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FACTORIZED DIFFERENCE SCHEME FOR TWO-DIMENSIONAL SUBDIFFUSION EQUATION IN NONHOMOGENEOUS MEDIA

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Abstract. A factorized finite-difference scheme for numerical approximation of initial-boundary value problem for two-dimensional subdiffusion equation in nonhomogeneous media is proposed. Its stability and convergence are investigated. The corresponding error bounds are obtained.

1. Introduction

Fractional partial differential equations have broad applications in mathematics, sciences and engineering (see [**9, 15, 16**]), such as anomalous transport in disordered systems, processes in viscoelastic and porous media, biological and social phenomena etc. The important characteristic of fractional differential equations is their non-local property. This means that the next state of the system depends not only upon its current state but also upon all of its previous states. Thus the fractional-order models are more realistic and it is the main reason of their popularity. On the other side, this feature makes the design of accurate and fast numerical methods difficult.

In this paper we consider the first initial-boundary value problem for two-dimensional fractional in time diffusion equation with variable coefficients. This equation is commonly called subdiffusion equation. The problem is approximated by factorized finite difference scheme, which belongs to the type of alternating direction implicit (ADI) schemes (see [**17**]). This scheme combines the advantages of explicit and implicit scheme–efficiency and stability. Analogous result for the problem with constant coefficients is obtained in [**6**]. In [**7**] the same problem is approximated by additive scheme–another type of ADI schemes. In [**3, 11**] multidimensional evolution equations with fractional in space derivatives are approximated using diverse ADI schemes.

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The paper is organized as follows. In Section 2 we introduce Riemann–Liouville and Caputo fractional derivatives. In Section 3 we define the first initial-boundary value problem for two-dimensional fractional in time diffusion equation and prove existence and uniqueness of its solution. In Section 4 we define factorized difference scheme approximating considered problem and prove its stability. In Section 5 we investigate the convergence of the proposed factorized difference scheme.

2. Fractional derivatives

Let u be a function defined on interval $[a,b]$ and $k-1\leqslant\sigma<k$, $k\in\mathbb{N}$. The left Riemann–Liouville fractional derivative of order σ is defined as

$$D^{\sigma}_{a+}u(t)=\frac{1}{\Gamma(k-\sigma)}\frac{d^k}{dt^k}\int_a^t\frac{u(s)}{(t-s)^{\sigma+1-k}}\,ds,\quad t\geqslant a, \tag{2.1}$$

where the $\Gamma(\cdot)$ denotes the Gamma function. The right Riemann–Liouville fractional derivative is defined analogously

$$D^{\sigma}_{b-}u(t)=\frac{(-1)^k}{\Gamma(k-\sigma)}\frac{d^k}{dt^k}\int_t^b\frac{u(s)}{(s-t)^{\sigma+1-k}}\,ds,\quad t\leqslant b.$$

For $\sigma=k-1$ from (2.1) immediately follows that $D^{k-1}_{a+}u(t)=u^{(k-1)}(t)$. Moreover, under some natural assumptions [**16**], we have $\lim_{\sigma\to k}D^{\sigma}_{a+}u(t)=u^{(k)}(t)$.

Commuting the derivative and integral in definition (2.1) one obtains the so-called Caputo fractional derivative

$${}^{C}D^{\sigma}_{a+}u(t)=\frac{1}{\Gamma(k-\sigma)}\int_a^t\frac{u^{(k)}(s)}{(t-s)^{\sigma+1-k}}\,ds.$$

The two definitions are linked by the following relationship

$$D^{\sigma}_{a+}u(t)={}^{C}D^{\sigma}_{a+}u(t)+\sum_{j=0}^{k-1}u^{(j)}(a)\frac{(x-a)^{j-\sigma}}{\Gamma(j-\sigma+1)}.$$

In particular, $D^{\sigma}_{a+}u(t)={}^{C}D^{\sigma}_{a+}u(t)$ if $u(a)=u'(a)=\cdots=u^{(k-1)}(a)=0$.

Unlike classical derivatives, fractional derivatives satisfy the semigroup property under certain additional assumptions [**16**], for example for continuous functions:

$$D^{\sigma}_{a+}D^{\varrho}_{a+}u(t)=D^{\sigma+\varrho}_{a+}u(t)\ \text{if}\ 0<\sigma,\ \varrho<1,\quad u(a)=0. \tag{2.2}$$

Let $0<\sigma<1$, and let $u(t)$ and $v(t)$ be continuously differentiable functions. Then, using the relationship between the Riemann–Liouville and Caputo fractional derivatives one easily obtains

$$(D^{\sigma}_{a+}u,v)_{L^2(a,b)}=(u,D^{\sigma}_{b-}v)_{L^2(a,b)}. \tag{2.3}$$

Let us mention another result that will be used in the sequel. Let $\sigma>0$ and let u be infinitely differentiable function in $\mathbb{R}$, with $\operatorname{supp}u\subset(a,b)$. Then (see [**5**]):

$$(D^{\sigma}_{a+}u,D^{\sigma}_{b-}u)_{L^2(a,b)}=\cos\pi\sigma\|D^{\sigma}_{a+}u\|^2_{L^2(a,+\infty)}. \tag{2.4}$$

For the functions of many variables, the partial fractional derivatives are defined in an analogous manner, for example

$$D^{\sigma}_{t,a+}u(x,t) = \frac{1}{\Gamma(k-\sigma)}\frac{\partial^k}{\partial t^k}\int_a^t \frac{u(x,s)}{(t-s)^{\sigma+1-k}}ds, \quad k-1<\sigma<k, \quad k\in\mathbb{N}.$$

3. Problem formulation

We shall consider the time fractional diffusion equation

$$D^{\alpha}_{t,0+}u + \mathcal{L}u = f(x,t), \quad 0<\alpha<1, \quad x=(x_1,x_2)\in\Omega, \quad t\in(0,T), \tag{3.1}$$

where $\Omega=(0,1)\times(0,1)$, $Q=\Omega\times(0,T)$ and

$$\mathcal{L}u = -\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) + au,$$

are subject to homogeneous boundary and initial conditions

$$u(x,t)=0, \qquad x\in\partial\Omega, \qquad t\in(0,T), \tag{3.2}$$

$$u(x,0)=0, \qquad x\in\bar{\Omega}. \tag{3.3}$$

We assume that the coefficients of the differential operator $\mathcal{L}u$ satisfy the standard ellipticity assumptions

$$\begin{gathered} a_{ij},\, a\in L^{\infty}(\Omega), \qquad a\geqslant 0, \qquad a_{ij}=a_{ji}, \\ \sum_{i,j=1}^{2}a_{ij}\xi_i\xi_j \geqslant c_0\sum_{i=1}^{2}\xi_i^2, \qquad x\in\Omega, \qquad \xi=(\xi_1,\xi_2)\in\mathbb{R}^2, \quad c_0>0. \end{gathered} \tag{3.4}$$

Initial-boundary value problem (3.1)–(3.3) is often called sub-diffusion problem.

With $C^k(\Omega)$ and $C^k(\bar{\Omega})$ we denote the spaces of k-fold differentiable functions in Ω and $\bar{\Omega}$, respectively. In particular, $\dot{C}^{\infty}(\Omega)=C_0^{\infty}(\Omega)$ stand for the space of infinitely differentiable functions with compact support in Ω. As usual, the space of measurable functions whose squares are Lebesgue integrable in Ω is denoted by $L^2(\Omega)$. We also use $W^{\sigma,p}(\Omega)$ to denote the Sobolev spaces [**1**]. By $\dot{W}^{\sigma,p}(\Omega)=W_0^{\sigma,p}(\Omega)$ we denote the closure of $\dot{C}^{\infty}(\Omega)$ with the respect to the norm of $W^{\sigma,p}(\Omega)$. In particular, for $p=2$ we set $H^{\sigma}(\Omega)=W^{\sigma,2}(\Omega)$ and $\dot{H}^{\sigma}(\Omega)=\dot{W}^{\sigma,2}(\Omega)$.

For $\sigma>0$ we set

$$|u|_{C^{\sigma}_{+}[a,b]} = \|D^{\sigma}_{a+}u\|_{C[a,b]}, \quad |u|_{C^{\sigma}_{-}[a,b]} = \|D^{\sigma}_{b-}u\|_{C[a,b]},$$

$$\|u\|_{C^{\sigma}_{\pm}[a,b]} = \big(\|u\|^2_{C^{[\sigma]^-}[a,b]} + |u|^2_{C^{\sigma}_{\pm}[a,b]}\big)^{1/2},$$

$$|u|_{H^{\sigma}_{+}(a,b)} = \|D^{\sigma}_{a+}u\|_{L^2(a,b)}, \quad |u|_{H^{\sigma}_{-}(a,b)} = \|D^{\sigma}_{b-}u\|_{L^2(a,b)},$$

$$\|u\|_{H^{\sigma}_{\pm}(a,b)} = \big(\|u\|^2_{H^{[\sigma]^-}(a,b)} + |u|^2_{H^{\sigma}_{\pm}(a,b)}\big)^{1/2},$$

where $[\sigma]^-$ denotes the largest integer $<\sigma$. Then we define $C^{\sigma}_{\pm}[a,b]$ as the space of functions $u\in C^{[\sigma]^-}[a,b]$ with the finite norms $\|u\|_{C^{\sigma}_{\pm}[a,b]}$. The space $H^{\sigma}_{\pm}(a,b)$ is defined analogously, while the space $\dot{H}^{\sigma}_{\pm}(a,b)$ is defined as the closure of $\dot{C}^{\infty}(a,b)$ with respect to the norm $\|\cdot\|_{H^{\sigma}_{\pm}(a,b)}$. Because for $\sigma=k\in\mathbb{N}\cup\{0\}$ fractional derivative reduces to standard k-th derivative, we have $C^k_{\pm}[a,b] = C^k[a,b]$ and $H^k_{\pm}(a,b) = H^k(a,b)$.

The following result holds:

LEMMA 3.1. (See [**12**]) *For* $\sigma > 0$, $\sigma \neq k + 1/2$, $k \in \mathbb{N}$, *the spaces* $\dot{H}^{\sigma}_{+}(a,b)$, $\dot{H}^{\sigma}_{-}(a,b)$ *and* $\dot{H}^{\sigma}(a,b)$ *are equal and their norms are equivalent.*

For the vector valued functions mapping real interval $[0,T]$ or $(0,T)$ into Banach space X we introduce the spaces $C^k([0,T],X)$, $k \in \mathbb{N} \cup \{0\}$ and $H^{\sigma}((0,T),X)$, $\sigma \geqslant 0$, in the usual way [**13**]. In analogous manner we define the spaces $C^{\sigma}_{\pm}([0,T],X)$ and $H^{\sigma}_{\pm}((0,T),X)$.

Taking the inner product of equation (3.1) with test function v and using partial integration and relations (2.2) and (2.3) one obtains the following weak formulation of the problem (3.1)–(3.3): find $u \in \dot{H}^{1,\alpha/2}(Q) = L^2((0,T),\dot{H}^1(\Omega)) \cap \dot{H}^{\alpha/2}((0,T),L^2(\Omega))$ such that

$$a(u,v) = l(v), \quad \forall v \in \dot{H}^{1,\alpha/2}(Q),$$

where

$$a(u,v) = \big(D^{\alpha/2}_{t,0+}u, D^{\alpha/2}_{t,T-}v\big)_{L^2(Q)} + \sum_{i,j=1}^{2}\Big(a_{ij}\frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i}\Big)_{L^2(Q)} + (au,v)_{L^2(Q)}$$

and

$$l(v) = (f,v)_{L^2(Q)}.$$

THEOREM 3.1. *Let* $\alpha \in (0,1)$, $f \in L^2(Q)$ *and let the assumptions* (3.4) *hold. Then the problem* (3.1)–(3.3) *is well posed in* $\dot{H}^{1,\alpha/2}(Q)$ *and its weak solution satisfies a priori estimate*

$$\|u\|_{H^{1,\alpha/2}(Q)} \leqslant C\|f\|_{L^2(Q)}.$$

The proof follows immediately using relation (2.4) and the Lax–Milgram lemma.

REMARK 3.1. From Theorem 3.1 immediately follows a priori estimate

$$\|u\|_{B^{1,\alpha/2}(Q)} \leqslant C\|f\|_{L^2(Q)} \tag{3.5}$$

in weaker norm [**13**]

$$\|u\|^2_{B^{1,\alpha/2}(Q)} = \int_0^T \big[(T-t)^{-\alpha}\|u(\cdot,t)\|^2_{L^2(\Omega)} + \|u(\cdot,t)\|^2_{H^1(\Omega)}\big]dt.$$

THEOREM 3.2. *Let the assumptions of Theorem 3.1 hold and let* $a_{ij} \in W^{1,\infty}(\Omega)$. *Then solution of the problem* (3.1)–(3.3) *belongs to the space* $H^{2,\alpha}_{+}(Q) \cap \dot{H}^{1,\alpha/2}(Q)$ *and the a priori estimate holds*

$$\|u\|_{H^{2,\alpha}_{+}(Q)} \leqslant C\|f\|_{L^2(Q)}.$$

The proof follows taking the inner product of (3.1) with $\mathcal{L}u$, using relation (2.4) and the so-called 'second fundamental inequality' [**10**]

$$\|u\|^2_{H^2(\Omega)} \leqslant C\big(\|\mathcal{L}u\|^2_{L^2(\Omega)} + \|u\|^2_{H^1(\Omega)}\big).$$

In the sequel, we shall assume that a_{ij}, $a \in C(\bar{\Omega})$ instead of $L^\infty(\Omega)$. Through the paper by C we shall denote the positive generic constant which may take different values in different formulas.

4. Finite difference approximation

In the area $\bar{Q} = [0,1]\times[0,1]\times[0,T]$, we define the uniform mesh $\bar{Q}_{h\tau} = \bar{\omega}_h \times \bar{\omega}_\tau$, where $\bar{\omega}_h = \{x = (n_1 h, n_2 h) \mid n_1, n_2 = 0,1,...,N;\ h = 1/N\}$ and $\bar{\omega}_\tau = \{t = t_m = m\tau \,|\, m = 0,1,\ldots,M;\ \tau = T/M\}$. We also define $\omega_h = \bar{\omega}_h \cap \Omega$, $\gamma_h = \bar{\omega}_h \setminus \omega_h$, $\omega_{0h} = \bar{\omega}_h \cap (0,1] \times (0,1]$, $\omega_{1h} = \bar{\omega}_h \cap (0,1] \times (0,1)$, $\omega_{2h} = \bar{\omega}_h \cap (0,1) \times (0,1]$, $\omega_\tau = \bar{\omega}_\tau \cap (0,T)$, $\omega_\tau^- = \bar{\omega}_\tau \cap [0,T)$ and $\omega_\tau^+ = \bar{\omega}_\tau \cap (0,T]$. We shall use the standard notation from the theory of the finite difference schemes (see [**17**]):

$$v = v(x,t), \quad \hat{v} = v(x,t+\tau), \quad \check{v} = v(x,t-\tau), \quad v^m = v(x,t_m), \quad x \in \bar{\omega}_h,$$

$$v_{x_i} = \frac{v(x+he_i,t) - v(x,t)}{h} = v_{\bar{x}_i}(x+he_i,t), \quad i = 1,2,$$

$$v_t = \frac{v(x,t+\tau) - v(x,t)}{\tau} = v_{\bar{t}}(x,t+\tau) = \hat{v}_{\bar{t}},$$

where e_i denotes the unit vector of the axis $0x_i$.

For a function u defined on $\bar{Q}$ which satisfies zero initial condition, we approximate the left Riemann–Liouville fractional derivative $D^\alpha_{t,0+}u(x,t_m)$, $0 < \alpha < 1$, by (see [**4**]):

$$(D^\alpha_{t,0+,\tau}u)^m = \frac{1}{\Gamma(2-\alpha)} \sum_{l=0}^{m-1} (t^{1-\alpha}_{m-l} - t^{1-\alpha}_{m-l-1}) u^l_t.$$

The following result holds:

LEMMA 4.1. (See [**18**]) *Suppose that* $\alpha \in (0,1)$, $u \in C^2([0,t], C(\bar{\Omega}))$ *and* $t \in \omega_\tau^+$. *Then*

$$|D^\alpha_{t,0+}u - D^\alpha_{t,0+,\tau}u| \leqslant \frac{\tau^{2-\alpha}}{1-\alpha}\left[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - (1+2^{-\alpha})\right] \max_{\bar{Q}_t} \left|\frac{\partial^2 u}{\partial t^2}\right|,$$

where denoted $Q_t = (0,t) \times \Omega$.

We approximate initial-boundary value problem (3.1)–(3.3) with the following factorized finite difference scheme:

$$\big((I+\theta\tau^\alpha A_1)(I+\theta\tau^\alpha A_2) D^\alpha_{t,0+,\tau} v\big)^m + \mathcal{L}_h v^{m-1} = \bar{f}^m, \quad x \in \omega_h, \tag{4.1}$$

$m = 1,2,\ldots,M,$ subject to zero boundary and initial conditions:

$$v(x,t) = 0, \quad (x,t) \in \gamma_h \times \omega_\tau^+, \tag{4.2}$$

$$v(x,0) = 0, \quad x \in \bar{\omega}_h, \tag{4.3}$$

where I is the identity operator, θ is positive parameter, $A_i v = -v_{x_i \bar{x}_i}$, $i = 1,2$, and

$$\mathcal{L}_h v = -\frac{1}{2} \sum_{i,j=1}^{2} \left[(a_{ij} v_{x_j})_{\bar{x}_i} + (a_{ij} v_{\bar{x}_j})_{x_i}\right] + av.$$

When the right-hand side f is a continuous function, we set $\bar{f} = f$, otherwise we must use some averaged value, for example $\bar{f} = T_1 T_2 f$, where T_i are Steklov averaging operators:

$$T_i f(x,t) = \int_{-1/2}^{1/2} f(x + hse_i, t)\, ds, \quad i = 1, 2.$$

Let us note, that the finite difference scheme (4.1)–(4.3) is numerically efficient, unlike the standard implicit scheme [**19**]. Indeed, to compute the values of solution v at the time layer $t = t_m$ it is necessary to invert the operators $(I + \theta\tau^\alpha A_1)$ and $(I + \theta\tau^\alpha A_2)$. By suitable ordering of mesh nodes in ω_h each of these operators can be represented by a tridiagonal matrix. In this way, the required values of solution are obtained by two applications of the Thomas algorithm.

We define the following discrete inner products and norms:

$$(v,w)_h = (v,w)_{L^2(\omega_h)} = h^2 \sum_{x\in\omega_h} vw, \quad \|v\|_h = \|v\|_{L^2(\omega_h)} = (v,v)_h^{1/2},$$

$$(v,w)_{ih} = (v,w)_{L^2(\omega_{ih})} = h^2 \sum_{x\in\omega_{ih}} vw, \quad \|v\|_{ih} = \|v\|_{L^2(\omega_{ih})} = (v,v)_{ih}^{1/2}, \quad i = 0,1,2,$$

$$|v|^2_{H^1(\omega_h)} = \sum_{i=1}^{2} \|v_{\bar{x}_i}\|^2_{ih}, \quad \|v\|^2_{H^1(\omega_h)} = |v|^2_{H^1(\omega_h)} + \|v\|^2_h,$$

$$|v|^2_{H^2(\omega_h)} = \sum_{i=1}^{2} \|v_{\bar{x}_i x_i}\|^2_h + 2\|v_{\bar{x}_1\bar{x}_2}\|^2_{0h}, \quad \|v\|^2_{H^2(\omega_h)} = |v|^2_{H^2(\omega_h)} + \|v\|^2_{H^1(\omega_h)},$$

$$\|v\|^2_{L^2(Q_{h\tau})} = \tau \sum_{m=1}^{M} \|v^m\|^2_h, \quad \|v\|^2_{L^2(Q_{ih\tau})} = \tau \sum_{m=1}^{M} \|v^m\|^2_{ih}, \quad i = 0,1,2,$$

$$\|v\|^2_{B^{1,\alpha/2}(Q_{h\tau})} = \tau \sum_{m=1}^{M} \Big[\big(D^\alpha_{t,0+,\tau}(\|v\|^2_h)\big)^m + \|v^m\|^2_{H^1(\omega_h)} \Big],$$

$$\|v\|^2_{H^{2,\alpha}_+(Q_{h\tau})} = \tau \sum_{m=1}^{M} \Big[\|(D^\alpha_{t,0+,\tau} v)^m\|^2_h + \|v^m\|^2_{H^2(\omega_h)} \Big].$$

For $\alpha \in (0,1)$ and every function $v(t)$ defined on the mesh $\bar{\omega}_\tau$, which satisfies the initial condition $v(0) = 0$, the following equality is valid (see [**4**])

$$\tau \sum_{m=1}^{M} \big(D^\alpha_{t,0+,\tau}(v^2)\big)^m = \frac{1}{\Gamma(2-\alpha)} \sum_{m=1}^{M} \big(t^{1-\alpha}_{M-m+1} - t^{1-\alpha}_{M-m}\big)(v^m)^2. \tag{4.4}$$

In particular, from here follows that the norm $\|v\|_{B^{1,\alpha/2}(Q_{h\tau})}$ is well defined.

LEMMA 4.2. (See [**7**]) *For $0 < \alpha < 1$ and any function $v(t)$ defined on the mesh $\bar{\omega}_\tau$ the following inequality is valid*

$$v^m (D^\alpha_{t,0+,\tau} v)^m \geqslant \frac{1}{2}(D^\alpha_{t,0+,\tau}(v^2))^m + \frac{\tau^{2-\alpha}(1-2^{-\alpha})}{\Gamma(2-\alpha)}(v_t^{m-1})^2, \tag{4.5}$$

for $\quad m = 1, 2, \ldots, M.$

THEOREM 4.1. *Let $\alpha \in (0,1)$ and $\theta \geqslant \frac{\Gamma(2-\alpha)}{1-2^{-\alpha}} \max_{ij} \|a_{ij}\|_{C(\bar\Omega)}$. Then for sufficiently small τ finite difference scheme* (4.1)–(4.3) *is absolutely stable and its solution satisfies the following a priori estimate:*

$$\|v\|_{B^{1,\alpha/2}(Q_{h\tau})} \leqslant C\|\bar f\|_{L^2(Q_{h\tau})}. \tag{4.6}$$

PROOF. Taking the inner product of (4.1) with v^m, we obtain

$$\big(v^m, BD^\alpha_{t,0+,\tau} v^m\big)_h + \big(v^m, \mathcal{L}_h v^m\big)_h - \big(v^m, \mathcal{L}_h(v^m - v^{m-1})\big)_h = \big(v^m, \bar f^m\big)_h,$$

where denoted $B = (I + \theta\tau^\alpha A_1)(I + \theta\tau^\alpha A_2)$. The operators $\mathcal{L}_h$ and B are positive and selfadjoint, so the corresponding energy norms (see [**17**]) $\|v\|_{\mathcal{L}_h} = (\mathcal{L}_h v, v)_h^{1/2}$ and $\|v\|_B = (Bv, v)_h^{1/2}$ are well defined.

Using inequality (4.5), Cauchy–Schwarz and ε inequalities and taking into account that $v^m = \frac12(v^m + v^{m-1}) + \frac12(v^m - v^{m-1})$, we obtain

$$\frac12 D^\alpha_{t,0+,\tau}\|v^m\|_B^2 + \frac{\tau^{2-\alpha}(1-2^{-\alpha})}{\Gamma(2-\alpha)}\|v_t^{m-1}\|_B^2 + \frac12\|v^m\|^2_{\mathcal{L}_h} + \frac12\|v^{m-1}\|^2_{\mathcal{L}_h} - \frac{\tau^2}{2}\|v_t^{m-1}\|^2_{\mathcal{L}_h} \leqslant \frac{1}{4\varepsilon}\|\bar f^m\|_h^2 + \varepsilon\|v^m\|_h^2, \quad \forall\varepsilon > 0.$$

Further we have

$$\|v\|_B^2 = \|v\|_h^2 + \theta\tau^\alpha\|v\|^2_{A_1+A_2} + \theta^2\tau^{2\alpha}\|v\|^2_{A_1A_2} = \|v\|_h^2 + \theta\tau^\alpha|v|^2_{H^1(\omega_h)} + \theta^2\tau^{2\alpha}\|v_{\bar x_1\bar x_2}\|^2_{0h}.$$

Using (3.4) one obtains

$$c_0|v|^2_{H^1(\omega_h)} \leqslant \|v\|^2_{\mathcal{L}_h} \leqslant c_1|v|^2_{H^1(\omega_h)} + c_2\|v\|_h^2,$$

where $c_1 = 2\max_{ij}\|a_{ij}\|_{C(\bar\Omega)}$ and $c_2 = \|a\|_{C(\bar\Omega)}$.

From the last three inequalities it follows

$$\frac12 D^\alpha_{t,0+,\tau}\|v^m\|_h^2 + \Big(\theta\frac{(1-2^{-\alpha})}{\Gamma(2-\alpha)} - \frac{c_1}{2}\Big)\tau^2|v_t^{m-1}|^2_{H^1(\omega_h)} + \Big(\frac{1-2^{-\alpha}}{\Gamma(2-\alpha)} - \frac{c_2}{2}\tau^\alpha\Big)\tau^{2-\alpha}\|v_t^{m-1}\|_h^2 + \frac{c_0}{2}|v^m|^2_{H^1(\omega_h)} + \frac{c_0}{2}|v^{m-1}|^2_{H^1(\omega_h)} \leqslant \frac{1}{4\varepsilon}\|\bar f^m\|_h^2 + \varepsilon\|v^m\|_h^2,$$

whereby for

$$\theta \geqslant \theta_0 = \frac{c_1\Gamma(2-\alpha)}{2(1-2^{-\alpha})} = \frac{\Gamma(2-\alpha)}{1-2^{-\alpha}}\max_{ij}\|a_{ij}\|_{C(\bar\Omega)}$$

and

$$\tau \leqslant \tau_0 = \Big(\frac{2(1-2^{-\alpha})}{c_2\Gamma(2-\alpha)}\Big)^{1/\alpha} = \Big(\frac{2(1-2^{-\alpha})}{\Gamma(2-\alpha)\|a\|_{C(\bar\Omega)}}\Big)^{1/\alpha}$$

one obtains

$$\frac12 D^\alpha_{t,0+,\tau}\|v^m\|_h^2 + \frac{c_0}{2}|v^m|^2_{H^1(\omega_h)} + \frac{c_0}{2}|v^{m-1}|^2_{H^1(\omega_h)} \leqslant \frac{1}{4\varepsilon}\|\bar f^m\|_h^2 + \varepsilon\|v^m\|_h^2.$$

Finally, using the discrete Poincaré inequality (see [**17**])

$$\|v\|_h \leqslant \frac14\,|v|_{H^1(\omega_h)},$$

setting $\varepsilon = 4c_0$ and summing for $m = 1, 2, \ldots, M$, we obtain a priori estimate (4.6) with

$$C = \sqrt{\frac{1}{8c_0 \min\{1, \frac{8c_0}{17}\}}} \qquad \square$$

Note that inequality (4.6) is discrete analogue of a priori estimate (3.5). The discrete version of Theorem 3.2 we shall prove in the special case when $\mathcal{L}_h$ is finite difference approximation of Laplace operator.

THEOREM 4.2. *Let $\alpha \in (0,1)$ and $\theta \geqslant \frac{\Gamma(2-\alpha)}{2(1-2^{-\alpha})}$. Then finite difference scheme (4.1)–(4.3) with $\mathcal{L}_h v = Av = (A_1 + A_2)v = -v_{x_1\bar{x}_1} - v_{x_2\bar{x}_2}$ is absolutely stable in the norm $H_+^{2,\alpha}(Q_{h\tau})$ and its solution satisfies a priori estimate*

$$\|v\|_{H_+^{2,\alpha}(Q_{h\tau})} \leqslant C\|\bar{f}\|_{L^2(Q_{h\tau})}. \tag{4.7}$$

PROOF. Taking the inner product of (4.1) with Av^m, we obtain

$$(Av^m, BD^\alpha_{t,0+,\tau} v^m)_h + (Av^m, Av^m)_h - \big(Av^m, A(v^m - v^{m-1})\big)_h = (Av^m, \bar{f}^m)_h,$$

whereby, analogously as in the proof of Lemma 4.1

$$\frac{1}{2} D^\alpha_{t,0+,\tau}\|v^m\|^2_{AB} + \frac{\tau^{2-\alpha}(1-2^{-\alpha})}{\Gamma(2-\alpha)}\|v_t^{m-1}\|^2_{AB} + \frac{1}{2}\|Av^m\|^2_h + \frac{1}{2}\|Av^{m-1}\|^2_h$$
$$- \frac{\tau^2}{2}\|Av_t^{m-1}\|^2_h \leqslant \frac{1}{4\varepsilon}\|\bar{f}^m\|^2_h + \varepsilon\|Av^m\|^2_h.$$

Further we have

$$\|v\|^2_{AB} \geqslant \|v\|^2_A = |v|^2_{H^1(\omega_h)},$$
$$\|v\|^2_{AB} \geqslant \theta\tau^\alpha\|v\|^2_{A^2},$$
$$\|v\|^2_{A^2} = \|Av\|^2_h = |v|^2_{H^2(\omega_h)}.$$

From the previous inequalities it follows

$$\frac{1}{2} D^\alpha_{t,0+,\tau}|v^m|^2_{H^1(\omega_h)} + \Big(\theta\frac{(1-2^{-\alpha})}{\Gamma(2-\alpha)} - \frac{1}{2}\Big)\tau^2|v_t^{m-1}|^2_{H^2(\omega_h)}$$
$$+ \frac{1}{2}|v^m|^2_{H^2(\omega_h)} + \frac{1}{2}|v^{m-1}|^2_{H^2(\omega_h)} \leqslant \frac{1}{4\varepsilon}\|\bar{f}^m\|^2_h + \varepsilon|v^m|^2_{H^2(\omega_h)},$$

whereby for $\theta \geqslant \frac{\Gamma(2-\alpha)}{2(1-2^{-\alpha})}$, setting $\varepsilon = 1/4$ and summing for $m = 1, 2, \ldots, M$, we obtain

$$2\tau\sum_{m=1}^{M} D^\alpha_{t,0+,\tau}|v^m|^2_{H^1(\omega_h)} + \tau\sum_{m=1}^{M}|v^m|^2_{H^2(\omega_h)} \leqslant 4\|\bar{f}^m\|^2_{L^2(Q_{h\tau})}.$$

The first sum on the left hand side is positive according to (4.4). From equation (4.1) it follows

$$\|(D^\alpha_{t,0+,\tau}v)^m\|_h \leqslant \|\mathcal{L}v^{m-1}\|_h + \|\bar{f}^m\|_h = \|Av^{m-1}\|_h + \|\bar{f}^m\|_h$$
$$= |v^{m-1}|_{H^2(\omega_h)} + \|\bar{f}^m\|_h.$$

Hence,

$$\tau\sum_{m=1}^{M}\|(D^{\alpha}_{t,0+,\tau}v)^m\|_h^2+\tau\sum_{m=1}^{M}|v^m|^2_{H^2(\omega_h)}\leqslant 14\|\bar f^m\|^2_{L^2(Q_{h\tau})}$$

and results follow using equivalence of seminorm $|v|_{H^2(\omega_h)}$ and norm $\|v\|_{H^2(\omega_h)}$ for mesh functions satisfying homogeneous boundary condition $v=0$ on γ_h (see [**17**]):

$$|v|_{H^2(\omega_h)}\leqslant\|v\|_{H^2(\omega_h)}\leqslant\sqrt{1+\frac{1}{4^2}+\frac{1}{16^2}}\,|v|_{H^2(\omega_h)}. \qquad \square$$

5. Convergence of the difference scheme

Let u be solution of the initial-boundary-value problem (3.1)–(3.3) and v the solution of the difference problem (4.1)–(4.3) with $\bar f=T_1T_2f$. The error $z=u-v$ is defined on the mesh $\bar\omega_h\times\bar\omega_\tau$. Putting $v=-z+u$ into (4.1)–(4.3) we conclude that the error z satisfies the following finite difference scheme:

$$\big((I+\theta\tau^\alpha A_1)(I+\theta\tau^\alpha A_2)D^{\alpha}_{t,0+,\tau}z\big)^m+\mathcal{L}_hz^{m-1}=\psi^m,\qquad x\in\omega_h,\quad m=1,2,\dots,M, \tag{5.1}$$

$$z=0,\quad x\in\gamma_h,\quad t\in\bar\omega_\tau, \tag{5.2}$$

$$z^0=z(x,0)=0,\quad x\in\omega_h, \tag{5.3}$$

where

$$\begin{aligned}\psi^m&=(I+\theta\tau^\alpha A_1)(I+\theta\tau^\alpha A_2)D^{\alpha}_{t,0+,\tau}u^m+\mathcal{L}_hu^{m-1}-T_1T_2f^m\\&=\xi^m+\eta^m+\sum_{i,j=1}^{2}(\eta^m_{ij,x_i}+\zeta^m_{ij,x_i})+\sum_{i=1}^{2}(\chi^m_{i,x_i}+\mu^m_{i,x_i}),\end{aligned}$$

and

$$\xi=D^{\alpha}_{t,0+,\tau}u-T_1T_2(D^{\alpha}_{t,0+}u),$$
$$\eta=a\check u-T_1T_2(au),$$
$$\eta_{ij}=T_{3-i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big)\Big|_{(x-0.5he_i,t)}-\frac12\Big[(a_{ij}u_{\bar x_j})\big|_{(x,t)}+(a_{ij}u_{x_j})\big|_{(x-he_i,t)}\Big],$$
$$\zeta_{ij}=\frac{\tau}{2}\Big[(a_{ij}u_{\bar x_j\bar t})\big|_{(x,t)}+(a_{ij}u_{x_j\bar t})\big|_{(x-he_i,t)}\Big],$$
$$\chi_i=-\theta\tau^\alpha D^{\alpha}_{t,0+,\tau}u_{\bar x_i},$$
$$\mu_i=\frac12\,\theta^2\tau^{2\alpha}D^{\alpha}_{t,0+,\tau}u_{\bar x_ix_{3-i}\bar x_{3-i}}.$$

THEOREM 5.1. *Under the assumptions of Theorem* 4.1 *finite difference scheme* (5.1)–(5.3) *is absolutely stable and the following a priori estimate holds:*

$$\begin{aligned}\|z\|_{B^{1,\alpha/2}(Q_{h\tau})}\leqslant C\Big[&\sum_{i,j=1}^{2}\big(\|\eta_{ij}\|_{L^2(Q_{ih\tau})}+\|\zeta_{ij}\|_{L^2(Q_{ih\tau})}\big)\\&+\sum_{i=1}^{2}\big(\|\chi_i\|_{L^2(Q_{ih\tau})}+\|\mu_i\|_{L^2(Q_{ih\tau})}\big)+\|\xi\|_{L^2(Q_{h\tau})}+\|\eta\|_{L^2(Q_{h\tau})}\Big].\end{aligned} \tag{5.4}$$

The proof is analogous to the proof of Theorem 4.1, where in the estimation of truncation error terms containing finite differences the partial summation is used.

In such a way, to obtain the error bound for finite difference scheme (4.1)–(4.3) it is sufficient to estimate the right-hand side terms in (5.4).

THEOREM 5.2. *Let the assumptions of Theorem* 4.1 *hold,* $a_{ij}, a \in H^2(\Omega)$ *and let the solution* u *of initial-boundary value problem* (3.1)–(3.3) *belongs to the space* $C^2([0,T], C(\bar{\Omega})) \cap C_+^\alpha([0,T], H^3(\Omega)) \cap H^1((0,T), H^2(\Omega))$. *Then the solution* v *of finite difference scheme* (4.1)–(4.3) *with* $\bar{f} = T_1T_2f$ *converges to* u *and the following convergence rate estimate holds:*

$$\|u - v\|_{B^{1,\alpha/2}(Q_{h\tau})} = O(h^2 + \tau^\alpha).$$

PROOF. Let us set $\xi = \xi_1 + \xi_2$, where

$$\xi_1 = D^\alpha_{t,0+,\tau}u - D^\alpha_{t,0+}u \quad \text{and} \quad \xi_2 = D^\alpha_{t,0+}u - T_1T_2D^\alpha_{t,0+}u.$$

From Lemma 4.1 immediately it follows

$$\|\xi_1\|_{L^2(Q_{h\tau})} \leqslant C\tau^{2-\alpha}\|u\|_{C^2([0,T],C(\bar{\Omega}))}.$$

From integral representation

$$u(x,t) - T_1T_2u(x,t) = \frac{1}{h^2}\int_{x_1-h/2}^{x_1+h/2}\int_{x_2-h/2}^{x_2+h/2}\bigg(\int_{x_1'}^{x_1}\int_{x_2'}^{x_2}\frac{\partial^2 u}{\partial x_1 \partial x_2}(x_1'', x_2'', t)dx_2''dx_1''$$
$$- \int_{x_1'}^{x_1}\int_{x_1''}^{x_1}\frac{\partial^2 u}{\partial x_1^2}(x_1''', x_2', t)dx_1'''dx_1'' - \int_{x_2'}^{x_2}\int_{x_2''}^{x_2}\frac{\partial^2 u}{\partial x_2^2}(x_1', x_2''', t)\,dx_2'''dx_2''\bigg)dx_2'dx_1'$$

one obtains

$$\|\xi_2\|_{L^2(Q_{h\tau})} = \left(\tau\sum_{m=1}^{M}\|\xi_2^m\|_h^2\right)^{1/2} \leqslant Ch^2\|u\|_{C_+^\alpha([0,T],H^2(\Omega))}.$$

Hence,

$$\|\xi\|_{L^2(Q_{h\tau})} \leqslant C(\tau^{2-\alpha}\|u\|_{C^2([0,T],C(\bar{\Omega}))} + h^2\|u\|_{C_+^\alpha([0,T],H^2(\Omega))}). \tag{5.5}$$

Let us set $\eta = \eta_1 + \eta_2$, where

$$\eta_1 = a\check{u} - T_1T_2(a\check{u}) \quad \text{and} \quad \eta_2 = T_1T_2(a\check{u}) - T_1T_2(au) = -\tau T_1T_2(au_{\bar{t}}).$$

The value of η_1 at the mesh node $x \in \omega_h$ is bounded linear functional of $a\check{u} \in H^2(e)$, where $e = e(x) = (x_1 - h/2, x_1 + h/2) \times (x_2 - h/2, x_2 + h/2)$, which vanishes when $a\check{u} = 1, x_1, x_2$. Using the Bramble-Hilbert lemma [**2**] one obtains

$$|\eta_1| \leqslant Ch|a\check{u}|_{H^2(e)}.$$

Summing this inequality over the mesh ω_h and using the properties of multipliers in the Sobolev spaces [**14**] we get

$$\|\eta_1\|_h \leqslant Ch^2\|a\check{u}\|_{H^2(\Omega)} \leqslant Ch^2\|a\|_{H^2(\Omega)}\|\check{u}\|_{H^2(\Omega)}$$

and finally, after summation over the mesh ω_τ^+:

$$\|\eta_1\|_{L^2(Q_{h\tau})} \leqslant Ch^2\|a\|_{H^2(\Omega)}\|u\|_{C([0,T],H^2(\Omega))}.$$

The term η_2 can be estimated directly:

$$\|\eta_2\|_{L^2(Q_{h\tau})} \leqslant C\tau\|a\|_{C(\bar\Omega)}\left\|\frac{\partial u}{\partial t}\right\|_{L^2(Q)} \leqslant C\tau\|a\|_{C(\bar\Omega)}\|u\|_{C^1([0,T],L^2(\Omega))}.$$

From the obtained inequalities it follows

$$\|\eta\|_{L^2(Q_{h\tau})} \leqslant C(h^2\|a\|_{H^2(\Omega)}\|u\|_{C([0,T],H^2(\Omega))} + \tau\|a\|_{C(\bar\Omega)}\|u\|_{C^1([0,T],L^2(\Omega))}). \tag{5.6}$$

Let us decompose η_{ij} in the following manner

$$\eta_{ij} = \eta_{ij1} + \eta_{ij2} + \eta_{ij3} + \eta_{ij4}, \quad \text{where}$$

$$\eta_{ij1} = T_{3-i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big)\Big|_{(x-0.5he_i,t)} - T_{3-i}(a_{ij})\Big|_{(x-0.5he_i)} T_{3-i}\Big(\frac{\partial u}{\partial x_j}\Big)\Big|_{(x-0.5he_i,t)},$$

$$\eta_{ij2} = T_{3-i}\,(a_{ij})\,\Big|_{(x-0.5he_i)}\Big[T_{3-i}\Big(\frac{\partial u}{\partial x_j}\Big)\Big|_{(x-0.5he_i,t)} - \frac{1}{2}\big(u_{\bar x_j}(x,t) + u_{x_j}(x-he_i,t)\big)\Big],$$

$$\eta_{ij3} = \frac{1}{2}\Big[T_{3-i}\,(a_{ij})\,\Big|_{(x-0.5he_i)} - \frac{1}{2}\Big(a_{ij}(x) + a_{ij}(x-he_i)\Big)\Big] \times \Big[u_{\bar x_j}(x,t) + u_{x_j}(x-he_i,t)\Big],$$

$$\eta_{ij4} = -\frac{1}{4}\Big[a_{ij}(x) - a_{ij}(x-he_i)\Big]\Big[u_{\bar x_j}(x,t) - u_{x_j}(x-he_i,t)\Big].$$

The value of η_{ij1} at the mesh node $x \in \omega_{ih}$ is bounded bilinear functional of $(a_{ij}, u) \in W^{1,q}(e_i) \times W^{2,2q/(q-2)}(e_i)$, where $e_i = e(x-0.5he_i)$ and $q > 2$, which vanishes when $a_{ij} = 1$ or $u = 1, x_1, x_2$. Using the bilinear version of the Bramble–Hilbert lemma and methodology presented in [**8**] one obtains

$$|\eta_{ij1}(x,t)| \leqslant Ch|a_{ij}|_{W^{1,q}(e_i)}|u(\cdot,t)|_{W^{2,2q/(q-2)}(e_i)}, \quad q > 2.$$

Summing this inequality over the mesh ω_{ih} and using the Sobolev imbedding theorems [**1**] we get

$$\|\eta_{ij1}(\cdot,t)\|_{ih} \leqslant Ch^2\|a_{ij}\|_{W^{1,q}(\Omega)}\|u(\cdot,t)\|_{W^{2,2q/(q-2)}(\Omega)} \leqslant Ch^2\|a_{ij}\|_{H^2(\Omega)}\|u(\cdot,t)\|_{H^3(\Omega)}.$$

Analogous inequalities hold for η_{ij2}, η_{ij3} and η_{ij4}. In such a way, after summation over the mesh ω_τ^+ we obtain:

$$\|\eta_{ij}\|_{L^2(Q_{ih\tau})} \leqslant Ch^2\|a_{ij}\|_{H^2(\Omega)}\|u\|_{C([0,T],H^3(\Omega))}. \tag{5.7}$$

The terms ζ_{ij}, χ_i and μ_i can be estimated directly:

$$\|\zeta_{ij}\|_{L^2(Q_{ih\tau})} \leqslant C\tau\|a_{ij}\|_{C(\bar\Omega)}\|u\|_{H^1((0,T),H^2(\Omega))}, \tag{5.8}$$

$$\|\chi_i\|_{L^2(Q_{ih\tau})} \leqslant C\tau^\alpha\|u\|_{C_+^\alpha([0,T],H^2(\Omega))}, \tag{5.9}$$

$$\|\mu_i\|_{L^2(Q_{ih\tau})} \leqslant C\tau^{2\alpha}\|u\|_{C_+^\alpha([0,T],H^3(\Omega))}. \tag{5.10}$$

The result follows from (5.4)–(5.10). □

In the case when $\mathcal{L}u = -\Delta u$ we shall derive error bound in the norm $H_+^{2,\alpha}(Q_{h\tau})$. Then, for $f \in C(\bar Q)$, equation (4.1) reduces to

$$((I + \theta\tau^\alpha A_1)(I + \theta\tau^\alpha A_2)D_{t,0+,\tau}^\alpha v)^m + Av^{m-1} = f^m, \quad x \in \omega_h, \tag{5.11}$$

where denoted $Av = (A_1 + A_2)v = -v_{x_1\bar{x}_1} - v_{x_2\bar{x}_2}$.

Let u be the solution of initial-boundary-value problem (3.1)–(3.3) with $\mathcal{L}u = -\Delta u$, and v the solution of difference problem (5.11), (4.2), (4.3). The error $z = u - v$ satisfies finite the difference scheme

$$((I + \theta\tau^\alpha A_1)(I + \theta\tau^\alpha A_2)D^\alpha_{t,0+,\tau}z\big)^m + Az^{m-1} = \bar{\psi}^m, \quad x \in \omega_h, \tag{5.12}$$

and homogeneous boundary and initial conditions (5.2)–(5.3). Here

$$\begin{aligned}\bar{\psi}^m &= (I + \theta\tau^\alpha A_1)(I + \theta\tau^\alpha A_2)D^\alpha_{t,0+,\tau}u^m - \Delta u^{m-1} - f^m \\ &= \bar{\xi}^m + \bar{\mu}^m + \sum_{i=1}^{2}(\bar{\eta}_i^m + \bar{\zeta}_i^m + \bar{\chi}_i^m),\end{aligned}$$

where denoted

$$\bar{\xi} = D^\alpha_{t,0+,\tau}u - (D^\alpha_{t,0+}u) = \xi_1, \quad \eta_i = \frac{\partial^2 u}{\partial x_i^2} - u_{x_i\bar{x}_i}, \quad \zeta_i = \tau\, u_{x_i\bar{x}_i\bar{t}},$$

$$\bar{\chi}_i = -\theta\tau^\alpha D^\alpha_{t,0+,\tau}u_{x_i\bar{x}_i} = \chi_{i,x_i}, \quad \bar{\mu} = \theta^2\tau^{2\alpha}D^\alpha_{t,0+,\tau}u_{\bar{x}_1\bar{x}_1x_2\bar{x}_2} = \mu_{1,x_1} + \mu_{2,x_2}.$$

THEOREM 5.3. *Let the assumptions of Theorem 4.2 hold and let the solution u of initial-boundary value problem* (3.1)–(3.3) *with $\mathcal{L}u = -\Delta u$ belong to the space $C^2([0,T], C(\bar{\Omega})) \cap C^\alpha_+([0,T], H^4(\Omega)) \cap H^1((0,T), H^2(\Omega))$. Then the solution v of finite difference scheme* (5.11), (4.2), (4.3) *converges to u and the following convergence rate estimate holds:*

$$\|u - v\|_{H^{2,\alpha}_+(Q_{h\tau})} = O(h^2 + \tau^\alpha).$$

PROOF. Note that under our assumptions the function $f(x,t)$ is continuous, so (5.11) is well defined.

Applying a priori estimate (4.7) to (5.12) we obtain:

$$\begin{aligned}&\|u - v\|_{H^{2,\alpha}_+(Q_{h\tau})} = \|z\|_{H^{2,\alpha}_+(Q_{h\tau})} \leqslant C\|\bar{\psi}\|_{L^2(Q_{h\tau})} \\ &\leqslant C\Big[\|\bar{\xi}\|_{L^2(Q_{h\tau})} + \|\bar{\mu}\|_{L^2(Q_{h\tau})} + \sum_{i=1}^{2}\big(\|\bar{\eta}_i\|_{L^2(Q_{h\tau})} + \|\bar{\zeta}_i\|_{L^2(Q_{h\tau})} + \|\bar{\chi}_i\|_{L^2(Q_{h\tau})}\big)\Big].\end{aligned} \tag{5.13}$$

From Lemma 4.1 immediately it follows:

$$\|\bar{\xi}\|_{L^2(Q_{h\tau})} \leqslant C\tau^{2-\alpha}\|u\|_{C^2([0,T],C(\bar{\Omega}))}. \tag{5.14}$$

The term $\bar{\eta}_i$ can be estimated using the Bramble–Hilbert lemma:

$$\|\bar{\eta}_i\|_{L^2(Q_{h\tau})} \leqslant Ch^2\|u\|_{C([0,T],H^4(\Omega))}. \tag{5.15}$$

The terms $\bar{\zeta}_i$, $\bar{\chi}_i$ and $\bar{\mu}$ can be estimated directly:

$$\|\bar{\zeta}_i\|_{L^2(Q_{h\tau})} \leqslant C\tau\|u\|_{H^1((0,T),H^2(\Omega))}, \tag{5.16}$$

$$\|\bar{\chi}_i\|_{L^2(Q_{h\tau})} \leqslant C\tau^\alpha\|u\|_{C^\alpha_+([0,T],H^2(\Omega))}, \tag{5.17}$$

$$\|\bar{\mu}\|_{L^2(Q_{h\tau})} \leqslant C\tau^{2\alpha}\|u\|_{C^\alpha_+([0,T],H^4(\Omega))}. \tag{5.18}$$

The result follows from (5.13)–(5.18). □

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