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# CURVATURE PROPERTIES OF SOME CLASS OF HYPERSURFACES IN EUCLIDEAN SPACES

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*Dedicated to Professor Makoto Yawata on his seventy-second birthday*

Abstract. We determine curvature properties of pseudosymmetry type of hypersurfaces in Euclidean spaces $\mathbb{E}^{n+1}$, $n \geqslant 5$, having three distinct nonzero principal curvatures $\lambda_1$, $\lambda_2$ and $\lambda_3$ of multiplicity 1, $p$ and $n-p-1$, respectively. For some hypersurfaces having this property the sum of $\lambda_1$, $\lambda_2$ and $\lambda_3$ is equal to the trace of the shape operator of $M$. We present an example of such hypersurface.

## 1. Introduction

Let $H$ be the second fundamental tensor of a hypersurface $M$ immersed isometrically in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, with signature $(s, n+1-s)$, $n \geqslant 4$, where $c = \frac{\widetilde{\kappa}}{n(n+1)}$ and $\widetilde{\kappa}$ is the scalar curvature of the ambient space. For precise definitions of the symbols used we refer to Section 2 of this paper and Sections 2 and 3 of [**16**] (see also [**3, 5, 14, 34, 50**]). Let $\mathcal{U}_H \subset M$ be the set of all points at which the tensor $H^2$ is not a linear combination of $H$ and the metric tensor $g$ of $M$. Curvature conditions of pseudosymmetry type on hypersurfaces $M$ in $N_s^{n+1}(c)$, $n \geqslant 4$, satisfying on $\mathcal{U}_H \subset M$ the equation

$$H^3 = \operatorname{tr}(H)H^2 + \psi H, \tag{1.1}$$

where $\psi$ is some function on $\mathcal{U}_H$, were investigated in several papers: [**1, 7, 8, 11, 12, 15, 16, 22, 23, 25, 26, 34, 37**]. For instance, the Cartan hypersurfaces satisfy (1.1) (see, e.g., [**12**, Theorem 4.3], [**16**, Example 5.1(iii)]). Examples of hypersurfaces in Euclidean spaces $\mathbb{E}^{n+1}$, $n \geqslant 5$, as well as in semi-Euclidean spaces $\mathbb{E}_s^{n+1}$, with signature $(s, n+1-s)$, $n \geqslant 5$, satisfying (1.1) are given in [**1**] and [**11**], respectively. For further examples we refer to [**15, 16, 23, 26, 27, 31, 35**].

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Curvature conditions of pseudosymmetry type on hypersurfaces $M$ in $N_s^{n+1}(c)$, $n \geqslant 4$, satisfying on $M \setminus \mathcal{U}_H$ the equation

$$H^2 = \psi H + \rho g, \tag{1.2}$$

for some functions $\psi$ and $\rho$ on this set, were investigated among others in [**2**,**7**,**9**,**18**, **27**, **33**, **35**, **45**, **47**, **54**]. Examples of hypersurfaces in spaces of constant curvature satisfying (1.2) are given among others in [**27**, **35**, **43**, **56**, **57**]. It is obvious that (1.1) is a special case of a more general equation

$$H^3 = \phi H^2 + \psi H + \rho g, \tag{1.3}$$

where $\phi$, $\psi$ and $\rho$ are some functions on $\mathcal{U}_H$. Hypersurfaces $M$ in $N_s^{n+1}(c)$, $n \geqslant 4$, satisfying (1.3) on $\mathcal{U}_H \subset M$ were investigated for instance in [**6**, **21**, **50**]. Here we investigate curvature conditions of pseudosymmetry type on hypersurfaces $M$ in $\mathbb{E}_s^{n+1}$, $n \geqslant 5$, satisfying (1.3) on $\mathcal{U}_H$. We can also consider (1.3) with $\phi = \mathrm{tr}(H)$ on $\mathcal{U}_H$, i.e., the equation

$$H^3 = \mathrm{tr}(H) H^2 + \psi H + \rho g, \tag{1.4}$$

where $\psi$ and $\rho$ are some functions on $\mathcal{U}_H$. Hypersurfaces $M$ in $N_s^{n+1}(c)$, $n \geqslant 4$, satisfying (1.4) on $\mathcal{U}_H \subset M$ were investigated in [**4**, **30**, **51**–**53**]. In [**30**, Proposition 2.1] it was proved that for every hypersurface $M$ in $N_s^5(c)$ equation (1.4) reduces on $\mathcal{U}_H \subset M$ to (1.1). Evidently, $\rho = 0$ on $\mathcal{U}_H$. The assumption that $\dim M = 4$ is essential. In Section 5 we present an example of a hypersurface $M$ in $\mathbb{E}^{n+1}$, $n \geqslant 5$, having at every point three distinct principal curvatures $\lambda_1$, $\lambda_2$ and $\lambda_3$ of multiplicity 1, $p$ and $q$, respectively, where $n = 1 + p + q$, satisfying (1.4) with nonzero function $\rho$. In [**50**, Proposition 4.1] it was shown that the tensors $R \cdot C$, $C \cdot R$ and $C \cdot C$ of a hypersurface $M$ in $N_s^{n+1}(c)$, $n \geqslant 4$, satisfying (1.3) on $\mathcal{U}_H \subset M$ are expressed on this set by a linear combinations of the Tachibana tensors $Q(g, R)$, $Q(S, R)$, $Q(S, G)$, $Q(H, G)$ and $Q(S, g \wedge H)$, and the tensors $g \wedge Q(H, H^2)$ and $H \wedge Q(g, H^2)$. In Section 3 we present these formulas in the case when $M$ is a hypersurface in $\mathbb{E}_s^{n+1}$, $n \geqslant 4$. Further, in the next section we present these formulas in the special case when $M$ is a hypersurface in $\mathbb{E}^{n+1}$, $n \geqslant 5$, and at every point of the set $\mathcal{U}_H$ of a hypersurface $M$ there are three distinct principal curvatures of multiplicity 1, $p$ and $p$, respectively, where $n = 2p + 1$. In Section 5 we present an example of such hypersurface.

## 2. Preliminaries

Throughout the paper all manifolds are assumed to be connected paracompact manifolds of class $C^\infty$. Let $(M, g)$ be an $n$-dimensional, $n \geqslant 3$, semi-Riemannian manifold and let $\nabla$ be its Levi-Civita connection and $\Xi(M)$ the Lie algebra of vector fields on $M$. We define on $M$ the endomorphisms $X \wedge_A Y$ and $\mathcal{R}(X, Y)$ of $\Xi(M)$ by

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y,$$
$$\mathcal{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where $A$ is a symmetric $(0,2)$-tensor on $M$ and $X, Y, Z \in \Xi(M)$. The Ricci tensor $S$, the Ricci operator $\mathcal{S}$, the tensor $S^2$ and the scalar curvature $\kappa$ of $(M,g)$ are defined by $S(X,Y) = tr\{Z \to \mathcal{R}(Z,X)Y\}$, $g(\mathcal{S}X,Y) = S(X,Y)$, $S^2(X,Y) = S(\mathcal{S}X,Y)$ and $\kappa = tr\mathcal{S}$, respectively. The endomorphism $\mathcal{C}(X,Y)$ we define by

$$\mathcal{C}(X,Y)Z = \mathcal{R}(X,Y)Z - \frac{1}{n-2}\Big(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y\Big)Z.$$

Further, we define the $(0,4)$-tensor $G$, the Riemann–Christoffel curvature tensor $R$ and the Weyl conformal curvature tensor $C$ of $(M,g)$ by

$$\begin{aligned} G(X_1,X_2,X_3,X_4) &= g((X_1 \wedge_g X_2)X_3,X_4),\\ R(X_1,X_2,X_3,X_4) &= g(\mathcal{R}(X_1,X_2)X_3,X_4),\\ C(X_1,X_2,X_3,X_4) &= g(\mathcal{C}(X_1,X_2)X_3,X_4), \end{aligned}$$

respectively, where $X_1, X_2, \dots \in \Xi(M)$.

Let $\mathcal{B}(X,Y)$ be a skew-symmetric endomorphism of $\Xi(M)$ and let $B$ be a $(0,4)$-tensor associated with $\mathcal{B}(X,Y)$ by

$$B(X_1,X_2,X_3,X_4) = g(\mathcal{B}(X_1,X_2)X_3,X_4). \tag{2.1}$$

The tensor $B$ is said to be a generalized curvature tensor [**44**] if

$$\begin{gathered} B(X_1,X_2,X_3,X_4) + B(X_2,X_3,X_1,X_4) + B(X_3,X_1,X_2,X_4) = 0,\\ B(X_1,X_2,X_3,X_4) = B(X_3,X_4,X_1,X_2). \end{gathered}$$

Let $\mathcal{B}(X,Y)$ be a skew-symmetric endomorphism of $\Xi(M)$ and let $B$ be the tensor defined by (2.1). We extend $\mathcal{B}(X,Y)$ to a derivation $\mathcal{B}(X,Y)\cdot$ of the algebra of tensor fields on $M$, by assuming that it commutes with contractions and $\mathcal{B}(X,Y)\cdot f = 0$, for any smooth function $f$ on $M$. Now for a $(0,k)$-tensor field $T$, $k \geqslant 1$, we can define the $(0,k+2)$-tensor $B \cdot T$ by

$$\begin{aligned} (B\cdot T)(X_1,\dots,X_k;X,Y) &= (\mathcal{B}(X,Y)\cdot T)(X_1,\dots,X_k;X,Y)\\ &= -T(\mathcal{B}(X,Y)X_1,X_2,\dots,X_k) - \dots - T(X_1,\dots,X_{k-1},\mathcal{B}(X,Y)X_k). \end{aligned}$$

If $A$ is a symmetric $(0,2)$-tensor then we define the $(0,k+2)$-tensor $Q(A,T)$ by

$$\begin{aligned} Q(A,T)(X_1,\dots,X_k;X,Y) &= (X \wedge_A Y \cdot T)(X_1,\dots,X_k;X,Y)\\ &= -T((X\wedge_A Y)X_1,X_2,\dots,X_k) - \dots - T(X_1,\dots,X_{k-1},(X\wedge_A Y)X_k). \end{aligned}$$

In this manner we obtain the $(0,6)$-tensors $B\cdot B$ and $Q(A,B)$. Setting in the above formulas $\mathcal{B} = \mathcal{R}$ or $\mathcal{B} = \mathcal{C}$, $T = R$ or $T = C$ or $T = S$, $A = g$ or $A = S$, we get the tensors $R\cdot R$, $R\cdot C$, $C\cdot R$, $C\cdot C$, $R\cdot S$, $C\cdot S$, $Q(g,R)$, $Q(S,R)$, $Q(g,C)$ and $Q(g,S)$. Let $A$ be a symmetric $(0,2)$-tensor and $T$ a $(0,p)$-tensor, $p \geqslant 2$. According to [**22**], the tensor $Q(A,T)$ is called the Tachibana tensor of $A$ and $T$, or the Tachibana tensor for short. We also remark that in some papers, the $(0,6)$-tensor $Q(g,R)$ is called the Tachibana tensor (see, e.g., [**39**–**41**,**46**,**55**]). For symmetric $(0,2)$-tensors

$E$ and $F$ we define their Kulkarni–Nomizu product $E \wedge F$ by

$$\begin{aligned}(E \wedge F)(X_1, X_2, X_3, X_4) = E(X_1, X_4)F(X_2, X_3) + E(X_2, X_3)F(X_1, X_4) \\ - E(X_1, X_3)F(X_2, X_4) - E(X_2, X_4)F(X_1, X_3).\end{aligned}$$

Clearly, the tensors $R$, $C$, $G$ and $E \wedge F$ are generalized curvature tensors. For a symmetric $(0,2)$-tensor $E$ we define the $(0,4)$-tensor $\overline{E}$ by $\overline{E} = \frac{1}{2}E \wedge E$. We have $\overline{g} = G = \frac{1}{2}g \wedge g$. We note that the Weyl tensor $C$ can be presented in the form

$$C = R - \frac{1}{n-2}g \wedge S + \frac{\kappa}{(n-2)(n-1)}G. \tag{2.2}$$

We also have (see, e.g., [**15**, Section 3])

$$Q(E, E \wedge F) = -Q(F, \overline{E}). \tag{2.3}$$

Now (2.2) and (2.3) yield $Q(g, C) = Q(g, R) + (1/(n-2))Q(S, G)$. For a symmetric $(0,2)$-tensor $E$ and a $(0,k)$-tensor $T$ $k \geqslant 2$, we define their Kulkarni–Nomizu product $E \wedge T$ by [**12**]

$$\begin{aligned}&(E \wedge T)(X_1, X_2, X_3, X_4; Y_3, \ldots, Y_k) \\ &\quad = E(X_1, X_4)T(X_2, X_3, Y_3, \ldots, Y_k) + E(X_2, X_3)T(X_1, X_4, Y_3, \ldots, Y_k) \\ &\qquad - E(X_1, X_3)T(X_2, X_4, Y_3, \ldots, Y_k) - E(X_2, X_4)T(X_1, X_3, Y_3, \ldots, Y_k).\end{aligned}$$

Using the above definitions we can prove

LEMMA 2.1. *[**11, 12**] Let $E_1$, $E_2$ and $F$ be symmetric $(0,2)$-tensors at a point $x$ of a semi-Riemannian manifold $(M, g)$, $n \geqslant 3$. Then at $x$ we have*

$$E_1 \wedge Q(E_2, F) + E_2 \wedge Q(E_1, F) = -Q(F, E_1 \wedge E_2).$$

*If $E = E_1 = E_2$, then*

$$E \wedge Q(E, F) = -Q(F, \overline{E}). \tag{2.4}$$

## 3. Hypersurfaces in semi-Euclidean spaces

Let $M$, $n \geqslant 3$, be a connected hypersurface isometrically immersed in a semi-Riemannian manifold $(N, g^N)$. We denote by $g$ the metric tensor induced on $M$ from $g^N$. Further, we denote by $\nabla$ and $\nabla^N$ the Levi-Civita connections corresponding to the metric tensors $g$ and $g^N$, respectively. Let $\xi$ be a local unit normal vector field on $M$ in $N$ and let $\varepsilon = g^N(\xi, \xi) = \pm 1$. We can write the Gauss formula and the Weingarten formula of $(M, g)$ in $(N, g^N)$ in the form: $\nabla^N_X Y = \nabla_X Y + \varepsilon H(X, Y)\xi$ and $\nabla^N_X \xi = -\mathcal{A}X$, respectively, where $X, Y$ are vector fields tangent to $M$, $H$ is the second fundamental tensor of $(M, g)$ in $(N, g^N)$, $\mathcal{A}$ is the shape operator and $H^k(X, Y) = g(\mathcal{A}^k X, Y)$, $k \geqslant 1$, $H^1 = H$ and $\mathcal{A}^1 = \mathcal{A}$. We denote by $R$ and $R^N$ the Riemann–Christoffel curvature tensors of $(M, g)$ and $(N, g^N)$, respectively. Let $x^r = x^r(y^k)$ be the local parametric expression of $(M, g)$ in $(N, g^N)$, where $y^k$ and $x^r$ are local coordinates of $M$ and $N$, respectively, and $h, i, j, k \in \{1, 2, \ldots, n\}$ and $p, r, t, u \in \{1, 2, \ldots, n+1\}$. The Gauss equation of $(M, g)$ in $(N, g^N)$ has the form

$$R_{hijk} = R^N_{prtu}B^p_h B^r_i B^t_j B^u_k + \varepsilon(H_{hk}H_{ij} - H_{hj}H_{ik}), \quad B^r_k = \frac{\partial x^r}{\partial y^k}, \tag{3.1}$$

where $R^N_{prtu}$, $R_{hijk}$ and $H_{hk}$ are the local components of the tensors $R^N$, $R$ and $H$, respectively. If $(N, g^N)$ is a conformally flat space then we have [**23**, Section 4]

$$C_{hijk} = \mu G_{hijk} + \varepsilon \overline{H}_{hijk} + \frac{\varepsilon}{n-2}\big(g \wedge (H^2 - \operatorname{tr}(H)H)\big)_{hijk},$$
$$\mu = \frac{1}{(n-2)(n-1)}\big(\kappa - 2\widetilde{S}_{rt}B^r_h B^t_k g^{hk} + \widetilde{\kappa}\big), \tag{3.2}$$

where $\widetilde{S}_{rt}$ are the local components of the Ricci tensor $\widetilde{S}$ of the ambient space, $G_{hijk}$ are the local components of the tensor $G$ and $\widetilde{\kappa}$ and $\kappa$ are the scalar curvatures of $(N, g^N)$ and $(M, g)$, respectively.

Let now $M$ be a hypersurface in $\mathbb{E}^{n+1}_s$, $n \geqslant 4$. Clearly, (3.1) and (3.2) read

$$R_{hijk} = \varepsilon \overline{H}_{hijk}, \quad \mu = \frac{\kappa}{(n-2)(n-1)}, \tag{3.3}$$

respectively. Contracting (3.3) with $g^{ij}$ and $g^{kh}$ we obtain

$$S_{hk} = \varepsilon(\operatorname{tr}(H)H_{hk} - H^2_{hk}), \quad \kappa = \varepsilon\big((\operatorname{tr}(H))^2 - \operatorname{tr}(H^2)\big), \tag{3.4}$$

respectively, where $\operatorname{tr}(H) = g^{hk}H_{hk}$, $\operatorname{tr}(H^2) = g^{hk}H^2_{hk}$ and $S_{hk}$ are the local components of the Ricci tensor $S$ of $M$. We recall that on every hypersurface $M$ in $\mathbb{E}^{n+1}_s$, $n \geqslant 3$, we have the following identity $R \cdot R = Q(S, R)$ [**32**]. We prove now that on $M$ in $\mathbb{E}^{n+1}_s$, $n \geqslant 3$, we also have

Proposition 3.1. *On every hypersurface $M$ in $\mathbb{E}^{n+1}_s$, $n \geqslant 3$, the following identities are satisfied*

$$g \wedge Q(H, H^2) = \varepsilon g \wedge Q(S, H), \tag{3.5}$$
$$H \wedge Q(g, H^2) = \varepsilon \operatorname{tr}(H)Q(g, R) - \varepsilon H \wedge Q(g, S). \tag{3.6}$$

Proof. From (3.4) we have

$$H^2 = \operatorname{tr}(H)H - \varepsilon S, \tag{3.7}$$

and this yields

$$g \wedge Q(H, H^2) = g \wedge Q(H, \operatorname{tr}(H)H - \varepsilon S) = \varepsilon g \wedge Q(S, H).$$

Thus (3.5) is proved. Further, using (2.4), (3.3) and (3.7) we obtain

$$\begin{aligned} H \wedge Q(g, H^2) &= \operatorname{tr}(H)H \wedge Q(g, H) - \varepsilon H \wedge Q(g, S) \\ &= -\operatorname{tr}(H)H \wedge Q(H, g) - \varepsilon H \wedge Q(g, S) \\ &= \operatorname{tr}(H)Q(g, \overline{H}) - \varepsilon H \wedge Q(g, S) \\ &= \varepsilon \operatorname{tr}(H)Q(g, R) - \varepsilon H \wedge Q(g, S). \end{aligned}$$

Our proposition is thus proved. □

Let now $M$ be a hypersurface in $\mathbb{E}_s^{n+1}$, $n \geqslant 4$, satisfying (1.3) on $\mathcal{U}_H \subset M$. We set (cf. [**50**, eq. (34)])

$$
\begin{aligned}
\beta_1 &= \varepsilon(\phi - \operatorname{tr}(H)),\\
\beta_2 &= -\frac{\varepsilon}{n-2}(\phi(2\operatorname{tr}(H) - \phi) - (\operatorname{tr}(H))^2 - \psi - (n-2)\varepsilon\mu),\\
\beta_3 &= \varepsilon\mu\operatorname{tr}(H) + \frac{1}{n-2}\big(\psi(2\operatorname{tr}(H) - \phi) + (n-3)\rho\big),\\
\beta_4 &= \beta_3 - \varepsilon\beta_2\operatorname{tr}(H),\\
\beta_5 &= \frac{\kappa}{n-1} + \varepsilon\psi + \beta_1\operatorname{tr}(H),\\
\beta_6 &= \beta_2,
\end{aligned}
\tag{3.8}
$$

where the functions $\phi$, $\psi$ and $\rho$ are defined by (1.3).

Proposition 3.2. *If $M$ is a hypersurface in $\mathbb{E}_s^{n+1}$, $n \geqslant 4$, satisfying* (1.3) *on $\mathcal{U}_H \subset M$, for some functions $\phi$, $\psi$ and $\rho$, then the following conditions are satisfied on this set*

$$S^2 = \gamma_2 S + \gamma_1 H + \gamma_0 g, \tag{3.9}$$

$$(n-2)R\cdot C = (n-2)Q(S,R) + \rho Q(H,G) - \beta_1 g \wedge Q(H,S), \tag{3.10}$$

$$
\begin{aligned}
(n-2)C\cdot R = {}&\Big(\frac{\kappa}{n-1} + \varepsilon\psi + \beta_1\operatorname{tr}(H)\Big)Q(g,R)\\
&+ (n-3)Q(S,R) - \beta_1 H \wedge Q(g,S),
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
(n-2)C\cdot C = {}&\beta_1 Q(S, g\wedge H) + \beta_4 Q(H,G)\\
&+ (n-3)Q(S,R) + \beta_5 Q(g,R) + \beta_2 Q(S,G),
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
\gamma_0 &= \rho(\phi - 2\operatorname{tr}(H)),\\
\gamma_1 &= \psi(\phi - 2\operatorname{tr}(H)) + \rho + \operatorname{tr}(H)(\phi^2 + \psi + (\operatorname{tr}(H))^2),\\
\gamma_2 &= -(\phi^2 + \psi + \operatorname{tr}(H)(\operatorname{tr}(H) - 2\phi)).
\end{aligned}
\tag{3.13}
$$

Proof. We denote by $S^2_{hk}$ the local components of the tensor $S^2$. Evidently, we have

$$S^2_{hk} = g^{ij}S_{hi}S_{kj} = H^4_{hk} - 2\operatorname{tr}(H)H^3_{hk} + (\operatorname{tr}(H))^2 H^2_{hk}.$$

Applying in this (1.3) we obtain

$$
\begin{aligned}
S^2 = {}&(\phi^2 + \psi + \operatorname{tr}(H)(\operatorname{tr}(H) - 2\phi))H^2\\
&+ \rho(\phi - 2\operatorname{tr}(H))g + (\psi(\phi - 2\operatorname{tr}(H)) + \rho)H.
\end{aligned}
$$

The last relation, by making use of (3.7) and (3.13), turns into (3.9). Further, we also have on $\mathcal{U}_H$ (cf. [**50**, Proposition 4.1]): (3.12) and

$$(n-2)R\cdot C = (n-2)Q(S,R) + \rho Q(H,G) + (\phi - \operatorname{tr}(H))g \wedge Q(H,H^2), \tag{3.14}$$

$$
\begin{aligned}
(n-2)C\cdot R = {}&\Big(\frac{\kappa}{n-1} + \varepsilon\psi\Big)Q(g,R) + (n-3)Q(S,R)\\
&+ (\phi - \operatorname{tr}(H))H \wedge Q(g,H^2),
\end{aligned}
\tag{3.15}
$$

where $\beta_1, \dots, \beta_5$ are defined by (3.8). Now (3.10) and (3.11) are an immediate consequence of (3.5), (3.6), (3.8), (3.14) and (3.15). □

## 4. Hypersurfaces with three principal curvatures

In this section we consider hypersurfaces $M$ in $\mathbb{E}^{n+1}$, $n \geqslant 5$, having at every point of the set $\mathcal{U}_H \subset M$ three distinct principal curvatures $\lambda_1$, $\lambda_2$ and $\lambda_3$. First we note that from (1.3) it follows that

$$\phi = \lambda_1 + \lambda_2 + \lambda_3, \quad \psi = -(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3), \quad \rho = \lambda_1\lambda_2\lambda_3. \tag{4.1}$$

Moreover we assume that $\lambda_1$, $\lambda_2$ and $\lambda_3$ are of multiplicity 1, $p$ and $p$, respectively. Evidently, $n = 2p+1$. Further, (3.3), (3.4) and (3.8) lead to

$$\begin{gathered}
\operatorname{tr}(H) = \lambda_1 + p(\lambda_2 + \lambda_3), \quad \operatorname{tr}(H^2) = \lambda_1^2 + p(\lambda_2^2 + \lambda_3^2),\\
\beta_1 = -\frac{n-3}{2}(\lambda_2 + \lambda_3), \quad \beta_2 = \frac{1}{n-3}\beta_1^2 = \psi + \frac{\kappa}{n-1}\\
\beta_3 = \frac{1}{n-2}\big(\operatorname{tr}(H)\beta_2 - \psi\beta_1 + (n-3)\rho\big),\\
\beta_4 = -\frac{1}{n-2}\big((n-3)\operatorname{tr}(H)\beta_2 + \psi\beta_1 - (n-3)\rho\big), \quad \beta_5 = \beta_2 + \operatorname{tr}(H)\beta_1.
\end{gathered} \tag{4.2}$$

Using now (3.9) and (4.1) we find

$$\begin{aligned}
\gamma_0 &= -\lambda_1\lambda_2\lambda_3(\lambda_1 + (2p-1)(\lambda_2 + \lambda_3)),\\
\gamma_1 &= p(p-1)^2(\lambda_2^3 + \lambda_3^3) + p(p-1)(\lambda_2^2 + \lambda_3^2)\lambda_1\\
&\quad + (3p^2(p-2) + 4p - 1)\lambda_2\lambda_3(\lambda_2 + \lambda_3) + (2p^2 - 2p + 1)\lambda_1\lambda_2\lambda_3,\\
\gamma_2 &= -(p-1)(\lambda_2^2 + \lambda_3^2) - (p-2)\lambda_1(\lambda_2 + \lambda_3) - (2p-3)\lambda_2\lambda_3.
\end{aligned} \tag{4.3}$$

From (3.4) and (4.2) it follows immediately that the eigenvalues $\rho_1$, $\rho_2$ and $\rho_3$ of the Ricci tensor $S$ of $M$ are expressed on $\mathcal{U}_H$ trought the following relations

$$\begin{aligned}
\rho_1 &= \lambda_1(\operatorname{tr}(H) - \lambda_1) = p\lambda_1(\lambda_2 + \lambda_3),\\
\rho_2 &= \lambda_2(\operatorname{tr}(H) - \lambda_2) = \lambda_2(\lambda_1 + (p-1)\lambda_2 + p\lambda_3),\\
\rho_3 &= \lambda_3(\operatorname{tr}(H) - \lambda_3) = \lambda_3(\lambda_1 + p\lambda_2 + (p-1)\lambda_3).
\end{aligned} \tag{4.4}$$

Now (4.4) yields

$$\begin{aligned}
(\rho_1 - \rho_2)(\rho_1 - \rho_3)(\rho_2 - \rho_3) = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)((p-1)\lambda_2 + p\lambda_3)\\
(p\lambda_2 + (p-1)\lambda_3)(\lambda_1 + (p-1)(\lambda_2 + \lambda_3)).
\end{aligned} \tag{4.5}$$

Proposition 4.1. *Let $M$ be a hypersurface in $\mathbb{E}^{n+1}$, $n = 2p+1 \geqslant 5$, having at every point of $\mathcal{U}_H \subset M$ three distinct principal curvatures $\lambda_1$, $\lambda_2$ and $\lambda_3$ of multiplicity* 1, *$p$ and $p$, respectively. We have*

(i) *The Ricci tensor $S$ of $M$ has at a point $x \in \mathcal{U}_H$ three distinct eigenvalues $\rho_1$, $\rho_2$ and $\rho_3$ if and only if at this point we have*

$$((p-1)\lambda_2 + p\lambda_3)(p\lambda_2 + (p-1)\lambda_3)(\lambda_1 + (p-1)(\lambda_2 + \lambda_3)) \neq 0.$$

(ii) *If the Ricci tensor $S$ of $M$ has at a point $x \in \mathcal{U}_H$ three distinct eigenvalues $\rho_1$, $\rho_2$ and $\rho_3$, then $\gamma_1$, defined by* (3.13)*, is nonzero at this point, and in a consequence $H = \gamma_1^{-1}(S^2 - \gamma_2 S - \gamma_0 g)$.*

Proof. (i) follows immediately from (4.5).

(ii) Suppose that $\gamma_1 = 0$ at $x$. Then from (3.9) it follows that $S$ has at $x$ only two distinct eigenvalues, a contradiction. □

The above results, together with (3.3), Lemma 2.1 and Proposition 3.2, imply

Theorem 4.1. *Let $M$ be a hypersurface in $\mathbb{E}^{n+1}$, $n = 2p + 1 \geqslant 5$, having at every point of $\mathcal{U}_H \subset M$ three distinct principal curvatures $\lambda_1$, $\lambda_2$ and $\lambda_3$ of multiplicity* 1*, $p$ and $p$, respectively. Let $\mathcal{U} \subset \mathcal{U}_H$ be the set of all points at which Ricci tensor $S$ of $M$ has three distinct eigenvalues $\rho_1$, $\rho_2$ and $\rho_3$. Then on this set we have*

$$R = \frac{1}{2}\gamma_1^{-2}(S^2 - \gamma_2 S - \gamma_0 g) \wedge (S^2 - \gamma_2 S - \gamma_0 g), \tag{4.6}$$

$$\begin{aligned}(n-2)R \cdot C = {} & (n-2)Q(S,R) + \frac{\rho}{\gamma_1}Q(S^2, G)\\ & + \Big(\beta_1 - \frac{\rho\gamma_2}{\gamma_1}\Big)Q(S,G) + \frac{\beta_1}{\gamma_1} g \wedge Q(S, S^2),\end{aligned} \tag{4.7}$$

$$\begin{aligned}(n-2)C \cdot R = {} & (n-3)Q(S,R) + \Big(\frac{\kappa}{n-1} + \varepsilon\psi + \beta_1 \operatorname{tr}(H)\Big)Q(g,R)\\ & + \frac{\beta_1\gamma_0}{\gamma_1}Q(S,G) - \frac{\beta_1\gamma_2}{\gamma_1}Q\Big(g, \frac{1}{2}S \wedge S\Big) - \frac{\beta_1}{\gamma_1}S^2 \wedge Q(g,S),\end{aligned} \tag{4.8}$$

$$\begin{aligned}(n-2)C \cdot C = {} & (n-3)Q(S,R) + \beta_5 Q(g,R)\\ & + \Big(\beta_2 - \frac{2\beta_1\gamma_0 + \beta_4\gamma_2}{\gamma_1}\Big)Q(S,G)\\ & + \frac{\beta_1\gamma_2}{\gamma_1}Q\Big(g, \frac{1}{2}S \wedge S\Big) + \frac{\beta_1}{\gamma_1}Q(S, g \wedge S^2) + \frac{\beta_4}{\gamma_1}Q(S^2, G).\end{aligned} \tag{4.9}$$

Remark 4.1. Let $M$ be the hypersurface considered in Theorem 4.1. By making use of (4.6) we state that the curvature tensor $R$ of $M$ is expressed on $\mathcal{U}_H \subset M$ by a linear combination of the Tachibana tensors:

$$G = \frac{1}{2}g \wedge g, \quad g \wedge S, \quad g \wedge S^2, \quad S \wedge S^2, \quad \overline{S} = \frac{1}{2}S \wedge S, \quad \overline{S}^2 = \frac{1}{2}S^2 \wedge S^2.$$

Results on hypersurfaces in $N_s^{n+1}(c)$, $n \geqslant 4$, with the curvature tensor $R$ having the above property are given in [**21**] and [**52**]. Hypersurfaces in $N_s^{n+1}(c)$, $n \geqslant 4$, with the curvature tensor $R$ which is expressed by a linear combination of the tensors $g \wedge g$, $g \wedge S$ and $S \wedge S$ were investigated in [**35**]. For instance, the Clifford torus $S^p\big(\sqrt{p/n}\big) \times S^{n-p}\big(\sqrt{(n-p)/n}\big)$, $2 \leqslant p \leqslant n-2$, $n \neq 2p$, has this property [**35**, Corollary 3.1]. We also mention that semi-Riemannian manifolds with the curvature tensor $R$ expressed by a linear combination of the tensors $g \wedge g$, $g \wedge S$ and $S \wedge S$ were introduced and investigated in [**10**]. For further results on this class of manifolds we refer to [**13, 17, 19, 20, 24, 27, 29, 31, 36, 42**].

## 5. Example

EXAMPLE 5.1. (i)(cf. [**48**, Section 2], [**49**, Section 2]) Let $\alpha_1 = \alpha_1(t)$ and $\alpha_2 = \alpha_2(t)$ be positive smooth functions defined on an interval $I = (0; t_0) \subset \mathbb{R}$, $t_0 > 0$, such that $\alpha_1' \neq 0$ and $\alpha_2' \neq 0$ for every $t \in I$, where $\alpha_1' = \frac{d\alpha_1}{dt}$ and $\alpha_2' = \frac{d\alpha_2}{dt}$. Let $x = x(t, u^1, \dots, u^p, v^1, \dots, v^q)$ be a parametric expression of a subset $M$ of an $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$, $n = p + q + 1$, $p \geqslant 2$, $q \geqslant 2$, defined by

$$x = \alpha_1 F_1 + \alpha_2 F_2, \tag{5.1}$$

$$F_1 = (\cos u^1, \sin u^1 \cos u^1, \dots, \sin u^1 \dots \sin u^{p-1} \cos u^p, \sin u^1 \dots \sin u^p, 0, \dots, 0),$$

$$F_2 = (0, \dots, 0, \cos v^1, \sin v^1 \cos v^1, \dots, \sin v^1 \dots \sin v^{q-1} \cos v^q, \sin v^1 \dots \sin v^q),$$

where $u^1, \dots, u^p, v^1, \dots, v^q \in (0, \frac{\pi}{2})$ and 0 occurs $(q+1)$-and $(p+1)$-times, respectively. We set

$$\xi = \beta\big(-\alpha_2' F_1 + \alpha_1' F_2\big), \quad \beta^{-1} = \sqrt{{\alpha_1'}^2 + {\alpha_2'}^2}. \tag{5.2}$$

Further, we have $\langle F_1, F_1 \rangle = \langle F_2, F_2 \rangle = \langle \xi, \xi \rangle = 1$, where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product of $\mathbb{E}^{n+1}$. Differentiating (5.1) we obtain

$$x_t' = \frac{\partial x}{\partial t} = \alpha_1' F_1 + \alpha_2' F_2,$$

$$x_k' = \frac{\partial x}{\partial u^k} = \alpha_1 \frac{\partial F_1}{\partial u^k} = \alpha_1 F_{1k}', \quad x_l' = \frac{\partial x}{\partial v^l} = \alpha_2 \frac{\partial F_2}{\partial v^l} = \alpha_2 F_{2l}', \tag{5.3}$$

where $k \in \{1, \dots, p\}$ and $l \in \{p+1, \dots, p+q\}$. Using (5.2) and (5.3) we can easy check that

$$\langle \xi, x_t' \rangle = \langle \xi, x_k' \rangle = \langle \xi, x_l' \rangle = 0. \tag{5.4}$$

We assume that at $x$ we have

$$\mu_0 x_t' + \mu_1 x_1' + \dots + \mu_p x_p' + \mu_{p+1} x_{p+1}' + \dots + \mu_{p+q} x_{p+q}' = 0,$$

where $\mu_0, \dots, \mu_{p+q} \in \mathbb{R}$. The last relation, by (5.3), turns into

$$\alpha_1\Big(\frac{\alpha_1' \mu_0}{\alpha_1} F_1 + \mu_1 F_{11}' + \dots + \mu_p F_{1p}'\Big) + \alpha_2\Big(\frac{\alpha_2' \mu_0}{\alpha_2} F_2 + \mu_{p+1} F_{2p+1}' + \dots + \mu_{p+q} F_{2p+q}'\Big) = 0.$$

This and the definitions of $F_1$ and $F_2$ lead to

$$\frac{\alpha_1' \mu_0}{\alpha_1} F_1 + \mu_1 F_{11}' + \dots + \mu_p F_{1p}' = 0,$$

$$\frac{\alpha_2' \mu_0}{\alpha_2} F_2 + \mu_{p+1} F_{2p+1}' + \dots + \mu_{p+q} F_{2p+q}' = 0.$$

Since the vectors $F_1, F_{11}', \dots, F_{1p}'$, resp. $F_2, F_{2p+1}', \dots, F_{2p+q}'$ are linearly independent vectors (see, e.g., [**38**, Example 2, pp.329–331]) at $x$ we have

$$\frac{\alpha_1' \mu_0}{\alpha_1} = \mu_1 = \dots = \mu_p = 0, \quad \frac{\alpha_2' \mu_0}{\alpha_2} = \mu_{p+1} = \dots = \mu_{p+q} = 0.$$

Thus the vectors $x'_t,\ x'_1, \dots, x'_p,\ x'_{p+1}, \dots, x'_{p+q}$ are linearly independent at every point of $M$. Therefore we can state that $M$ is immersed isometrically in $\mathbb{E}^{n+1}$. In addition, from (5.4) it follows that $\xi$ is the unit normal vector field of $M$. Further, differentiating (5.2) we obtain

$$\xi'_t = \frac{\partial \xi}{\partial t} = -(\alpha'_2\beta)' F_1 + (\alpha'_1\beta)' F_2,$$

$$\xi'_k = \frac{\partial \xi}{\partial u^k} = -\alpha'_2\beta F_{1k}, \quad \xi'_l = \frac{\partial \xi}{\partial v^l} = \alpha'_1\beta F_{2l}, \tag{5.5}$$

where $\alpha''_1 = \frac{d\alpha'_1}{dt}$ and $\alpha''_2 = \frac{d\alpha'_2}{dt}$. From (5.3) and (5.5) we obtain the Weingarten formula for $M$

$$\xi'_t = (\alpha''_1\alpha'_2 - \alpha'_1\alpha''_2)\beta^3 x'_t = \frac{\alpha''_1\alpha'_2 - \alpha'_1\alpha''_2}{{\alpha'_1}^2 + {\alpha'_2}^2}\beta x'_t,$$

$$\xi'_k = -\alpha_1^{-1}\alpha'_2\beta x'_k, \quad \xi'_l = \alpha_2^{-1}\alpha'_1\beta x'_l.$$

Thus we have

$$\lambda_1 = (\alpha'_1\alpha''_2 - \alpha''_1\alpha'_2)\beta^3, \quad \lambda_2 = \alpha_1^{-1}\alpha'_2\beta, \quad \lambda_3 = -\alpha_2^{-1}\alpha'_1\beta.$$

(ii) It is easy to see that if at every point of $M$ we have

$$(p-1)\lambda_2 = -(q-1)\lambda_3 \tag{5.6}$$

then the second fundamental tensor $H$ of $M$ satisfies (1.4) on $\mathcal{U}_H \subset M$. Evidently, (5.6) yields $(p-1)\alpha_2\alpha'_2 = (q-1)\alpha_1\alpha'_1$, which is equivalent to

$$\alpha_2 = \sqrt{c + \frac{q-1}{p-1}\alpha_1^2},$$

where $c$ is a constant. Note that from (4.1) and (5.6) we get easily

$$\begin{aligned}\operatorname{tr}(H) &= \lambda_1 + p\lambda_2 + q\lambda_3 \\ &= \lambda_1 + \lambda_2 + \lambda_3 + (p-1)\lambda_2 + (q-1)\lambda_3 = \lambda_1 + \lambda_2 + \lambda_3 = \phi.\end{aligned}$$

Thus (1.3) turns into (1.4).

(iii) We consider the case: $p = q \geqslant 2$. Now (5.6) gives $\lambda_2 = -\lambda_3$. Thus (4.1)–(4.4) and (3.4) yield

$$\phi = \lambda_1, \quad \psi = \lambda_2^2, \quad \rho = -\lambda_1\lambda_2^2,$$

$$\operatorname{tr}(H) = \lambda_1, \quad \operatorname{tr}(H^2) = \lambda_1^2 + (n-1)\lambda_2^2,$$

$$\beta_1 = \beta_2 = \beta_5 = 0, \quad \beta_3 = \beta_4 = \frac{n-3}{n-2}\rho,$$

$$\gamma_0 = \lambda_1^2\lambda_2^2, \quad \gamma_1 = -\lambda_1\lambda_2^2, \quad \gamma_2 = -\lambda_2^2,$$

$$\rho_1 = 0, \quad \rho_2 = \lambda_2(\lambda_1 - \lambda_2), \quad \rho_3 = -\lambda_2(\lambda_1 + \lambda_2),$$

$$\kappa = \operatorname{tr}(H))^2 - \operatorname{tr}(H^2) = -(n-1)\lambda_2^2 = -(n-1)\psi.$$

We also have

$$S^3 = \frac{2\kappa}{n-1} S^2 - \frac{\kappa}{n-1}\Big(\frac{\kappa}{n-1} + (\mathrm{tr}(H))^2\Big) S,$$
$$(\mathrm{tr}(H))^2 = -\frac{(n-1)\,\mathrm{tr}(S^3)}{\kappa^2} + \frac{2\,\mathrm{tr}(S^2)}{\kappa} - \frac{\kappa}{n-1}.$$

Conditions (4.7)–(4.9), by making use of the above presented formulas, turn into

$$R \cdot C = Q(S,R) + \frac{1}{n-2} Q\Big(S^2 - \frac{\kappa}{n-1} S, G\Big), \tag{5.7}$$

$$C \cdot R = \frac{n-3}{n-2} Q(S,R), \tag{5.8}$$

$$C \cdot C = \frac{n-3}{n-2}\bigg(Q(S,R) + \frac{1}{n-2} Q\Big(S^2 - \frac{\kappa}{n-1} S, G\Big)\bigg), \tag{5.9}$$

respectively. From (5.7) and (5.8) we get immediately

$$(n-2)(R \cdot C - C \cdot R) = Q(S,R) + \frac{1}{n-2} Q\Big(S^2 - \frac{\kappa}{n-1} S, G\Big). \tag{5.10}$$

Thus the difference tensor $R \cdot C - C \cdot R$ is expressed by a linear combination of some Tachibana tensors. We mention that hypersurfaces in spaces of of constant curvature with the tensor $R \cdot C - C \cdot R$ expressed by a linear combination of certain Tachibana tensors were investigated among others in [**16**, **22**, **26**, **28**, **51**]. We also note that (5.9) and (5.10) yield $(n-3)(R \cdot C - C \cdot R) = C \cdot C$. Thus the difference tensor $R \cdot C - C \cdot R$ of $M$ is a conformal invariant.

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