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BOREL SETS AND COUNTABLE MODELS

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Abstract. We show that certain families of sets and functions related to a countable structure $\mathbb{A}$ are analytic subsets of a Polish space. Examples include sets of automorphisms, endomorphisms and congruences of $\mathbb{A}$ and sets of the combinatorial nature such as coloring of countable plain graphs and domino tiling of the plane. This implies, without any additional set-theoretical assumptions, i.e., in ZFC alone, that cardinality of every such uncountable set is $2^{\aleph_0}$.

1. Introduction

Works of Kueker [**6**], Reyes [**10**], Barwise [**1**], Makkai [**8**] and others, show that certain sets $\mathcal{S}$ of model-theoretic objects related to a countable structure $\mathbb{A}$, as $\operatorname{Aut}\mathbb{A}$ for example, behave as analytic subsets of the Cantor discontinuum. Here we present a method for coding some notions related to countable first-order structures by theories of propositional calculus L_{ω_1}. After that, we consider countable models and we discus the topological and cardinal properties of the set of valuations satisfying a formula of the infinitary logic $L_{\omega_1\omega}$. Finally, we transfer those considerations to the models that are not necessarily countable. Then, we use the Suslin theorem on the cardinality of analytic subsets in a Polish space, to derive various known theorems in a uniform way.

Let X be a Polish space. The family of the Borel subsets of X is the smallest σ algebra on $P(X)$ which contains closed subsets of X. Analytic sets are continuous images of Borel sets. The following theorem, see [**9**], will have the important role in the following considerations.

Theorem 1.1 (Suslin). *Let X be an infinite analytic subset of a Polish space. Then either $|X| = \aleph_0$ or $|X| = 2^{\aleph_0}$.*

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We will say that a collection $\mathcal{X}$ satisfies CH (Continuum hypothesis) if every uncountable member $\mathcal{S}$ of $\mathcal{X}$ has the cardinality of continuum, $2^{\aleph_0}$. In proofs that $\mathcal{X}$ satisfies CH, the idea of our approach is to interpret the members $\mathcal{S}$ of $\mathcal{X}$ by analytic subsets of Cantor space, or more generally of a Polish space.

2. Coding into $\mathcal{L}_{\omega_1}^{\mathcal{P}}$

In this section we present a method for coding the first order properties of countable structures by valuations of the sentences of the infinitary propositional logic $\mathcal{L}_{\omega_1}$. Let us first review some definitions and notions of this logic.

The logic $\mathcal{L}_{\omega_1\omega}$ is an extension of the classical first order predicate logic $\mathcal{L}_{\omega\omega}$. Besides the usual logical symbols it admits also countable conjunctions ($\bigwedge$) and disjunctions ($\bigvee$), see [**4**]. For instance, if the language contains countably many constant symbols c_n, $n \in \omega$, then the formula of this logic $\forall x \bigvee_{n\in\omega} x = c_n$ asserts that the domain of the structure is at most countable. On the other hand, the logic $\mathcal{L}_{\omega_1}^{\mathcal{P}}$ is an extension of the classical propositional calculus with particular set of propositional letters $\mathcal{P}$ which allows countable conjunctions and disjunctions. The set of formulas $\mathcal{F}_{cP}$ of $\mathcal{L}_{\omega_1}^{\mathcal{P}}$ is defined by recursion as follows:

$$F_0 = \mathcal{P},$$
$$F_{n+1} = F_n \cup \{\neg\varphi \mid \varphi \in F_n\} \cup \{\bigwedge S \mid S \in [F_n]^{\leqslant\omega}\} \cup \{\bigvee S \mid S \in [F_n]^{\leqslant\omega}\},$$
$$\mathcal{F}_{cP} = \bigcup_{n\in\omega} F_n.$$

Here, the symbol $[X]^{\leqslant\omega}$ denotes the set of at most countable subsets of X. If $S = \{\varphi_n \mid n \in \omega\}$, then $\bigwedge S$ and $\bigvee S$ are denoted also by $\bigwedge_{n\in\omega}\varphi_n$ and $\bigvee_{n\in\omega}\varphi_n$ respectively. A map $\mu : \mathcal{P} \mapsto 2$ is called a valuation and the value of the formula in the valuation μ is defined by induction on complexity of formulas as follows:

$p[\mu] = \mu(p)$, $p \in \mathcal{P}$,
$(\bigwedge_{n\in\omega}\varphi_n)[\mu] = \prod_{n\in\omega}\varphi_n[\mu]$, where $\prod_{n\in\omega}\varphi_n[\mu]$ is an infimum of the set $\{\varphi_n[\mu] \mid n \in \omega\}$ in the boolean algebra $\mathbf{2} = (2, \cdot, +, ', 0, 1)$,
$(\bigvee_{n\in\omega}\varphi_n)[\mu] = \sum_{n\in\omega}\varphi_n[\mu]$, where $\sum_{n\in\omega}\varphi_n[\mu]$ is a supremum of the set $\{\varphi_n[\mu] \mid n \in \omega\}$ in $\mathbf{2}$,
$(\neg\varphi)[\mu] = \varphi[\mu]'$.

Therefore we have defined for each formula φ a function $\hat{\varphi} : 2^cP \to 2$ such that $\hat{\varphi}(\mu) = \varphi[\mu]$ for all $\mu : cP \to 2$. If φ is finite, observe that $\hat{\varphi}$ is a continuous function. We say that a valuation μ is a model of the formula φ if $\hat{\varphi}(\mu) = 1$. Let us assume discrete topology on the set 2. Then $2^{\mathcal{P}}$ is the Cantor space and its domain is the set of all valuations of the propositional variables, i.e., models of the propositional calculus. Since $\hat{\varphi}$ is a Borel function, the set $\mathfrak{M}(\varphi)$ of all models of φ is a Borel subset of the Cantor space $2^{\mathcal{P}}$. Thus we proved

Theorem 2.1. *Assume T is a theory in $\mathcal{L}_{\omega_1}^{\mathcal{P}}$ over a countable set cP of propositional letters. Then $\mathfrak{M}(T)$ is a Borel subset of the Cantor space $2^{\mathcal{P}}$.*

By Theorem 1.1 we have immediately

Corollary 2.1. *CH holds for $\mathfrak{M}(\varphi)$.*

The essence of the method we are proposing now is based on the Corollary and is stated as follows: Let $\mathcal{X}$ be a class of certain sets. If every $X \in \mathcal{X}$ is coded by a set of all models of a theory of $\mathcal{L}^{\mathcal{P}}_{\omega_1}$, then CH is true for a class $\mathcal{X}$.

2.1. Map $*$. Let $\mathbb{A} = (A, \dots)$ be a countable first-order structure of a countable language L, $L_A = L \cup \{\underline{a} \mid a \in A\}$, and $(\mathbb{A}, a)_{a\in A}$ the simple expansion of $\mathbb{A}$ to L_A. We define the set cP of propositional letters as

$$cP = \big\{p_{F,a_1,\dots,a_n,b} \mid a_i, b \in A,\ F \text{ is a function symbol of } L\big\} \cup \big\{q_{R,a_1,\dots,a_n} \mid a_i \in A,\ R \text{ is a relation symbol of } L\big\}.$$

The map $*$ from the set Sent_{L_A} of all $\mathcal{L}_{\omega_1\omega}$-sentences of L_A into the set of infinitary propositional formulas of $\mathcal{L}^{\mathcal{P}}_{\omega_1}$ over the set $\mathcal{P}$ is defined recursively as follows:

$$\begin{aligned}
&(F(\underline{a}_1, \dots, \underline{a}_n) = \underline{b})^* = p_{F,a_1,\dots,a_n,b},\\
&(R(\underline{a}_1, \dots, \underline{a}_n))^* = q_{R,a_1,\dots,a_n},\\
&(\neg\theta)^* = \neg\theta^*,\ (\textstyle\bigwedge_{n\in\omega}\theta_n)^* = \bigwedge_{n\in\omega}\theta_n^*, \quad (\bigvee_{n\in\omega}\theta_n)^* = \bigvee_{n\in\omega}\theta_n^*,\\
&(\forall x\theta)^* = \textstyle\bigwedge_{a\in A}\theta(\underline{a})^*, \quad (\exists x\theta)^* = \bigvee_{a\in A}\theta(\underline{a})^*,\\
&(F(t_1(\underline{a}_{i1}, \dots, \underline{a}_{im}), \dots, t_n(\underline{a}_{i1}, \dots, \underline{a}_{im})) = \underline{b})^* =\\
&\quad \textstyle\bigwedge_{(b_1,\dots,b_n)\in A^n}(\bigwedge_{i=1}^n(\underline{b}_i = t_i(\underline{a}_{i1}, \dots, \underline{a}_{im}))^* \to p_{F,b_1,\dots,b_n,b}),\\
&(R(t_1(\underline{a}_{i1}, \dots, \underline{a}_{im}), \dots, t_n(\underline{a}_{i1}, \dots, \underline{a}_{im})))^* =\\
&\quad \textstyle\bigwedge_{(b_1,\dots,b_n)\in A^n}((\bigwedge_{i=1}^n t_i(\underline{a}_{i1}, \dots, \underline{a}_{im}) = \underline{b}_i)^* \wedge q_{R,b_1,\dots,b_n}).
\end{aligned}$$

Theorem 2.2. *Let $\mathbb{A} = (A, \dots)$ be a countable model of a countable language L, L' is a countable expansion of L and T be a theory of L' in $\mathcal{L}_{\omega_1\omega}$. Then the set of all L'-expansions $\mathbb{A}'$ of $\mathbb{A}$ that are models of T is coded by a Borel subset of the Cantor space.*

Proof. With the notation as above, let $T^* = \{\varphi^* \mid \varphi \in T\}$. Then $\mathfrak{M}(T^*) = \bigcap_{\varphi\in T}\mathfrak{M}(\varphi^*)$ is a Borel set as a countable intersection of Borel sets. It remains to see that there is a one-to-one and onto correspondence between valuations that are the models of T^* and expansions $\mathbb{A}'$ of $\mathbb{A}$ in L' that are models of T. The function h which assigns to each $\mu \in \mathfrak{M}(T^*)$ an expansion $h(\mu) = \mathbb{A}_\mu$ of $\mathbb{A}$ is defined as follows:

If $F \in L'$ is a function symbol, then

$$F^{\mathbb{A}_\mu}(a_1, \dots, a_n) = b \quad \text{iff} \quad \big(F(\underline{a}_1, \dots, \underline{a}_n) = \underline{b}\big)[\mu] = 1,$$
$$\text{i.e.,}\ \ F^{\mathbb{A}_\mu}(a_1, \dots, a_n) = b \quad \text{iff} \quad \mu(p_{F,a_1,\dots,a_n,b}) = 1.$$

If $R \in L'$ is a predicate symbol, then

$$R^{\mathbb{A}_\mu}(t_1, \dots, t_n) \quad \text{iff} \quad (R(\underline{t}_1, \dots, \underline{t}_n))[\mu] = 1,$$
$$\text{i.e.,}\ \ R^{\mathbb{A}_\mu}(t_1, \dots, t_n) \quad \text{iff} \quad \mu(q_{R,t_1,\dots,t_n}) = 1.$$

By induction on the complexity of the formula φ, it is easy to prove that $\mathbb{A}_\mu$ is a model of T and if $\mu \neq \nu$, then $\mathbb{A}_\mu \neq \mathbb{A}_\nu$. Hence the mapping h is one-to-one. Conversely, let $\mathbb{A}'$ be an L'-expansion of $\mathbb{A}$ that is a model of T. Then we define a valuation $\mu_{\mathbb{A}'}$ of as follows:

$$\mu_{\mathbb{A}'}(p_{F,a_1,\dots,a_n,b}) = 1 \quad \text{iff} \quad \mathbb{A}' \models F(\underline{a}_1,\dots,\underline{a}_n) = \underline{b}.$$
$$\mu_{\mathbb{A}'}(q_{R,t_1,\dots,t_n}) = 1 \quad \text{iff} \quad \mathbb{A}' \models R(t_1,\dots,t_n).$$

Since $\mathbb{A}'$ is a model of T, $\mu_{\mathbb{A}'}$ is a model of T^*, i.e., $h(\mu_{\mathbb{A}'}) = \mathbb{A}'$. Thus h is onto. The mapping $c = h^{-1}$ codes expansions $\mathbb{A}'$ and this proves the theorem. □

Under the assumptions of the previous theorem by Suslin's theorem we have immediately the following

COROLLARY 2.2. *If the set of all L'-expansions $\mathbb{A}'$ of $\mathbb{A}$ that are models of T is uncountable, then it has the cardinality $2^{\aleph_0}$.*

From Theorem 2.2 we can easily deduce a variant of Makkai's theorem [**8**, Theorem 9.1.]. Let ψ be a Σ^1_1 sentence of the logic $\mathcal{L}_{\omega_1\omega}$, i.e., ψ is a second order formula of the form $\exists\bar{R}\,\varphi(\bar{R})$, where $\varphi(\bar{R})$ is of $\mathcal{L}_{\omega_1\omega}$ in a language $L\cup\{\bar{R}\}$, $\bar{R}$ is a finite or infinite set of predicate (and/or function) symbols not belonging to L.

THEOREM 2.3. *Let $\mathbb{A} = (A,\dots)$ be a countable model of a countable language L, L' is a countable expansion of L and $\psi = \exists\bar{R}\,\varphi(\bar{R})$ be a Σ^1_1 sentence of L' in $\mathcal{L}_{\omega_1\omega}$ (the sequence $\bar{R}$ does not belong to L'). Then the set of all L'-expansions $\mathbb{A}'$ of $\mathbb{A}$ that are models of ψ is coded by an analytic subset of the Cantor space.*

PROOF. Let $\mathcal{S}$ be the set of all expansions $\mathbb{A}'' = (\mathbb{A}',\bar{R})$ that satisfy the sentence $\varphi(\bar{R})$, where $\mathbb{A}'$ is an expansion of $\mathbb{A}$ to the language L'. By Theorem 2.2, $\mathcal{S}$ is coded by a Borel subset B of 2^{cP}, where cP is the set of propositional letters introduced by the inductive definition of the map $*$. Let cP_1 be the set of propositional letters in cP which do not contain in their indices the symbols which occur in the block $\exists\bar{R}$ and cP_2 be the set of propositional letters in cP which do contain in their indices the symbols which occur in the block $\exists\bar{R}$. Hence $cP = cP_1\cup cP_2$ and $cP_1\cap cP_2 = \emptyset$. So, B is a Borel subset of $2^{cP_1}\times 2^{cP_2}$ and its projection B' to 2^{cP}_1 is the code set of $\mathcal{S}'$, the set of all expansions $\mathbb{A}'$ satisfying ψ. Hence B' is the analytic subset of B. □

2.2. Examples. Though some of the following examples can be seen as special cases of Theorem 2.2, for the sake of simplicity we will give a direct application of $*$. Also, we believe that in this way certain combinatorial problems are easier to be coded directly.

EXAMPLE 2.1 (Kueker's theorem, revisited). Let $\mathbb{A} = (A,\dots)$ be a countable algebra of a countable language L. We shall show that CH is true for the number of automorphisms of $\mathbb{A}$, see [**6**]. First, we describe the suitable propositional theory that codes the notion of an automorphism of the model $\mathbb{A}$.

Let $f\in\operatorname{Aut}\mathbb{A}$ and $\mathcal{P} = \{p_{ab}\mid a,b\in A\}$ be a set of propositional letters. Here, p_{ab} intuitively means that $f(a) = b$, or, more formally, according to definition of the map $*$, $(f(a) = b)^* = p_{ab}$. The theory T that we shall define now will exactly describe this situation. The theory T is defined as follows:

- $T_1 = \{\neg(p_{ab_1} \wedge p_{ab_2}) \mid b_1 \neq b_2, a, b_1, b_2 \in A\}$.
  Observe that $\mathfrak{M}(T_1) = \bigcap_{\varphi \in T_1} \hat{\varphi}^{-1}[1]$, so $\mathfrak{M}(T_1)$ is a closed subset of the Cantor space $2^{\mathcal{P}}$. Also observe that T_1 codes the notion of the function.
- $T_2 = \{\bigvee_b \varphi_{ab} \mid a \in A\}$, where $\varphi_{ab} = p_{ab}$.
  Note that $\mathfrak{M}(T_2) = \bigcap_a \bigcup_b \hat{\varphi}_{ab}^{-1}[1]$, thus $\mathfrak{M}(T_2)$ is G_δ in $2^{\mathcal{P}}$. Also note that T_2 says that A is the domain of a function.
- $T_3 = \{\neg(p_{a_1 b} \wedge p_{a_2 b}) \mid a_1 \neq a_2, a_1, a_2, b \in A\}$.
  It is easy to see that $\mathfrak{M}(T_3)$ is closed in $2^{\mathcal{P}}$. Observe that T_3 codes the notion of injection.
- $T_4 = \{\bigvee_a p_{ab} \mid b \in A\}$.
  Similarly as in the case of the set of all models of T_2, $\mathfrak{M}(T_4)$ is G_δ subset of $2^{\mathcal{P}}$. Note that T_4 codes the notion of surjection.
- Let F be an n-ary function symbol of L. Define the theory T_F by

$$T_F = \left\{\neg(p_{a_1 b_1} \wedge \cdots \wedge p_{a_n b_n}) \vee p_{F^{\mathbb{A}}(a_1,\ldots,a_n) F^{\mathbb{A}}(b_1,\ldots,b_n)} \mid a_i, b_j \in A\right\}$$

  and define T_5 by $T_5 = \bigcup_F T_F$. Clearly, $\mathfrak{M}(T_5)$ is F_σ in $2^{\mathcal{P}}$. Finally, let us remark that T_5 provides the compatibility of the function f with the operation F. Then the theory T_5 says that f is an endomorphism of the algebra $\mathbb{A}$.

Let $T = T_1 \cup \cdots \cup T_5$. We see that the theory T says that f is an automorphism of the algebra $\mathbb{A}$. Hence, $\mathfrak{M}(T) = \bigcap_{i=1}^{5} \mathfrak{M}(T_i)$ is a Borel subset of $2^{\mathcal{P}}$ since it is a finite intersection of Borel sets. Therefore CH holds for $\mathfrak{M}(T)$.

Finally, the map $H : \mathfrak{M}(T) \to \operatorname{Aut} \mathbb{A}$ defined by $H(\mu)(a) = b$ iff $\mu(p_{ab}) = 1$ is a bijection, thus the number of automorphisms of an countable algebra satisfies CH. If $\mathbb{A}$ is an countable algebra with relations, i.e., a model of the first order predicate calculus, a similar proposition holds for $\operatorname{Aut} \mathbb{A}$. It is obtained by adding to the theory T the set of appropriate axioms for relation symbols of the language of the model $\mathbb{A}$.

Example 2.2 (Graph coloring). A graph is a pair (Γ, R), where Γ is a nonempty set and R is a binary relation on Γ. We say that the elements of the set Γ are nodes of the graph. If a and b are nodes and if $a\,R\,b$ is true, then we say that nodes a and b are connected.

It is well known that there is a coloring of nodes of a countable planar graph in four colors such that connected nodes are colored in different colors. We will show that the number of such colorings satisfies CH.

Let $\mathcal{P} = \{p_{ab} \mid a, b \in \Gamma\} \cup \{q_a^n \mid a \in \Gamma,\ n \in 4\}$, where p_{ab} means that nodes a and b are connected ($(aRb)^* = p_{ab}$), and q_a^n means that the node a is colored in n-th color($(R_i(a))^* = q_a^i$, where R_i is the relation “i-th color"). We define the theory T as follows:

$T_1 = \{\bigvee_{n \in 4} q_a^n \mid a \in \Gamma\}$,
$T_2 = \{\neg q_a^m \vee \neg q_a^n \mid a \in \Gamma,\ m, n \in 4,\ m \neq n\}$,
$T_3 = \{\neg p_{ab} \vee \neg q_a^n \vee \neg q_b^n \mid a, b \in \Gamma,\ n \in 4\}$,
$T = T_1 \cup T_2 \cup T_3$.

Notice that each model of the theory T defines a unique coloring of the graph (Γ, R) in four colors, and vice versa, each coloring defines a unique model of the theory T. Thus the number of such colorings is equal to the number of models of the theory T, so it satisfies CH.

EXAMPLE 2.3 (Wang dominoes). This example is connected with the problem of domino tiling of a plane [**11**], see also [**5**, Vol., pp. 381–384]. Suppose we have countably many dominoes, where each domino is a unit square divided into four triangles by its diagonals and each triangle is enumerated by some natural number.

The type of the domino is a quadruple of natural numbers (a, b, c, d), where a is a label of the lower triangle, b is a label of the left triangle, c is a label of the upper triangle and d is a label of the right triangle. Let S be a finite set of domino types.

We cover the plane by dominoes as follows: The vertices of dominoes should have integers as coordinates. The position of a domino are coordinates of the upper right vertex. We combine dominoes in the usual way. Using the Transfer Principle from the nonstandard analysis we can show that the existence of a covering of the first quadrant implies existence of a covering of the entire plane. Namely, suppose that there is a covering of the first quadrant. Then, by the Transfer Principle, there is a covering of the nonstandard first quadrant. If H is an infinite nonstandard positive integer, then any covering of the galaxy of (H, H) is also a covering of the standard plane. Now we shell discuss the number of these coverings.

Let $\mathcal{P} = \{p^{m,n}_{abcd} \mid (a, b, c, d) \in S,\ m, n \in \omega\}$ be the set of propositional letters, where $p^{m,n}_{abcd}$ means "the domino of the type (a, b, c, d) is on the position (m, n)". The suitable propositional theory T which will describe the covering of the plane is defined as follows:

$$\begin{aligned}
T_1 &= \{\textstyle\bigvee_S p^{m,n}_{abcd} \mid m, n \in \mathbb{Z}\},\\
T_2 &= \{\neg p^{m,n}_{abcd} \vee \neg p^{m,n-1}_{pqrs} \mid a \neq r, p^{m,n}_{abcd}, p^{m,n-1}_{pqrs} \in S\},\\
T_3 &= \{\neg p^{m,n}_{abcd} \vee \neg p^{m+1,n}_{pqrs} \mid b \neq s, p^{m,n}_{abcd}, p^{m+1,n}_{pqrs} \in S\},\\
T_4 &= \{\neg p^{m,n}_{abcd} \vee \neg p^{m,n+1}_{pqrs} \mid c \neq p, p^{m,n}_{abcd}, p^{m,n+1}_{pqrs} \in S\},\\
T_5 &= \{\neg p^{m,n}_{abcd} \vee \neg p^{m-1,n}_{pqrs} \mid q \neq d, p^{m,n}_{abcd}, p^{m-1,n}_{pqrs} \in S\},\\
T &= T_1 \cup \cdots \cup T_5.
\end{aligned}$$

Observe that T_1 codes the notion of the covering of the plain, while T_2, T_3, T_4, T_5 code other properties of the domino tiling. Further, it is easy to see that $\mathfrak{M}(T) = \bigcap_{\varphi \in T} \hat{\varphi}^{-1}[1]$, thus $\mathfrak{M}(T)$ is a Borel subset of the Cantor space $2^{\mathcal{P}}$, so CH holds for $\mathfrak{M}(T)$. In addition, we have a one-to-one and onto correspondence between the set of all coverings of the plane and the set of all models of the theory T. Thus, the number of coverings satisfies CH.

Of course, many other examples fall in this category. For example, CH is true for:

(1) The number of countable linear extensions of a countable partially ordered set.

(2) The number of congruences of a countable algebra (Burris and Kwaitinetz).
(3) The number of maximal (prime) ideals of countable ring, i.e., Zariski space of a countable ring. In particular, CH holds for the Stone space of a countable Boolean algebra $\mathbb{B}$.
(4) The numbers of maximal chains and antichains in a countable partially ordered set. In particular, CH holds for the number of branches of a countable tree.
(5) The number of ordered fields $(\mathbb{F}, \leqslant)$, for every countable real closed field $\mathbb{F}$.

3. On the number of valuations

In this section we discus the topological and cardinal properties of the set of valuations satisfying a formula of the infinitary logic $L_{\omega_1\omega}$. The basic assumption is that the domain of the considered model is equipped with a certain topology. We note that our consideration in this section is not limited only to the countable models.

3.1. Valuations in countable models. Let L be a language with a countable set of variables V and let $\mathbb{A} = (A, \dots)$ be a countable model of language L. If we consider A as a discrete topological space, then the set of all valuations A^V is homeomorphic to the Baire space.

Let $\varphi(x_1, \dots, x_n)$ be an arbitrary first-order formula of the language L. We define the map $\hat{\varphi} : A^V \to 2$ by

$$\hat{\varphi}(\mu) = \begin{cases} 1, & \mathbb{A} \models \varphi[\mu] \\ 0, & \mathbb{A} \models \neg\varphi[\mu]. \end{cases}$$

Lemma 3.1. *The function $\hat{\varphi}$ is continuous.*

Proof. The set

$$\begin{aligned} \hat{\varphi}^{-1}[\{1\}] &= \big\{\mu \in A^V \mid \mathbb{A} \models \varphi[\mu]\big\} \\ &= \bigcup \big\{\pi_1^{-1}[\{a_1\}] \cap \dots \cap \pi_n^{-1}[\{a_n\}] \mid a_i \in A \wedge \mathbb{A} \models \varphi[a_1, \dots, a_n]\big\}, \end{aligned}$$

is open as the union of open sets. Similarly, $\hat{\varphi}^{-1}[\{0\}]$ is open, so $\hat{\varphi}$ is a continuous map. □

Let $t(x_1, \dots, x_n) = \{\varphi_m(x_1, \dots, x_n) \mid m \in \omega\}$ be a countable type in variables $x_1, \dots, x_n$ of the language L. We define the function $\hat{t} : A^V \to 2^\omega$ by

$$\hat{t}(\mu) = \langle \hat{\varphi}_n[\mu] \mid n \in \omega \rangle.$$

Since for each m we have $\pi_m \circ \hat{t} = \hat{\varphi}_m$, and each function $\hat{\varphi}_m$ is continuous, we obtain the following

Corollary 3.1. *The function $\hat{t}$ is continuous.*

The set of all valuations that satisfy the type $t(x_1, \dots, x_n)$ is equal to the set $\hat{t}^{-1}[\{\langle 1 \mid n \in \omega \rangle\}]$, so we have the following fact.

Corollary 3.2. *The set of all valuations that satisfy the type $t(x_1, \dots, x_n)$ is a closed subset of the Baire space A^V.*

Next we shall consider formulas $\varphi(x_1, x_2, \dots)$ and types $t(x_1, x_2, \dots)$ of $L_{\omega_1\omega}$ with countably many free variables, together with corresponding functions $\hat{\varphi}$ and $\hat{t}$.

Theorem 3.1. *For arbitrary formula $\varphi(x_1, x_2, \dots)$ of $L_{\omega_1\omega}$ the function $\hat{\varphi}$ is Borel.*

Proof. We use the induction on the complexity of φ. If φ is an atomic formula, then by the argument stated above the function $\hat{\varphi}$ is continuous and hence it is Borel.

Suppose that φ is the formula $\bigvee_{n\in\omega} \varphi_n$. Then $\hat{\varphi}^{-1}[\{1\}] = \bigcup_{n\in\omega} \hat{\varphi}_n^{-1}[\{1\}]$, so it is a Borel set as a countable union of Borel sets.

Suppose that φ is the formula $\neg\psi$. Then $\hat{\varphi}^{-1}[\{1\}] = \hat{\psi}^{-1}[\{0\}]$, so it is a Borel set by the induction hypothesis.

Suppose that φ is the formula $\exists x\, \psi(x, x_1, x_2, \dots)$. Then

$$\begin{aligned}\hat{\varphi}^{-1}[\{1\}] &= \big\{\mu \in A^V \mid \mathbb{A} \models \exists x \psi(x, x_1, x_2, \dots)[\mu]\big\}\\ &= \bigcup_{b\in A} \big\{\mu \in A^V \mid \mathbb{A} \models \psi(b, x_1, x_2, \dots)[\mu]\big\},\end{aligned}$$

so it is a Borel set as a countable union of Borel sets. □

Corollary 3.3. *The set of all valuations which satisfy the type $t(x_1, x_2, \dots)$ is a Borel subset of the Baire space, thus CH is true for this set as well.*

Let $\mathbb{A}$ and $\mathbb{B}$ be countable models of the language L and let $\mathbb{A}$ be a submodel of $\mathbb{B}$. Then $\mathbb{A}$ is an elementary submodel of $\mathbb{B}$ if for each first-order formula φ the function $\hat{\varphi}_{\mathbb{A}} : A^V \to 2$ has a continuous extension $\hat{\varphi}_{\mathbb{B}} : B^V \to 2$.

Example 3.1. The number of cuts of a countable linear ordering satisfies CH. Really, the notion of the cut can be coded by the conjunction of the following formulas:

$$\forall x \bigvee_{i\in\omega} x = x_i, \quad \bigwedge_{i\in\omega} x_{2i} < x_{2i+2}, \quad \bigwedge_{i\in\omega} x_{2i+3} < x_{2i+1},$$
$$\bigwedge_{i,j\in\omega} x_{2i} < x_{2j+1}, \quad \bigwedge_{i\in\omega}\bigvee_{j\in\omega} x_{2i} < x_{2j}, \quad \bigwedge_{i\in\omega}\bigvee_{j\in\omega} x_{2j+1} < x_{2i+1}.$$

3.2. Valuations in uncountable models. In this subsection we shall transfer some results from the previous subsection to the models that are not necessarily countable. Let $\mathbb{A} = (A, \dots)$ be a model of the language L, where A is a Polish space and suppose that all of functions and relations of the model $\mathbb{A}$ are Borel. For simplicity such models we shall call Borel models. Note that if A is countable, then all these relations and functions are anyway Borel. In a Borel model $\mathbb{A}$ we can identify valuations from A^V with elements of the Polish space A^ω. In what follows, we shall use the following two basic facts:

LEMMA 3.2. *Let A be a Polish space and let B be a Borel subset of A^n. If π is a permutation of the set $\{0,\dots,n-1\}$, then $\big\{(a_{\pi(0)},\dots,a_{\pi(n-1)}) \mid (a_0,\dots,a_{n-1}) \in B\big\}$ is a Borel set.*

LEMMA 3.3. *Let A be a Polish space and let B be a Borel subset of A^n. Then the sets $B \times A^k$ and $B \times A^\omega$ are Borel subsets of A^{n+k} and A^ω, respectively.*

Let $\mathbb{A}$ be a Borel model of a language L and $t = t(x_1, x_2, \dots, x_m)$ be a term of L, where $x_1, x_2, \dots, x_m$ is the list of all variables occurring in t. The term mapping $f_t \colon A^m \to A$ associated to t is naturally defined by $f_t(a_1, a_2, \dots, a_m) = t^{\mathbb{A}}[a_1, a_2, \dots, a_m]$. The following lemma shows that in Borel models term mappings are Borel functions.

LEMMA 3.4. *Let $\mathbb{A} = (A, \dots)$ be a Borel model of the language L and t be a term of L. Then f_t is the Borel function.*

PROOF. We prove the statement by induction on the complexity of terms. Obviously, terms of the complexity 0 define unary functions that are identities or constant functions.

Let $t = g(t_1, \dots, t_n)$, where g is a function symbol and $t_1, \dots, t_n$ are terms of the lower complexity. Further, let $x_1, x_2, \dots, x_m$ be the list of all variables occurring in t. The terms $t_1, \dots, t_n$ define Borel term functions $h_1, \dots, h_n$ by the induction hypothesis. To prove that the function $f = g^{\mathbb{A}} \circ (h_1, \dots, h_n)$ is Borel, it suffices to show that the function $(h_1, \dots, h_n) \colon A^m \to A^n$ is Borel. The graph of the function $(h_1, \dots, h_n)$ is the set

$$\begin{aligned}&\big\{(a_0,\dots,a_m,b_1,\dots,b_n) \in A^{m+n+1} \mid h_i(a_0,\dots,a_m) = b_i, i \in \{1,\dots,n\}\big\}\\ &\quad= \bigcap_{1\leqslant i\leqslant n} \big\{(a_0,\dots,a_m,b_1,\dots,b_n) \in A^{m+n+1} \mid h_i(a_0,\dots,a_m) = b_i\big\}.\end{aligned}$$

Further, the set

$$\begin{aligned}&\big\{(a_0,\dots,a_m,b_1,\dots,b_n) \in A^{m+n+1} \mid h_1(a_0,\dots,a_m) = b_1\big\}\\ &\quad= \big\{(a_0,\dots,a_m,b_1) \in A^{m+2} \mid h_1(a_0,\dots,a_m) = b_1\big\} \times A^{n-1}\end{aligned}$$

is Borel, according to the previous lemma. Similarly, the other terms of the intersection are Borel sets, therefore the function $(h_1, \dots, h_n)$ is Borel. □

THEOREM 3.2. *Let $\mathbb{A} = (A, \dots)$ be a Borel model. If φ is a quantifier-free formula of the language $L_{\omega_1\omega}$, then the set of valuations which satisfy formula φ is a Borel subset of A^V.*

PROOF. Since

$$\big\{\mu \in A^V \mid \mathbb{A} \models \textstyle\bigwedge_{n\in\omega}\varphi_n\big\} = \bigcap_{n\in\omega} \big\{\mu \in A^V \mid \mathbb{A} \models \varphi_n\big\},$$

$$\big\{\mu \in A^V \mid \mathbb{A} \models \neg\varphi\big\} = \big\{\mu \in A^V \mid \mathbb{A} \models \varphi\big\}^c,$$

it suffices to prove the statement for the atomic formulas.

Let $R(t_1(x_0, \dots, x_m), \dots, t_n(x_0, \dots, x_m))$ be an atomic formula. We show that the set

$$S = \{(a_0, \dots, a_m) \in A^{m+1} \mid R^{\mathbb{A}}(t_1^{\mathbb{A}}[a_0, \dots, a_m], \dots, t_n^{\mathbb{A}}[a_0, \dots, a_m])\}$$

is Borel. Notice that $S = \pi(S_1)$, where

$$S_1 = \{(a_0, \dots, a_m, b_1, \dots, b_n) \in A^{m+n+1} \mid R^{\mathbb{A}}(b_1, \dots, b_n), b_i = t_n^{\mathbb{A}}[a_0, \dots, a_m]\}$$

and $\pi : A^{m+n+1} \longrightarrow A^{m+1}$ is the projection

$$\pi((a_0, \dots, a_m, b_1, \dots, b_n)) = (a_0, \dots, a_m).$$

Then,

$$\begin{aligned} S_1 = &\{(a_0, \dots, a_m, b_1, \dots, b_n) \in A^{m+n+1} \mid R^{\mathbb{A}}(b_1, \dots, b_n)\} \\ &\cap \bigcap_{1 \leqslant i \leqslant n} \{(a_0, \dots, a_m, b_1, \dots, b_n) \in A^{m+n+1} \mid b_i = t_i^{\mathbb{A}}[a_0, \dots, a_m]\}. \end{aligned}$$

By our assumption and Lemma 3.3, the set

$$\begin{aligned} &\{(a_0, \dots, a_m, b_1, \dots, b_n) \in A^{m+n+1} \mid R^{\mathbb{A}}(b_1, \dots, b_n)\} \\ &\qquad = A^{m+1} \times \{(b_1, \dots, b_n) \in A^n \mid R^{\mathbb{A}}(b_1, \dots, b_n)\} \end{aligned}$$

is Borel. Using lemmas 3.2, 3.3 and 3.4 we conclude that the set

$$\begin{aligned} &\{(a_0, \dots, a_m, b_1, \dots, b_n) \in A^{m+n+1} \mid b_1 = t_1^{\mathbb{A}}[a_0, \dots, a_m]\} \\ &\qquad = \{(a_0, \dots, a_m, b_1) \in A^{m+2} \mid b_1 = t_1^{\mathbb{A}}[a_0, \dots, a_m]\} \times A^{n-1} \end{aligned}$$

is Borel, as well as the other members of the intersection. Hence, S_1 is a Borel set.

The projection π is continuous; notice that its restriction $\pi|_{S_1}$ is injective. Thus, S is a Borel set as an injective image of the Borel set S_1 (see [**3**, Theorem 15.1]). □

Since the continuous image of a Borel set is analytic, we have

COROLLARY 3.4. *Assume that the model $\mathbb{A} = (A, \dots)$ of the language L satisfies the conditions of the previous theorem. If φ is an $L_{\omega_1\omega}$ formula of the form $\exists x_1 \dots \exists x_n \, \psi$, where ψ is quantifier free, CH holds for the number of valuations which satisfy φ.*

In the same manner as in the proof of Theorem 3.2, we can prove the following theorem:

THEOREM 3.3. *Let $\mathbb{A} = (A, \dots)$ be a model of the language L, where the set A is a Polish space and all of the functions and relations of $\mathbb{A}$ are projective. If φ is an arbitrary formula of $L_{\omega_1\omega}$, then the set of valuations satisfying φ is the projective subset of A^V.*

We have immediately from [**3**, Theorem 38.17] the following result.

COROLLARY 3.5. *Let $\mathbb{A} = (A, \dots)$ be a model of the language L, where the set A is a Polish space and all of the functions and relations of $\mathbb{A}$ are projective and assume PD (the axiom of projective determinacy). If φ is an arbitrary formula of $L_{\omega_1\omega}$, then CH holds for the number of valuations satisfying φ.*

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