an open subset V_i of X such that $p_i \in V_i$ and, if $u \in V_i$, then there exists $Q \in C(X)$ such that $u \in Q \in \mathcal{U}_i$.

Fix a point $x \in X - A$. Let α be an order arc in $C(X)$ joining $\{x\}$ to X. Notice that $\alpha \subset \mathcal{E}$. Let $E_0 \in \alpha$ be the first element of α containing A, that is $E_0 \in \alpha$, $A \subset E_0$ and, if $E \subsetneq E_0$ and $E \in \alpha$, then $A \not\subseteq E$. Since the set $\langle V_1, \dots, V_k, X\rangle \cap C(X)$ is an open subset of $C(X)$ containing E_0, there exists $E \in \alpha$ such that $E \subsetneq E_0$ and $E \in \langle V_1, \dots, V_k, X\rangle$. For each $i \in \{1, \dots, k\}$, choose a point $u_i \in E \cap V_i$. By the choice of V_i, there exists $Q_i \in C(X)$ such that $u_i \in Q_i \in \mathcal{U}_i$. Let $D = \{u_1, \dots, u_k\}$.

Since $x \in E$, $E \not\subseteq A$. Since X is hereditarily indecomposable and $E \in C(X)$, $E \cap A = \emptyset$. Thus, $D \cap A = \emptyset$. Let $Q = Q_1 \cup \cdots \cup Q_k$, since $D \subset Q$, we have that $Q \in \mathcal{E}$. It is easy to show that $Q \in \mathcal{U}$. This completes the proof of the lemma. □

Theorem 4.4. *If X is a hereditarily indecomposable Hausdorff continuum, then X has unique hyperspaces 2^X and $C_n(X)$, for each n.*

Proof. Let Y be a Hausdorff continuum. Let $\mathcal{K}(X)$ denote one of the hyperspaces 2^X or $C_n(X)$ of X and let $\mathcal{K}(Y)$ be the correspondig hyperspace of Y. Suppose that $\mathcal{K}(X)$ is homeomorphic to $\mathcal{K}(Y)$. Let $h : \mathcal{K}(X) \to \mathcal{K}(Y)$ be a homeomorphism. Since $F_1(X)$ (respectively $F_1(Y)$) is homeomorphic to X (respectively Y) it is enough to show that $h(F_1(X)) = F_1(Y)$.

Using Proposition 6.3 of [**9**], it is easy to show that every proper subcontinuum of X has empty interior.

Claim 1. Let $A \in C(X) - (\{X\} \cup F_1(X))$. Let $\mathcal{E}$ be the g-arcwise component of $\mathcal{K}(X) - \{A\}$ that contains X. Then $\mathcal{E}$ is the only dense g-arcwise component of $\mathcal{K}(X) - \{A\}$.

We prove Claim 1. By Lemma 4.3, Claim 1 holds for the case that $\mathcal{K}(X) = C_n(X)$. Thus, in the proof of Claim 1, we assume that $\mathcal{K}(X) = 2^X$. Let $\mathcal{F}$ be a g-arcwise component of $2^X - \{A\}$ such that $\mathcal{F} \neq \mathcal{E}$. Given $B \in 2^X - \{A\}$ such that $B \not\subseteq A$, taking an order arc from B to X, we obtain that $B \in \mathcal{E}$. This implies that $\mathcal{F} \subset 2^A$. Since 2^A is compact and $2^A \neq 2^X$, we conclude that $\mathcal{F}$ is not dense in $2^X - \{A\}$. Now we check that $\mathcal{E}$ is dense in $2^X - \{A\}$. Let $B \in 2^X - \{A\}$ and let $\mathcal{U} = \langle U_1, \dots, U_m\rangle$ be a basic set in 2^X such that $B \in \mathcal{U}$, where each U_i is a nonempty open subset of X. Since $\mathrm{int}_X(A) = \emptyset$, for each $i \in \{1, \dots, m\}$, $U_i \not\subseteq A$.

Given $i \in \{1, \dots, m\}$, fix a point $p_i \in U_i - A$. Let $G = \{p_1, \dots, p_m\}$. Then $G \in \mathcal{U} \cap \mathcal{E}$. This completes the proof of Claim 1.

Claim 2. $h(C(X)) \subset C(Y)$.

In order to prove Claim 2, take an element $A \in C(X) - (\{X\} \cup F_1(X))$. By Lemma 4.1, $\mathcal{K}(X) - \{A\}$ is not g-arcwise connected. By Lemma 4.2, if $h(A)$ is not connected, then $\mathcal{K}(Y) - \{h(A)\}$ is g-arcwise connected. Since h is a homeomorphism, we conclude that $h(A)$ is connected. That is, $h(A) \in C(Y)$. Therefore, $h(C(X) - (\{X\} \cup F_1(X))) \subset C(Y)$. Since $\mathrm{cl}_{2^X}(C(X) - (\{X\} \cup F_1(X))) = C(X)$ and $C(Y)$ is compact, we conclude that $h(C(X)) \subset C(Y)$.

Claim 3. Let $A \in C(X) - \{X\}$ and let $B \in \mathcal{K}(X)$ be such that $B \subset A$. Then $h(B) \subset h(A)$.

We prove Claim 3. In the case that $A \in F_1(X)$, $B = A$ and $h(B) = h(A)$. Thus, suppose that $A \notin F_1(X)$. By Claim 2, $h(A)$ is connected. We can suppose that $h(A) \neq Y$. Let $\mathcal{E}$ (respectively, $\mathcal{F}$) be the g-arcwise component of $\mathcal{K}(X) - \{A\}$ (respectively, $\mathcal{K}(Y) - \{h(A)\}$) that contains X (respectively, Y). By Lemma 4.1, $B \notin \mathcal{E}$. If $h(B) \not\subset h(A)$, then taking an order arc from $h(B)$ to Y in $\mathcal{K}(Y)$ we can prove that $h(B) \in \mathcal{F}$. Since h is a homeomorphism, $h(\mathcal{E}) \neq \mathcal{F}$, so $h(\mathcal{E}) \cap \mathcal{F} = \emptyset$. Since $\mathcal{K}(Y) - (2^{h(A)} \cap \mathcal{K}(Y)) \subset \mathcal{F}$, $h(\mathcal{E}) \subset 2^{h(A)} \cap \mathcal{K}(Y)$. Since $h(A) \neq Y$, $2^{h(A)} \cap \mathcal{K}(Y)$ is a proper compact subset of $\mathcal{K}(Y)$. Thus, $h(\mathcal{E})$ is not dense in $\mathcal{K}(Y) - \{h(A)\}$. This implies that $\mathcal{E}$ is not dense in $\mathcal{K}(X) - \{A\}$, contrary to Claim 1. This proves that $h(B) \subset h(A)$.

Claim 4. $h(F_1(X)) \subset F_1(Y)$.

Suppose, contrary to Claim 4, that there exists a point $p \in X$ such that $h(\{p\}) \notin F_1(Y)$. Since $F_1(Y)$ is compact, there exists an open subset U of X such that $p \in U$ and $\langle U \rangle \subset \mathcal{K}(X) - h^{-1}(F_1(Y))$. Taking an order arc from $\{p\}$ to X in $C(X)$, it is possible to find two nondegenerate proper subcontinua A and B of X such that $p \in A \subsetneq B \subset U$. Let $\mathcal{E}$ be the g-arcwise connected component of $\mathcal{K}(X) - \{B\}$ such that $\{p\}, A \in \mathcal{E}$. By Lemma 4.1, $\mathcal{E} \subset \mathcal{K}(X) \cap 2^B \subset \langle U \rangle$. Thus $h(\mathcal{E}) \subset \mathcal{K}(Y) - F_1(Y)$. Since h is a homeomorphism, $h(\mathcal{E})$ is a g-arcwise component of $\mathcal{K}(Y) - \{h(B)\}$. By Claims 2 and 3, $h(A) \in C(Y)$ and $h(A) \subsetneq h(B)$. Fix a point $y \in h(A)$ and take an order arc β from $\{y\}$ to $h(A)$ in $C(Y)$. Notice that $\beta \subset \mathcal{K}(Y) - \{h(B)\}$. Since $h(A) \in h(\mathcal{E})$, $\beta \subset h(\mathcal{E})$. Hence, $\{y\} \in h(\mathcal{E}) \cap F_1(Y)$, a contradiction. Claim 4 is proved.

Claim 5. Let $A \in C(X)$. Then $h(A) = \bigcup\{h(\{p\}) \in \mathcal{K}(Y) : p \in A\}$.

Let $B = \bigcup\{h(\{p\}) \in \mathcal{K}(Y) : p \in A\}$. By [**1**, Lemma 2.1], B is a subcontinuum of Y. In the case that $A \neq X$, by Claim 3, $B \subset h(A)$. Now, we see that, in the case that $A = X$, we also have that $B \subset h(A)$. It is enough to show that, if $p \in X$, then $h(\{p\}) \subset h(X)$. Suppose to the contrary that there exists a point $y \in h(\{p\}) - h(X)$. Let $\mathcal{W} = h^{-1}(\langle Y - \{y\} \rangle)$. Then $\mathcal{W}$ is an open subset of $\mathcal{K}(X)$ such that $X \in \mathcal{W}$. Let λ be an order arc from $\{p\}$ to X in $C(X)$. Then there exists $E \in \lambda$ such that $E \neq X$ and $E \in \mathcal{W}$. By Claim 3, $h(\{p\}) \subset h(E)$. Thus, $y \in h(E) \in \langle Y - \{y\} \rangle$. This contradiction proves that, in every case, $B \subset h(A)$.

Suppose that $B \neq h(A)$. Let $\mathcal{V} = \mathcal{K}(Y) - (2^B \cap \mathcal{K}(Y))$ and $\mathcal{U} = h^{-1}(\mathcal{V})$. Then $\mathcal{U}$ is an open subset of $\mathcal{K}(X)$ such that $A \in \mathcal{U}$. Fix a point $p_0 \in A$ and let α be an order arc from $\{p_0\}$ to A in $C(X)$. Then there exists $C \in \alpha - \{A, \{p_0\}\}$ such that $C \in \mathcal{U}$. Then $\{p_0\} \subsetneq C \subsetneq A$. Fix a point $q_0 \in A - C$. Since $h(\{p_0\}), h(\{q_0\}) \subset B$, taking order arcs in $C(Y)$, it is possible to construct a generalized arc β joining $h(\{p_0\})$ and $h(\{q_0\})$ in $2^B \cap \mathcal{K}(Y)$. Thus, $h^{-1}(\beta)$ is a generalized arc in $\mathcal{K}(X)$ joining $\{p_0\}$ and $\{q_0\}$. By Lemma 4.1, $C \in h^{-1}(\beta)$. Thus, $h(C) \in \beta \subset 2^B \cap \mathcal{K}(Y)$. Hence, $h(C) \in 2^B \cap \mathcal{K}(Y) \cap \mathcal{V}$, a contradiction. We have shown that $B = h(A)$. This ends the proof of Claim 5.

CLAIM 6. Let $A \in \mathcal{K}(X)$. Then $h(A) \subset h(X)$.

In order to prove claim 6, we may assume that $A \neq X$. By Claim 2, $h(X)$ is connected. In order to prove Claim 6, since $2^{h(X)} \cap \mathcal{K}(Y)$ is compact, it is enough to show that $h(A) \in \mathrm{cl}_{2^Y}(2^{h(X)} \cap \mathcal{K}(Y))$. Let $\mathcal{V}$ be an open subset of $\mathcal{K}(Y)$ such that $h(A) \in \mathcal{V}$. Let $\mathcal{U} = \langle U_1, \ldots, U_m \rangle \cap \mathcal{K}(X)$ be a basic open subset of $\mathcal{K}(X)$ such that $A \in \mathcal{U} \subset h^{-1}(\mathcal{V})$ and $X \notin \mathcal{U}$, where each U_i is open in X. Note that $X \nsubseteq U_1 \cup \cdots \cup U_m$. Fix a point $p \in X$ and let α be an order arc from $\{p\}$ to X in $C(X)$. We analyze two cases:

Case 1. $\mathcal{K}(X) = 2^X$. Since $X \in \langle U_1, \ldots, U_m, X \rangle$, there exists $B \in \alpha \cap \langle U_1, \ldots, U_m, X \rangle$ such that $B \neq X$. For each $i \in \{1, \ldots, m\}$, choose a point $p_i \in B \cap U_i$. Let $C = \{p_1, \ldots, p_m\}$. Then $C \subset B$, $C \in \mathcal{U}$ and $h(C) \in \mathcal{V}$. By Claim 3, $h(C) \subset h(B)$. By Claim 5, $h(B) = \bigcup\{h(\{b\}) \in 2^Y : b \in B\} \subset \bigcup\{h(\{x\}) \in 2^Y : x \in X\} = h(X)$. Hence, $h(C) \in 2^{h(X)}$. Hence, $h(A) \in \mathrm{cl}_{2^Y}(2^{h(X)} \cap \mathcal{K}(Y)) = 2^{h(X)} \cap \mathcal{K}(Y)$.

Case 2. $\mathcal{K}(X) = C_n(X)$. Let $A_1, \ldots, A_k$ be the components of A. Then $k \leqslant n$. For each $i \in \{1, \ldots, k\}$, choose a point $a_i \in A_i$, let $F_i = \{j \in \{1, \ldots, m\} : A_i \cap U_j \neq \emptyset\}$ and $\mathcal{U}_i = \{C \in C(X) : C \subset \bigcup\{U_j : j \in F_i\}$ and $C \cap U_j \neq \emptyset$ for each $j \in F_i\}$. Then $\mathcal{U}_i$ is open in $C(X)$ and $A_i \in \mathcal{U}_i$. By Theorem 3.1, there exists an open subset V_i of X such that $a_i \in V_i$ and, if $u \in V_i$, then there exists $Q \in C(X)$ such that $u \in Q \in \mathcal{U}_i$. Since $X \in \langle V_1, \ldots, V_k, X \rangle \cap \mathcal{K}(X)$, there exists $B \in (\alpha - \{X\}) \cap \langle V_1, \ldots, V_k, X \rangle$. For each $i \in \{1, \ldots, k\}$, fix a point $b_i \in B \cap V_i$ and let $Q_i \in C(X)$ be such that $b_i \in Q_i \in \mathcal{U}_i$. Let $Q = Q_1 \cup \cdots \cup Q_k$. Then $Q \in C_n(X)$ and $Q \in \mathcal{U}$. Since $Q \subset U_1 \cup \cdots \cup U_m$, $Q \neq X$. Since X is hereditarily indecomposable and $B \neq X$, we obtain that $B \cup Q \neq X$. Notice that $B \cup Q$ is a subcontinuum of X. By Claim 3, $h(Q) \subset h(B \cup Q)$ and, by Claim 5, $h(B \cup Q) = \bigcup\{h(\{b\}) \in 2^Y : b \in B \cup Q\} \subset \bigcup\{h(\{x\}) \in 2^Y : x \in X\} = h(X)$. Thus, $h(Q) \subset h(X)$ and $h(Q) \in \mathcal{V}$. Therefore, $h(A) \in \mathrm{cl}_{2^Y}(2^{h(X)} \cap \mathcal{K}(Y)) = 2^{h(X)} \cap \mathcal{K}(Y)$. Claim 6 is proved.

We are ready to show that $h(F_1(X)) = F_1(Y)$. Let $y \in Y$ and $B = h^{-1}(\{y\})$. By Claim 6, $\{y\} = h(B) \subset h(X)$. By Claim 5, $h(X) = \bigcup\{h(\{x\}) \in 2^Y : x \in X\}$. Thus, there exists $x \in X$ such that $y \in h(\{x\})$. By Claim 4, $h(\{x\})$ is a one-point set. Hence, $\{y\} = h(\{x\}) \in h(F_1(X))$. We have shown that $F_1(Y) \subset h(F_1(X))$. Thus, Claim 4 implies that $h(F_1(X)) = F_1(Y)$. This ends the proof of the theorem. □

QUESTION 1. (see [**3**, Problem 37]) Let X be a hereditarily indecomposable Hausdorff continuum. Is it true that X has a unique hyperspace $F_n(X)$, for each

$n \geqslant 2$? That is, suppose that Y is a Hausdorff continuum such that $F_n(X)$ is homeomorphic to $F_n(Y)$, does it follows that X is homeomorphic to Y?

The answer to this question is not known even for the case that X is a metric continuum and $n = 2$.

Acknowledgment. The author wishes to thank the participants in the Third Workshop of Research in Continua and Hyperspaces (Toluca City, July, 2009) for useful discussions on the topic of this paper, particularly, Gerardo Acosta, Mauricio E. Chacón, Jorge M. Martínez, Rodrigo Hernández, Antonio Peláez and Alicia Santiago. The author also thanks Marianela Vélez for her technical help during the preparation of this paper.

References

1. J. J. Charatonik and A. Illanes, *Local connectedness in hyperspaces*, Rocky Mountain J. Math. **36** (2006), 811–856.
2. R. Hernández-Gutiérrez, A. Illanes and V. Martínez-de-la-Vega, *Uniqueness of hyperspaces for Peano continua*, to appear in Rocky Mountain J. Math.
3. A. Illanes, *Hyperspaces of Continua*; in: Elliott Pearl (ed.), *Open Problems in Topology II*, Elsevier, 2007, 279–288.
4. A. Illanes, *Dendrites with unique hyperspace $C_2(X)$, II*, Topology Proc. **34** (2009), 77–96.
5. A. Illanes and S. B. Nadler, Jr., *Hyperspaces: Fundamentals and Recent Advances*, Monographs and Textbooks Pure Appl. Math. 216, Marcel Dekker, New York and Basel, 1999.
6. I. Lončar, *Nonmetric rim-metrizable continua and unique hyperspace*, Publ. Inst. Math., Nouv. Sér. **73(87)** (2003), 97–113.
7. S. Macías, *Hereditarily indecomposable continua have unique hyperspace 2^X*, Bol. Soc. Mat. Mexicana **5**(3) (1999), 415–418.
8. S. Macías, *On the hyperspaces $C_n(X)$ of a continuum X, II*, Topology Proc. **25** (2000), 255–276.
9. S. B. Nadler, Jr., *Hyperspaces of sets: A text with research questions*, Monographs and Textbooks Pure Appl. Math. 49, Marcel Dekker, New York and Basel, 1978.
10. S. B. Nadler, Jr., *Continuum Theory: An Introduction*, Monographs and Textbooks Pure Appl. Math. 158, Marcel Dekker, New York, Basel and Hong Kong, 1992.
11. A. Peláez, *On the uniqueness of the hyperspaces 2^X and $C_n(X)$ of rim-metrizable continua*, Topology Proc. **30** (2006), 565–576.

Instituto de Matemáticas
Universidad Nacional Autónoma de México
Area de la Investigación Científica
Circuito Exterior
Ciudad Universitaria
México, 04510, D.F.
México
illanes@matem.unam.mx

(Received 22 04 2010)