PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 99(113) (2016), 125–137 DOI: 10.2298/PIM1613125J

ON AVAKUMOVIĆ'S THEOREM FOR GENERALIZED THOMAS–FERMI DIFFERENTIAL EQUATIONS

Jaroslav Jaroš and Kusano Takaŝi

Abstract. For the generalized Thomas–Fermi differential equation

$$(|x'|^{\alpha-1}x')' = q(t)|x|^{\beta-1}x,$$

it is proved that if $1 \leqslant \alpha < \beta$ and $q(t)$ is a regularly varying function of index μ with $\mu > -\alpha - 1$, then all positive solutions that tend to zero as $t \to \infty$ are regularly varying functions of one and the same negative index ρ and their asymptotic behavior at infinity is governed by the unique definite decay law. Further, an attempt is made to generalize this result to more general quasilinear differential equations of the form $(p(t)|x'|^{\alpha-1}x')' = q(t)|x|^{\beta-1}x$.

1. Introduction

In this paper the generalized Thomas–Fermi differential equation

$$(|x'|^{\alpha-1}x')' = q(t)|x|^{\beta-1}x \tag{A}$$

is considered under the assumption that

(a) α and β are positive constants such that $\alpha < \beta$;
(b) $q : [a, \infty) \to (0, \infty)$ is a continuous function.

This equation may well be called a super-half-linear generalized Thomas–Fermi differential equation.

We are interested in positive solutions of (A) which are defined in a neighborhood of infinity and decrease to zero as $t \to \infty$. Such solutions are often referred to as *strongly decreasing solutions* of (A). It is known that (A) has strongly decreasing solutions if and only if

$$\int_a^\infty q(t)\,dt = \infty \quad \text{or else} \quad \int_a^\infty \Big(\int_t^\infty q(s)\,ds\Big)^{\frac{1}{\alpha}} dt = \infty. \tag{1.1}$$

For the proof of this result see e.g. Theorems 2.2, 2.3 and 5.1 in [**10**].

2010 *Mathematics Subject Classification:* Primary 34C11; Secondary 26A12.
Key words and phrases: generalized Thomas–Fermi differential equation, Avakumović's theorem, positive solutions, asymptotic behavior, regularly varying functions.
The first author was supported by Grant 1/0071/14 of the Slovak Grant Agency VEGA.
Communicated by Gradimir Milovanović.

A natural question then arises: Is it possible to determine precisely the asymptotic behavior of possible strongly decreasing solutions of (A)? This question seems to be very difficult to answer for equation (A) with a general positive continuous coefficient $q(t)$ even in the special case $\alpha = 1$, that is, for the superlinear Thomas–Fermi differential equation

$$x'' = q(t)|x|^{\beta-1}x, \tag{A$_0$}$$

where $\beta > 1$ and $q(t) > 0$ is continuous. A hint as to what to do in such circumstances can be found in the pioneering work of Avakumović [**1**] who analyzed the differential equation (A_0) by means of regularly varying functions (in the sense of Karamata) and proved the following theorem

Theorem A_0. *Let $\beta > 1$. Assume that $q(t)$ is a regularly varying function of index $\mu > -2$. Then all strongly decreasing solutions $x(t)$ of equation* (A_0) *are regularly varying functions of index $\rho = -\frac{\mu+2}{\beta-1}$ and their asymptotic behavior is governed by the unique formula*

$$x(t) \sim \Big[\frac{t^2 q(t)}{(-\rho)(1-\rho)}\Big]^{-\frac{1}{\beta-1}}, \quad t \to \infty.$$

Here the symbol $\sim$ denotes the asymptotic equivalence of two positive functions:

$$f(t) \sim g(t),\ t \to \infty \ \Leftrightarrow\ \lim_{t\to\infty} \frac{g(t)}{f(t)} = 1.$$

The above result (referred to as Avakumović's property) is remarkable in that all strongly decreasing solutions of equation (A_0) with regularly varying coefficient $q(t)$ belong to one and the same definite class of regularly varying functions and moreover obey the unique precise decay law depending on $q(t)$ as $t \to \infty$. Theorem A_0 has been generalized to the equation $x'' = q(t)\,\phi(x)$ in three papers of Marić and Tomić [**7, 8, 9**].

In view of a growing tendency in recent years to the in-depth analysis of quasilinear differential equations such as (A), it is natural to expect that Avakumović's property of (A_0) could be shared by the more general equation (A). The objective of this paper is to demonstrate the truth of this expectation by proving the following theorem in Section 3.

Theorem A. *Let $1 \leqslant \alpha < \beta$. Assume that $q(t)$ is a regularly varying function of index $\mu > -\alpha-1$. Then all strongly decreasing solutions $x(t)$ of equation* (A) *are regularly varying functions of index $\rho = -\frac{\mu+\alpha+1}{\beta-\alpha}$, and their asymptotic behavior is governed by the unique formula*

$$x(t) \sim \Big[\frac{t^{\alpha+1} q(t)}{\alpha(1-\rho)(-\rho)^{\alpha}}\Big]^{-\frac{1}{\beta-\alpha}}, \quad t \to \infty.$$

It is natural to ask whether Theorem A may be generalized to more general differential equations of the form

$$(p(t)|x'|^{\alpha-1}x')' = q(t)|x|^{\beta-1}x, \tag{B}$$

where α, β and $q(t)$ are as in (A) and $p(t) > 0$ is a positive continuous function on $[a, \infty)$ satisfying

$$\int_a^\infty p(t)^{-\frac{1}{\alpha}} dt = \infty. \tag{1.2}$$

In Section 4 we will show that Avakumović property for (B) is best understood in the framework of *generalized regularly varying functions* introduced by the present authors [**4**] and give a precise formulation of the extended version of Theorem A by means of *regularly varying functions with respect to the function* $P(t)$ *defined by*

$$P(t) = \int_a^t p(s)^{-\frac{1}{\alpha}} ds. \tag{1.3}$$

The result thus obtained for (B) can be specialized to equation (B) with regularly varying $p(t)$ and $q(t)$ to characterize the existence and asymptotic behavior of strongly decreasing solutions of (B) which are regularly varying.

The definition and some basic properties of regularly varying functions are summarized in Section 2, and the definition of generalized regularly varying functions is given at the beginning of Section 4.

2. Regularly varying functions

For the reader's benefit we state here the definition and some basic properties of regularly varying functions (in the sense of Karamata).

Definition 2.1. A measurable function $f : (0, \infty) \to (0, \infty)$ is called *regularly varying of index* $\rho \in \mathbf{R}$ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad \text{for each } \lambda > 0,$$

or equivalently if $f(t)$ is expressed in the form

$$f(t) = c(t) \exp\left\{ \int_{t_0}^t \frac{\delta(s)}{s} ds \right\}, \quad t \geqslant t_0, \tag{2.1}$$

for some measurable functions $c(t)$ and $\delta(t)$ such that

$$\lim_{t\to\infty} c(t) = c_0 \in (0, \infty) \quad \text{and} \quad \lim_{t\to\infty} \delta(t) = \rho.$$

In case $c(t) \equiv c_0$ in (2.1) $f(t)$ is said to be a normalized regularly varying function of index ρ.

The totality of regularly varying functions of index ρ is denoted by $\mathrm{RV}(\rho)$. Use is made of the symbol $\mathrm{RV} = \bigcup_{\rho\in\mathbf{R}} \mathrm{RV}(\rho)$. If in particular $\rho = 0$, then the symbol SV is used for $\mathrm{RV}(0)$ and members of SV are referred to as *slowly varying functions*. By definition $f \in \mathrm{RV}(\rho)$ is expressed as $f(t) = t^\rho g(t)$ with $g \in \mathrm{SV}$, and so the class SV of slowly varying functions is of fundamental importance in the theory of regular variation. Typical examples of slowly varying functions are: all functions tending to positive constants as $t \to \infty$,

$$\prod_{k=1}^{N} (\log_n t)^{\alpha_k}, \quad \alpha_k \in \mathbf{R}, \quad \text{and} \quad \exp\left\{ \prod_{k=1}^{N} (\log_n t)^{\beta_k} \right\}, \quad \beta_k \in (0,1),$$

where $\log_n t$ denotes the nth iteration of the logarithm. The function

$$L(t) = \exp\left\{ (\log t)^{\theta} \cos(\log t)^{\theta} \right\}, \quad \theta \in (0, \tfrac{1}{2}),$$

is a slowly varying function which is oscillating in the sense that

$$\limsup_{t\to\infty} L(t) = \infty, \qquad \liminf_{t\to\infty} L(t) = 0.$$

The following result concerns operations which preserve slow variation.

Proposition 2.1. *If $L(t)$, $L_1(t)$ and $L_2(t)$ are slowly varying, then $L(t)^{\alpha}$ for every real α, $L_1(t) + L_2(t)$, $L_1(t)L_2(t)$ and $L_1(L_2(t))$ (if $L_2(t) \to \infty$ as $t \to \infty$) are slowly varying.*

A slowly varying function may decay to zero or grow to infinity a $t \to \infty$. However, its order of decay or growth is severely limited as explained in the following

Proposition 2.2. *If $L(t)$ is slowly varying, then for any $\varepsilon > 0$*

$$\lim_{t\to\infty} t^{\varepsilon} L(t) = \infty, \qquad \lim_{t\to\infty} t^{-\varepsilon} L(t) = 0.$$

We quote the following result, Karamata's integration theorem, which is of the highest importance in the application of slowly and regularly varying functions.

Proposition 2.3. *Let $L(t) \in \mathrm{SV}$. We have,*

(i) *if $\alpha > -1$, then $\int_a^t s^{\alpha} L(s)\, ds \sim \frac{1}{\alpha+1} t^{\alpha+1} L(t)$, $t \to \infty$;*

(ii) *if $\alpha < -1$, then $\int_t^{\infty} s^{\alpha} L(s)\, ds \sim -\frac{1}{\alpha+1} t^{\alpha+1} L(t)$, $t \to \infty$;*

(iii) *if $\alpha = -1$, then*

$$l(t) = \int_a^t \frac{L(s)}{s} ds \in \mathrm{SV} \quad \textit{and} \quad \lim_{t\to\infty} \frac{L(t)}{l(t)} = 0,$$

$$m(t) = \int_t^{\infty} \frac{L(s)}{s} ds \in \mathrm{SV} \quad \textit{and} \quad \lim_{t\to\infty} \frac{L(t)}{m(t)} = 0,$$

provided $L(t)/t$ is integrable near the infinity in the latter case.

We conclude this section with two propositions which are crucial in the proof of our main result, Theorem 3.1.

Proposition 2.4. *A continuously differentiable function $f(t)$ is normalized regularly varying of index ρ if and only if $\lim_{t\to\infty} t\frac{f'(t)}{f(t)} = \rho$.*

Proposition 2.5. [**2**, Theorem 1.8.2] *Let $f(t)$ be regularly varying of index ρ. There exist two functions $f_1(t)$ and $f_2(t)$ such that $f_1(t) \sim f_2(t)$, $t \to \infty$, and $f_1(t) \leqslant f(t) \leqslant f_2(t)$ for all large t and such that the functions $\psi_i(\tau) := \log f_i(e^\tau)$, $i = 1, 2$, are C^∞ on a neighborhood of infinity and satisfy*

$$\frac{d}{d\tau}\psi_i(\tau) \to \rho, \qquad \frac{d^n}{d\tau^n}\psi_i(\tau) \to 0, \quad n \geqslant 2, \quad \textit{as} \quad \tau \to \infty,$$

for $i = 1, 2$.

For a complete exposition of theory of regular variation and its applications to various branches of mathematical analysis the reader is referred to Bingham et al. [**2**]. See also Seneta [**11**]. A comprehensive survey of results up to the year 2000 on the asymptotic analysis of positive solutions of second order differential equations can be found in Marić [**6**].

3. Generalization of Avakumovic's theorem

The purpose of this section is to prove the following theorem which, when specialized to equation (A_0), strengthens the assertion of Avakumović's theorem: Theorem A_0.

Theorem 3.1. *Let $1 \leqslant \alpha < \beta$ and suppose that $q(t)$ is regularly varying of index μ. All strongly decreasing solutions of equation* (A) *are regularly varying functions of negative index ρ if and only if $\mu > -\alpha - 1$, in which case ρ is given by $\rho = -\frac{\mu+\alpha+1}{\alpha-\beta}$, and the asymptotic behavior of any such solution $x(t)$ is governed by the unique formula*

$$x(t) \sim \Big[\frac{t^{\alpha+1}q(t)}{\alpha(1-\rho)(-\rho)^\alpha}\Big]^{-\frac{1}{\beta-\alpha}}, \quad t \to \infty. \tag{3.1}$$

Proof. We begin by noting that any positive strongly decreasing solution $x(t)$ satisfies the differential equation

$$x''(t) = \alpha^{-1}q(t)\,x(t)^\beta(-x'(t))^{1-\alpha}, \tag{3.2}$$

and the integral equation

$$x(t) = \int_t^\infty \Big(\int_s^\infty q(r)\,x(r)^\beta dr\Big)^{\frac{1}{\alpha}} ds. \tag{3.3}$$

(Proof of the "only if" part) Suppose that $q(t)$ is expressed in the form $q(t) = t^\mu m(t)$, $m \in \mathrm{SV}$. Let $x(t)$ be a strongly increasing solution of (A) on $[T, \infty)$ belonging to $\mathrm{RV}(\rho)$ with $\rho < 0$. We use the expression $x(t) = t^\rho \xi(t)$, $\xi \in \mathrm{SV}$.

The convergence of the integral

$$\int_t^\infty q(s)\,x(s)^\beta ds = \int_t^\infty s^{\mu+\rho\beta} m(s)\,\xi(s)^\beta ds \tag{3.4}$$

implies that $\mu + \rho\beta \leqslant -1$. If $\mu + \rho\beta = -1$, then from (3.4) we have

$$\Big(\int_t^\infty q(s)\,x(s)^\beta ds\Big)^{\frac{1}{\alpha}} = \Big(\int_t^\infty s^{-1} m(s)\,\xi(s)^\beta ds\Big)^{\frac{1}{\alpha}} \in \mathrm{SV}\,. \tag{3.5}$$

Since no slowly varying function is integrable in a neighborhood of infinity, (3.5) is not integrable on $[T,\infty)$, which means that $x(t)$ cannot satisfy (3.3), an impossibility. Therefore, only the case $\mu+\rho\beta<-1$ is possible. In this case, applying Karamata's integration theorem to (3.4), we obtain

$$\Big(\int_t^\infty q(s)x(s)^\beta ds\Big)^{\frac{1}{\alpha}} \sim \frac{t^{\frac{\mu+\rho\beta+1}{\alpha}} m(t)^{\frac{1}{\alpha}}\xi(t)^{\frac{\beta}{\alpha}}}{[-(\mu+\rho\beta+1)]^{\frac{1}{\alpha}}}, \quad t\to\infty. \tag{3.6}$$

From the integrability of (3.6) on $[T,\infty)$ we have $\frac{\mu+\rho\beta+1}{\alpha}\leqslant -1$, but the equality must be excluded. In fact, if the equality holds, then integrating (3.6) and using (3.3) we see that

$$x(t)\sim \alpha^{-\frac{1}{\alpha}}\int_t^\infty s^{-1}m(s)^{\frac{1}{\alpha}}\xi(s)^{\frac{\beta}{\alpha}}ds \in \mathrm{SV},$$

i.e., $\rho=0$, which is impossible. Let $\frac{\mu+\rho\beta+1}{\alpha}<-1$. Then, integrating (3.6) from t to ∞, we obtain via Karamata's integration theorem and (3.3)

$$x(t)\sim \frac{t^{\frac{\mu+\rho\beta+1}{\alpha}+1} m(t)^{\frac{1}{\alpha}}\xi(t)^{\frac{\beta}{\alpha}}}{[-(\mu+\rho\beta+1)]^{\frac{1}{\alpha}}\big[-\big(\frac{\mu+\rho\beta+1}{\alpha}+1\big)\big]}, \quad t\to\infty. \tag{3.7}$$

This shows that $x(t)$ is regularly varying of index $\frac{\mu+\rho\beta+1}{\alpha}+1$, that is,

$$\rho=\frac{\mu+\rho\beta+1}{\alpha}+1 \ \Rightarrow\ \rho=-\frac{\mu+\alpha+1}{\beta-\alpha}.$$

The requirement $\rho<0$ implies that $\mu>-\alpha-1$.

It is easy to check that the denominator of the right-hand side of (3.7) equals $[\alpha(1-\rho)]^{\frac{1}{\alpha}}(-\rho)$. Using this fact and noting that (3.7) is transformed into

$$x(t)\sim\frac{t^{\frac{\alpha+1}{\alpha}}q(t)^{\frac{1}{\alpha}}x(t)^{\frac{\beta}{\alpha}}}{[\alpha(1-\rho)]^{\frac{1}{\alpha}}(-\rho)}, \quad t\to\infty,$$

we conclude that the asymptotic behavior of $x(t)$ must obey the decay law (3.1).

(Proof of the "if" part) The proof is based on an extended adaptation of the method of Geluk [**3**] used in proving Theorem A_0. Suppose that $q\in\mathrm{RV}(\mu)$ with $\mu>-\alpha-1$. Since $q(t)$ satisfies (1.1) equation (A) possesses strongly decreasing solutions. Let $x(t)$ be any such solution on $[T,\infty)$. Put

$$v(s)=\log[x(e^s)^{\alpha-\beta}], \qquad \psi(s)=\log\Big[\frac{(\beta-\alpha)^\alpha}{\alpha}e^{(\alpha+1)s}q(e^s)\Big]. \tag{3.8}$$

Let us differentiate $v(s)$ twice with respect to s. In what follows $\cdot$ denotes differentiation with respect to s. First we have

$$\dot v(s)=-(\beta-\alpha)e^s\frac{x'(e^s)}{x(e^s)}>0, \tag{3.9}$$

and then using (3.2) and

$$-x'(t)=\frac{1}{\beta-\alpha}e^{-s}x(e^s)\dot v(s)$$

(a rewritten form of (3.9)), we easily obtain

$$\ddot{v}(s) - \dot{v}(s) - \gamma\dot{v}(s)^2 = -\dot{v}(s)^{1-\alpha}\exp\{\psi(s) - v(s)\},$$

which can be expressed as

$$\alpha^{-1}(\dot{v}(s)^\alpha)^{\cdot} - \dot{v}(s)^\alpha - \gamma(\dot{v}(s)^\alpha)^{1+\frac{1}{\alpha}} = -\exp\{\psi(s) - v(s)\}, \tag{3.10}$$

where $\gamma = 1/(\beta - \alpha) > 0$.

We now define the function $w(s)$ by $w(s) = v(s) - \psi_1(s)$, where $\psi_1(s)$ is a C^∞-function such that

$$\psi_1(s) \leqslant \psi(s), \quad \psi_1(s) \to \psi(s), \quad \text{and} \quad \dot{\psi}_1(s) \to \mu + \alpha + 1, \quad \ddot{\psi}_1(s) \to 0, \tag{3.11}$$

as $s \to \infty$. The existence of such a function $\psi_1(s)$ is guaranteed by Proposition 2.5. It is elementary to see that $w(s)$ satisfies

$$\begin{aligned}\ddot{w}(s) - \delta(s)\dot{w}(s) - \gamma\dot{w}(s)^2 \\ = \phi(s) - (\dot{w}(s) + \dot{\psi}_1(s))^{1-\alpha}\exp\{\psi(s) - \psi_1(s) - w(s)\},\end{aligned} \tag{3.12}$$

where $\delta(s) = 1 + 2\gamma\dot{\psi}_1(s)$, and $\phi(s) = -\ddot{\psi}_1(s) + \dot{\psi}_1(s) + \gamma\dot{\psi}_1(s)^2$. Notice that as $s \to \infty$

$$\delta(s) \to 1 + s\gamma(\mu + \alpha + 1) > 0, \quad \phi(s) \to (\mu + \alpha + 1)(1 + \gamma(\mu + \alpha + 1)) > 0. \tag{3.13}$$

We claim that the limit $\lim_{s\to\infty} w(s)$ exists and is finite. The following three cases are distinguished:

(a) $\dot{w}(s) \geqslant 0$ for large s;
(b) $\dot{w}(s) \leqslant 0$ for large s;
(c) $\dot{w}(s)$ changes sign infinitely often as $s \to \infty$.

Let case (a) hold. Assume that $\lim_{s\to\infty} w(s) = \infty$ and let s tend to infinity in (3.12). Note that since $(\dot{w}(s) + \dot{\psi}_1(s))^{1-\alpha}$ is bounded because of $\alpha \geqslant 1$ and (3.11), it follows that

$$(\dot{w}(s) + \dot{\psi}_1(s))^{1-\alpha}\exp\{\psi(s) - \psi_1(s) - w(s)\} \to 0, \quad s \to \infty.$$

Using this in (3.12) and taking (3.13) into account, we find that

$$\ddot{w}(s) \geqslant \frac{l}{2} \quad \text{for all large } s, \quad l = (\mu + \alpha + 1)(1 + \gamma(\mu + \alpha + 1)),$$

so that $\dot{w}(s) \to \infty$ as $s \to \infty$. Divide (3.12) by $\dot{w}(s)^2$ and let $s \to \infty$. Then,

$$\frac{\ddot{w}(s)}{\dot{w}(s)^2} \sim \gamma \;\Rightarrow\; -\frac{1}{\dot{w}(s)} \sim \gamma s,$$

which implies that $\dot{w}(s) < 0$ for all large s, a contradiction. Therefore, $w(s)$ must increase to a finite limit as $s \to \infty$.

Let case (b) hold. Assume that $\lim_{s\to\infty} w(s) = -\infty$. From (3.10) rewritten as

$$-\alpha^{-1}(\dot{v}(s)^\alpha)^{\cdot} + \dot{v}(s)^\alpha + \gamma(\dot{v}(s)^\alpha)^{1+\frac{1}{\alpha}} = \exp\{\psi(s) - \psi_1(s) - w(s)\},$$

we see that

$$-\alpha^{-1}(\dot{v}(s)^\alpha)^{\cdot} + \dot{v}(s)^\alpha + \gamma(\dot{v}(s)^\alpha)^{1+\frac{1}{\alpha}} \to \infty, \quad s \to \infty.$$

This implies that $\dot{v}(s)$ is unbounded as $s \to \infty$, because otherwise from the above it follows that $(\dot{v}(s)^\alpha)^{\cdot} \to -\infty$, and hence $\dot{v}(s)^\alpha \to -\infty$ as $s \to \infty$, which contradicts the positivity of $\dot{v}(s)$. Consequently, there exists $\{s_n\}$ such that $s_n \to \infty$ and $\dot{v}(s_n) \to \infty$ as $n \to \infty$. Then, $\dot{w}(s_n) = \dot{v}(s_n) - \dot{\psi}_1(s_n) \to \infty$, $n \to \infty$, which is impossible.

Let case (c) hold. Let $\{s_n\}$ be a sequence tending to infinity along which $\dot{w}(s)$ changes sign. Each s_n is a point at which $w(s)$ takes a local minimum or a local maximum. We may assume that $\{s_{2m-1}\}$ and $\{s_{2m}\}$ are, respectively, the points of local maxima and those of local minima of $w(s)$. It is clear that $\ddot{w}(s_{2m-1}) \leqslant 0$ and $\ddot{w}(s_{2m}) \geqslant 0$ for $m = 1, 2, \ldots$.

Let $s = s_{2m-1}$ in (3.12). Then,

$$\begin{aligned} 0 &\geqslant \ddot{w}(s_{2m-1}) \\ &= \phi(s_{2m-1}) - (\dot{\psi}_1(s_{2m-1}))^{1-\alpha} \exp\{\psi(s_{2m-1}) - \psi_1(s_{2m-1}) - w(s_{2m-1})\}, \end{aligned}$$

from which we have

$$w(s_{2m-1}) \leqslant \psi(s_{2m-1}) - \psi_1(s_{2m-1}) - \log[\phi(s_{2m-1})\dot{\psi}_1(s_{2m-1})^{\alpha-1}], \quad m = 1, 2, \ldots.$$

Letting $m \to \infty$ in the above, we find that

$$\limsup_{m\to\infty} w(s_{2m-1}) \leqslant -\log(l(\mu+\alpha+1)^{\alpha-1}) \Rightarrow \limsup_{s\to\infty} w(s) \leqslant -\log(l(\mu+\alpha+1)^{\alpha-1}). \tag{3.14}$$

Likewise, considering (3.12) along $\{s_{2m}\}$, we obtain

$$\liminf_{m\to\infty} w(s_{2m}) \geqslant -\log(l(\mu+\alpha+1)^{\alpha-1}) \Rightarrow \liminf_{s\to\infty} w(s) \geqslant -\log(l(\mu+\alpha+1)^{\alpha-1}). \tag{3.15}$$

From (3.14) and (3.15) it follows that

$$\lim_{s\to\infty} w(s) = c := -\log(l(\mu+\alpha+1)^{\alpha-1}),$$

which implies that

$$\lim_{s\to\infty} (v(s) - \psi(s)) = \lim_{s\to\infty} [w(s) + \psi_1(s) - \psi(s)] = c. \tag{3.16}$$

Note that (3.16) can be expressed as

$$x(e^s)^{\alpha-\beta} \sim e^c \frac{(\beta-\alpha)^\alpha}{\alpha} e^{(\alpha+1)s} q(e^s), \quad s \to \infty,$$

(cf. (3.8)) or equivalently

$$x(t)^{\alpha-\beta} \sim e^c \frac{(\beta-\alpha)^\alpha}{\alpha} t^{\alpha+1} q(t), \quad t \to \infty. \tag{3.17}$$

Since $t^{\alpha+1}q(t) \in \mathrm{RV}(\mu+\alpha+1)$, this shows that $x(t)$ is a regularly varying function of index $\rho = -\frac{\mu+\alpha+1}{\beta-\alpha} < 0$. $\square$

Remark 3.1. It can be shown that the solution $x(t)$ obtained above is a normalized regularly varying function. In fact, rewriting (3.17) as

$$x(t) \sim k^{\frac{1}{\alpha-\beta}} t^{\rho} m(t)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty, \quad \text{where} \quad k = e^{c}\frac{(\beta-\alpha)^{\alpha}}{\alpha}, \tag{3.18}$$

and using Karamata's integration theorem, we obtain

$$-x'(t) = \Big(\int_t^{\infty} q(s)x(s)^{\beta}ds\Big)^{\frac{1}{\alpha}} \sim \frac{k^{\frac{\beta}{\alpha(\alpha-\beta)}}}{[\alpha(1-\rho)]^{\frac{1}{\alpha}}} t^{\rho-1} m(t)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty. \tag{3.19}$$

Combining (3.18) with (3.19), we see that

$$-\lim_{t\to\infty} t\frac{x'(t)}{x(t)} = \frac{1}{[k\alpha(1-\rho)]^{\frac{1}{\alpha}}},$$

which implies that $x(t)$ is a normalized regularly varying function of index $-1/[k\alpha(1-\rho)]^{\frac{1}{\alpha}}$. Since we already know that $x \in \mathrm{RV}(\rho)$, we find that

$$\frac{1}{[k\alpha(1-\rho)]^{\frac{1}{\alpha}}} = -\rho \ \Rightarrow\ k = \frac{1}{\alpha(1-\rho)(-\rho)^{\alpha}}$$

which combined with (3.18) yields the asymptotic formula (3.1) for $x(t)$. This provides another way of establishing the unique asymptotic formula for $x(t)$.

Example 3.1. Consider equation (A) with $q(t)$ defined by

$$q(t) \sim t^{-100\alpha+99\beta-1}\exp\big\{k(\log t)^{\theta}\cos(\log t)^{\theta}\big\}, \quad t \to \infty,$$

where $\theta \in (0, \frac{1}{2})$ and $k > 0$ are constants. Here, $q(t)$ is regularly varying of index $\mu = -100\alpha + 99\beta - 1$ and we have $\rho = (\mu + \alpha + 1)/(\alpha - \beta) = -99$. Therefore, by Theorem 3.1 all strongly decreasing solutions of (A) belong to the class $\mathrm{RV}(-99)$ and obey the decay law

$$x(t) \sim (100\alpha{\cdot}99^{\alpha})^{\frac{1}{\beta-\alpha}} t^{-99}\exp\Big\{-\frac{k}{\beta-\alpha}(\log t)^{\theta}\cos(\log t)^{\theta}\Big\}, \quad t \to \infty.$$

Example 3.2. The differential equation

$$(|x'|^{\alpha-1}x')' = q(t)|x|^{2\alpha-1}x, \qquad q(t) = \frac{\alpha}{t^{\alpha+1}}\Big(1 + \frac{2}{\log t}\Big),$$

has a slowly varying solution $x(t) = 1/\log t \in \mathrm{SV} = \mathrm{RV}(0)$ which is strongly decreasing. This example shows that equation (A) may possess strongly decreasing slowly varying solutions which cannot be covered by Theorem 3.1.

4. More general equations of Thomas–Fermi type

Our aim here is to show that Theorem 3.1 can be generalized in a natural way to equations of the form (B) with $p(t)$ satisfying (1.2) if the analysis is made in the framework of generalized regular variation introduced by the present authors in [**4**].

4.1. Generalized regularly varying functions. Let $\Phi(t)$ be a positive C^1-function on $(0,\infty)$ such that $\Phi'(t) > 0$, $t > 0$, and $\lim_{t\to\infty} \Phi(t) = \infty$. Let Φ^{-1} denote the inverse of Φ. (Do not confuse $\Phi^{-1}(t)$ with $\Phi(t)^{-1} = 1/\Phi(t)$.)

Definition 4.1. A measurable function $f\colon [t_0,\infty) \to (0,\infty)$ is said to be a *slowly varying function with respect to* Φ if $f \circ \Phi^{-1}(t) = f(\Phi^{-1}(t))$ is slowly varying (as defined in Section 2), or equivalently, if $f(t)$ is expressed in the form $f(t) = L(\Phi(t))$ for some slowly varying function L. $f(t)$ is called a *regularly varying function of index* ρ *with respect to* Φ if it is expressed as $f(t) = \Phi(t)^\rho g(t)$ for some slowly varying function g with respect to Φ, or as $f(t) = \Phi(t)^\rho L(\Phi(t))$ for some slowly varying function L.

The set of all slowly varying functions (or regularly varying functions of index ρ with respect to Φ) is denoted by SV_Φ (or $\mathrm{RV}_\Phi(\rho)$). We use the symbol $\mathrm{RV}_\Phi = \bigcup_{\rho\in\mathbf{R}} \mathrm{RV}_\Phi(\rho)$. It is shown that most of the basic properties of ordinary regularly varying functions can be transplanted to generalized regularly varying functions (see [**4**]), but they will not be reproduced here.

Example 4.1. **1.** Let $\Phi \in \mathrm{RV}(m)$, $m > 0$. Then, $\Phi^{-1} \in \mathrm{RV}(1/m)$ and hence

$$f \in \mathrm{RV}(\rho) \;\Rightarrow\; f \in \mathrm{RV}_\Phi(\rho/m).$$

2. Let $\Phi(t) = e^t$. Then, $\Phi^{-1}(t) = \log t$.

(i) Consider $f(t) = \exp(t^\gamma)$, $\gamma > 0$:
  (a) If $\gamma < 1$, then $f \in \mathrm{SV}_{e^t}$;
  (b) If $\gamma = 1$, then $f \in \mathrm{RV}_{e^t}(1)$;
  (c) If $\gamma > 1$, then f is rapidly varying, so that $f \notin \mathrm{RV}_{e^t}$.
(ii) If $f \in \mathrm{RV}(\rho)$, then $f \in \mathrm{SV}_{e^t}$.

3. Let $\Phi(t) = \log t$. Since $\Phi^{-1}(t) = e^t$, we see that

(i) if $f(t) = (\log\log t)^\lambda$, $\lambda \neq 0$, then $f \in \mathrm{SV}_{\log t}$;
(ii) if $f(t) = (\log t)^\mu$, $\mu \neq 0$, then $f \in \mathrm{RV}_{\log t}(\mu)$;
(iii) if $f(t) = t^\nu$, $\nu \neq 0$, then f is rapidly varying, so that $f \notin \mathrm{RV}_{\log t}$.

4.2. Avakumović's property of equation (B). Consider equation (B) for which it is assumed that $0 < \alpha < \beta$, $p(t)$ and $q(t)$ are positive continuous functions on $[a,\infty)$ and $p(t)$ satisfies condition (1.2). The object of our investigation are positive solutions of (B) which decrease to zero as $t \to \infty$. Such solutions are also called strongly decreasing solutions of (B).

It is a matter of elementary computation to check that by the change of variables $(t,x) \to (\tau, X)$ given by

$$\tau = P(t) = \int_a^t p(s)^{-\frac{1}{\alpha}}\,ds, \quad X(\tau) = x(t),$$

equation (B) can be transformed into

$$(|\dot{X}|^{\alpha-1}\dot{X})^{\dot{}} = Q(\tau)|X|^{\beta-1}X, \quad Q(\tau) = p(t)^{\frac{1}{\alpha}} q(t), \tag{4.1}$$

where $\cdot$ denotes differentiation with respect to τ. Equation (4.1) is of the same type as equation (A), and so it has strongly decreasing solutions (in τ) if and only if

$$\int_{P(a)}^{\infty} Q(\tau)\,d\tau = \infty \quad \text{or else} \quad \int_{P(a)}^{\infty} \Big(\int_{\tau}^{\infty} Q(\sigma)\,d\sigma\Big)^{1/\alpha} d\tau = \infty. \tag{4.2}$$

Since (4.2) is equivalent to

$$\int_{a}^{\infty} q(t)\,dt = \infty \quad \text{or else} \quad \int_{a}^{\infty} \Big(\frac{1}{p(s)}\int_{s}^{\infty} q(r)\,dr\Big)^{1/\alpha} ds = \infty, \tag{4.3}$$

it follows that equation (B) has strongly decreasing solutions if and only if (4.3) holds. Furthermore, if Theorem 3.1 is applied to equation (B) with $Q \in \mathrm{RV}(\mu)$ as a function of τ, then then it is concluded that if $1 \leqslant \alpha < \beta$, all strongly decreasing solutions $X(\tau)$ are regularly varying of negative index ρ as functions of τ if and only if $\mu > -\alpha - 1$, in which case ρ is given by $\rho = -(\mu + \alpha + 1)/(\beta - \alpha)$ and any such solution enjoys the asymptotic behavior

$$X(\tau) \sim \Big[\frac{\tau^{\alpha+1} Q(\tau)}{\alpha(1-\rho)(-\rho)^{\alpha}}\Big]^{-\frac{1}{\beta-\alpha}}, \quad \tau \to \infty.$$

Observe that to each strongly decreasing solution $X(\tau)$ of (4.1) there corresponds a unique strongly decreasing solution $x(t) = X(\tau)$ of (B) and vice versa, and that in view of Definition 4.1 a regularly varying solution (in τ) of (4.1) determines a unique regularly varying solution with respect to $P(t)$ (in t) of (B). Thus what is described above for (4.1) can be translated into the following theorem which is a natural generalization of Theorem 3.1 to equation (B).

THEOREM 4.1. *Let $1 \leqslant \alpha < \beta$. Suppose that $p(t)$ and $q(t)$ are regularly varying functions with respect to P of indices λ and μ, respectively. All strongly decreasing solutions of equation* (B) *are regularly varying of negative index ρ with respect to P if and only if $\frac{\lambda}{\alpha} + \mu > -\alpha - 1$, in which case ρ is given by*

$$\rho = -\frac{\frac{\lambda}{\alpha} + \mu + \alpha + 1}{\beta - \alpha},$$

and any such solution $x(t)$ obeys the unique decay law

$$x(t) \sim \Big[\frac{P(t)^{\alpha+1} p(t)^{\frac{1}{\alpha}} q(t)}{\alpha(1-\rho)(-\rho)^{\alpha}}\Big]^{-\frac{1}{\beta-\alpha}}, \quad t \to \infty. \tag{4.4}$$

EXAMPLE 4.2. Consider equation (B) with $p(t)$ and $q(t)$ such that

$$p(t) \sim \exp(-\alpha t), \qquad q(t) \sim k\exp(\gamma t + \delta\sqrt{t})(t\log t)^{\varepsilon}, \quad t \to \infty,$$

where $k > 0$, γ, δ and ε are constants. Clearly $p(t)$ satisfies (1.2) and the function $P(t)$ defined by (1.3) can be taken to be $P(t) = e^{t}$. As easily checked, $p \in \mathrm{RV}_{e^t}(-\alpha)$ and $q \in \mathrm{RV}_{e^t}(\gamma)$. Applying Theorem 4.1 to this case, we see that all strongly decreasing solutions of (A) are regularly varying functions (with respect to e^{t}) of negative index $\rho = -(\alpha + \gamma)/(\beta - \alpha)$ and enjoy the asymptotic behavior

$$x(t) \sim \exp(\rho t)\Big[\frac{k\exp(\delta\sqrt{t})(t\log t)^{\varepsilon}}{\alpha(1-\rho)(-\rho)^{\alpha}}\Big]^{-\frac{1}{\beta-\alpha}}, \quad t \to \infty.$$

Let us now consider equation (B) in which $p(t)$ and $q(t)$ are ordinary regularly varying functions. In this case one may ask if (B) enjoys Avakumović's property within the class of regularly varying functions. The affirmative answer is given in the following result which is actually a corollary of Theorem 4.1.

THEOREM 4.2. *Let* $1 \leqslant \alpha < \beta$. *Suppose that* $p \in \mathrm{RV}(\lambda)$, $\lambda < \alpha$, *and* $q \in \mathrm{RV}(\mu)$. *All strongly decreasing solutions of equation* (B) *are regularly varying of negative index* ρ *if and only if*

$$-\lambda + \mu > -\alpha - 1, \tag{4.5}$$

in which case ρ *is given by*

$$\rho = -\frac{-\lambda + \mu + \alpha + 1}{\beta - \alpha}, \tag{4.6}$$

and any such solution $x(t)$ *obeys the unique decay law*

$$x(t) \sim \left[\frac{t^{\alpha+1} p(t)^{-1} q(t)}{\alpha\big(1 - \frac{\lambda}{\alpha} - \rho\big)(-\rho)^{\alpha}}\right]^{-\frac{1}{\beta-\alpha}}, \quad t \to \infty. \tag{4.7}$$

PROOF. We use the following expressions for $p(t)$ and $q(t)$:

$$p(t) = t^{\lambda} l(t), \qquad q(t) = t^{\mu} m(t), \quad l, m \in \mathrm{SV}.$$

Since $\lambda < \alpha$, $p(t)$ satisfies (1.2) and

$$P(t) = \int_a^t s^{-\frac{\lambda}{\alpha}} l(s)^{-\frac{1}{\alpha}} ds \sim \frac{\alpha}{\alpha - \lambda} t^{\frac{\alpha-\lambda}{\alpha}} l(t)^{-\frac{1}{\alpha}} \in \mathrm{RV}\Big(\frac{\alpha - \lambda}{\alpha}\Big). \tag{4.8}$$

(Note that we have excluded the case $\lambda = \alpha$ from our discussions because of computational difficulty.)

In view of (4.8) $p(t)$ and $q(t)$ can be considered as regularly varying functions with respect to P (cf. Example 4.1): $p \in \mathrm{RV}_P\big(\frac{\alpha\lambda}{\alpha-\lambda}\big)$, $q \in \mathrm{RV}_P\big(\frac{\alpha\mu}{\alpha-\lambda}\big)$.

Therefore, Theorem 4.1 can be applied to this case, allowing us to conclude that (4.5) is a necessary and sufficient condition for all strongly decreasing solutions of (B) to be regularly varying (with respect to P) of negative index ρ' given by

$$\rho' = -\frac{\frac{\lambda}{\alpha-\lambda} + \frac{\alpha\mu}{\alpha-\lambda} + \alpha + 1}{\beta - \alpha} = -\frac{\alpha}{\alpha - \lambda} \cdot \frac{-\lambda + \mu + \alpha + 1}{\beta - \alpha}.$$

This means all strongly solutions of (B) are (ordinary) regularly varying solutions of negative index ρ given by (4.6) if and only if (4.5) holds. A straightforward calculation shows that the Avakumović formula (4.4) (with ρ replaced by ρ') as RV_P-functions is transformed into the formula (4.7) as RV-functions. This completes the proof. $\square$

EXAMPLE 4.3. Consider the equation (B) with $p(t)$ and $q(t)$ defined by

$$p(t) \sim t^{-\frac{\alpha}{3}} \log t, \qquad q(t) \sim k t^{\frac{-5\alpha+\beta-3}{3}} (\log t \cdot \log\log t)^2, \quad t \to \infty,$$

where $k > 0$ is a constant. Here $\lambda = -\frac{\alpha}{3}$ and $\mu = \frac{-5\alpha+\beta-3}{3}$, which clearly satisfy (4.5) and determine the regularity index $\rho = -\frac{1}{3}$ of all strongly decreasing regularly varying solutions of this equation (B). When specialized to this case, the asymptotic formula (4.7) governing any such solutions reduces to

$$x(t) \sim \Big(\frac{5\alpha}{3^{\alpha+1}}\Big)^{\frac{1}{\beta-\alpha}} t^{-\frac{1}{3}} (k \log t(\log\log t)^2)^{-\frac{1}{\beta-\alpha}}, \quad t \to \infty.$$

REMARK 4.1. This paper is concerned exclusively with strongly decreasing solutions of equations (A). Equation (A) may have *strongly increasing solutions* $x(t)$ such that $\lim_{t\to\infty} x(t)/t = \infty$. Neither the regularity nor the precise asymptotics of such increasing solutions of (A) with regularly varying coefficient $q(t)$ seems to have been investigated in the literature.

REMARK 4.2. Consider equations of the form (A) in which the exponents α and β satisfy $\alpha > \beta > 0$. Such equations are referred to as sub-half-linear generalized Thomas–Fermi differential equations. It is not known if an analogue of Avakumović's property is possessed by the sub-half-linear equations (A), but instead the existence and asymptotic behavior of all possible regularly varying solutions of such equations with regularly varying $q(t)$ have been analyzed in some detail. See, for example, the paper [**5**].

References

1. V. G. Avakumović, *Sur l'équation différentielle de Thomas–Fermi*, Acad. Serbe Sci., Publ. Inst. Math. **1** (1947), 101–113.
2. N. H. Bingham, C. M. Goldie, J. L. Teugels, *Regular Variation*, Encycl. Math. Appl. **27** Cambridge University Press, 1987.
3. J. L. Geluk, *Note on a theorem of Avakumović*, Proc. Am. Math. Soc. **112** (1991), 429–431.
4. J. Jaroš, T. Kusano, *Self-adjoint differential equations and generalized Karamata functions*, Bull. **129** Acad. Serbe Sci. Arts, Cl. Sci. Math. Nat. Sci. Math. **29** (2004), 25–60.
5. T. Kusano, V. Marić, T. Tanigawa, *An asymptotic analysis of positive solutions of generalized Thomas–Fermi differential equation-The sub-half-linear case*, Nonlinear Anal. **75** (2012), 2474–2485.
6. V. Marić, *Regular Variation and Differential Equations*, Lect. Notes Math. **1726**, Springer-Verlag, Berlin–Heidelberg, 2000.
7. V. Marić, M. Tomić, *Asymptotic properties of solutions of the equation $y'' = f(x)\phi(y)$*, Math. Z. **149** (1976), 261–266.
8. ———, *Regular variation and asymptotic properties of solutions of nonlinear differential equations*, Publ. Inst. Math., Nouv. Sér. **21**(35) (1977), 119–129.
9. ———, *Asymptotic properties of a generalized Thomas–Fermi equation*, J. Differ. Equations **35** (1980), 36–44.
10. M. Mizukami, M. Naito, H. Usami, *Asymptotic behavior of solutions of a class of second order quasilinear differential equations*, Hiroshima Math. J. **32** (2002), 51–78.
11. E. Seneta, *Regularly Varying Functions*, Lect. Notes Math. **508**, Springer-Verlag, Berlin–Heidelberg, 1976.

Department of Mathematical Analysis and Numerical Mathematics
Comenius University, Bratislava, Slovakia
Jaroslav.Jaros@fmph.uniba.sk

(Received 07 02 2013)

Professor Emeritus at:
Department of Mathematics, Faculty of Science, Hiroshima University
Higashi-Hiroshima, Japan
kusanot@zj8.so-net.ne.jp