PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 99(113) (2016), 237–242 DOI: 10.2298/PIM141026009J

A DOUBLE INEQUALITY FOR THE COMBINATION OF TOADER MEAN AND THE ARITHMETIC MEAN IN TERMS OF THE CONTRAHARMONIC MEAN

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Abstract. We find the greatest value λ and the least value μ such that the double inequality

$$C(\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a) < \alpha A(a,b) + (1-\alpha)T(a,b) < C(\mu a + (1-\mu)b, \mu b + (1-\mu)a)$$

holds for all $\alpha \in (0,1)$ and $a, b > 0$ with $a \neq b$, where $C(a,b)$, $A(a,b)$, and $T(a,b)$ denote respectively the contraharmonic, arithmetic, and Toader means of two positive numbers a and b.

1. Introduction

For $p \in \mathbb{R}$ and $a, b > 0$, the contraharmonic mean $C(a,b)$, the p-th power mean $M_p(a,b)$, and Toader mean $T(a,b)$ are respectively defined by

$$C(a,b) = \frac{a^2+b^2}{a+b}, \quad M_p(a,b) = \begin{cases} \big((a^p+b^p)/2\big)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0, \end{cases}$$

$$T(a,b) = \frac{2}{\pi}\int_0^{\pi/2} \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,d\theta = \begin{cases} \frac{2a}{\pi}\mathcal{E}\big(\sqrt{1-(b/a)^2}\big), & a > b, \\ \frac{2b}{\pi}\mathcal{E}\big(\sqrt{1-(a/b)^2}\big), & a < b, \\ a, & a = b, \end{cases}$$

where $\mathcal{E} = \mathcal{E}(r) = \int_0^{\pi/2}\sqrt{1-r^2\sin^2\theta}\,d\theta$ for $r \in [0,1]$ is the complete elliptic integral of the second kind. For more information on complete elliptic integrals, see [**11**, **13**–**15**] and plenty of references therein.

Recently, the Toader mean has attracted attention of several researchers. In particular, many remarkable inequalities for $T(a,b)$ can be found in the literature [**6**, **7**, **9**, **10**, **18**]. It was conjectured in [**17**] that

$$M_{3/2}(a,b) < T(a,b) \tag{1.1}$$

2010 *Mathematics Subject Classification:* Primary 26E60; Secondary 33E05.
Key words and phrases: bound; contraharmonic mean; arithmetic mean; Toader mean; complete elliptic integrals.
The second author was partially supported by the NNSF of China under Grant No. 11361038.
Communicated by Gradimir V. Milovanović.

for all $a, b > 0$ with $a \neq b$. This conjecture was proved in [**3**, **16**] respectively. In [**1**], the best possible upper bound for the Toader mean was presented by

$$T(a,b) < M_{\ln 2/\ln(\pi/2)}(a,b)$$

for all $a, b > 0$ with $a \neq b$.

It is not difficult to verify that

$$C(a,b) > M_2(a,b) = \sqrt{(a^2+b^2)/2} \tag{1.2}$$

for all $a, b > 0$ with $a \neq b$. From (1.1) to (1.2) one has $A(a,b) < T(a,b) < C(a,b)$ for all $a, b > 0$ with $a \neq b$.

For positive numbers $a, b > 0$ with $a \neq b$, let

$$J(x) = C\big(xa + (1-x)b, xb + (1-x)a\big)$$

on $\left[\frac{1}{2}, 1\right]$. It is not difficult to verify that $J(x)$ is continuous and strictly increasing on $\left[\frac{1}{2}, 1\right]$. Note that $J\big(\frac{1}{2}\big) = A(a,b) < T(a,b)$ and $J(1) = C(a,b) > T(a,b)$.

In [**8**] it was proved that the double inequality

$$C\big(\alpha a + (1-\alpha)b, \alpha b + (1-\alpha)a\big) < T(a,b) < C\big(\beta a + (1-\beta)b, \beta b + (1-\beta)a\big)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leqslant \frac{3}{4}$ and $\beta \geqslant \frac{1}{2} + \frac{\sqrt{4\pi - \pi^2}}{2\pi}$.

The main purpose of the paper is to find the greatest value λ and the least value μ such that the double inequality

$$\begin{aligned} C(\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a) &< \alpha A(a,b) + (1-\alpha)T(a,b) \\ &< C(\mu a + (1-\mu)b, \mu b + (1-\mu)a) \end{aligned}$$

holds for all $\alpha \in (0,1)$ and $a, b > 0$ with $a \neq b$. As applications, we also present new bounds for the complete elliptic integral of the second kind.

2. Preliminaries and lemmas

In order to establish our main result, we need several formulas and lemmas below.

For $0 < r < 1$ and $r' = \sqrt{1-r^2}$, Legendre's complete elliptic integrals of the first and second kinds are defined in [**4**, **5**] respectively by

$$\begin{cases} \mathcal{K} = \mathcal{K}(r) = \displaystyle\int_0^{\pi/2} \frac{d\theta}{\big(1 - r^2 \sin^2\theta\big)^{1/2}}, \\ \mathcal{K}' = \mathcal{K}'(r) = \mathcal{K}(r'), \\ \mathcal{K}(0) = \dfrac{\pi}{2}, \quad \mathcal{K}(1) = \infty \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{E} = \mathcal{E}(r) = \displaystyle\int_0^{\pi/2} \big(1 - r^2 \sin^2\theta\big)^{1/2} d\theta, \\ \mathcal{E}' = \mathcal{E}'(r) = \mathcal{E}(r'), \\ \mathcal{E}(0) = \dfrac{\pi}{2}, \quad \mathcal{E}(1) = 1. \end{cases}$$

For $0 < r < 1$, the formulas

$$\frac{d\mathcal{K}}{dr} = \frac{\mathcal{E} - (r')^2\mathcal{K}}{r(r')^2}, \quad \frac{d\mathcal{E}}{dr} = \frac{\mathcal{E} - \mathcal{K}}{r}, \quad \frac{d\big(\mathcal{E} - (r')^2\mathcal{K}\big)}{dr} = r\mathcal{K},$$

$$\frac{d(\mathcal{K} - \mathcal{E})}{dr} = \frac{r\mathcal{E}}{(r')^2}, \quad \mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right) = \frac{2\mathcal{E} - (r')^2\mathcal{K}}{1+r}$$

were presented in [**2**, Appendix E, pp. 474–475].

LEMMA 2.1. [**2**, Theorem 3.21(1) and 3.43 Exercise 13(a)] *The function* $\frac{\mathcal{E}-(r')^2\mathcal{K}}{r^2}$ *is strictly increasing from* $(0,1)$ *onto* $\left(\frac{\pi}{4},1\right)$ *and the function* $2\mathcal{E}-(r')^2\mathcal{K}$ *is increasing from* $(0,1)$ *onto* $\left(\frac{\pi}{2},2\right)$.

LEMMA 2.2. *Let* $u,\alpha\in(0,1)$ *and*

$$f_{u,\alpha}(r)=ur^2-(1-\alpha)\Big\{\frac{2}{\pi}\big[2\mathcal{E}(r)-\big(1-r^2\big)\mathcal{K}(r)\big]-1\Big\}.$$

Then $f_{u,\alpha}>0$ *for all* $r\in(0,1)$ *if and only if* $u\geqslant(1-\alpha)\left(\frac{4}{\pi}-1\right)$ *and* $f_{u,\alpha}<0$ *for all* $r\in(0,1)$ *if and only if* $u\leqslant\frac{1-\alpha}{4}$.

PROOF. It is clear that

$$f_{u,\alpha}\big(0^+\big)=0, \tag{2.1}$$

$$f_{u,\alpha}(1^-)=u-(1-\alpha)(4/\pi-1), \tag{2.2}$$

$$f'_{u,\alpha}(r)=2r[u-(1-\alpha)g(r)], \tag{2.3}$$

where $g(r)=\frac{1}{\pi}\frac{\mathcal{E}-(r')^2\mathcal{K}}{r^2}$.

When $u\geqslant\frac{1-\alpha}{\pi}$, from (2.3) and Lemma 2.1 and by the monotonicity of $g(r)$, it follows that $f_{u,\alpha}(r)$ is strictly increasing on $(0,1)$. Therefore, $f_{u,\alpha}(r)>0$ for all $r\in(0,1)$.

When $u\leqslant\frac{1-\alpha}{4}$, from (2.3) and Lemma 2.1 and by the monotonicity of $g(r)$, we obtain that $f_{u,\alpha}(r)$ is strictly decreasing on $(0,1)$. Therefore, $f_{u,\alpha}(r)<0$ for all $r\in(0,1)$.

When $\frac{1-\alpha}{4}<u\leqslant(1-\alpha)\left(\frac{4}{\pi}-1\right)$, from (2.2) and (2.3) and by the monotonicity of $g(r)$, we see that there exists $\lambda\in(0,1)$ such that $f_{u,\alpha}(r)$ is strictly increasing in $(0,\lambda]$ and strictly decreasing in $[\lambda,1)$ and

$$f_{u,\alpha}(1^-)\leqslant 0. \tag{2.4}$$

Therefore, making use of equation (2.1), inequality (2.4), and the piecewise monotonicity of $f_{u,\alpha}(r)$ lead to the conclusion that there exists $0<\lambda<\eta<1$ such that $f_{u,\alpha}(r)>0$ for $r\in(0,\eta)$ and $f_{u,\alpha}(r)<0$ for $r\in(\eta,1)$.

When $(1-\alpha)\left(\frac{4}{\pi}-1\right)\leqslant u<\frac{1-\alpha}{\pi}$, by (2.2), it follows that

$$f_{u,\alpha}(1^-)\geqslant 0. \tag{2.5}$$

From (2.2) and (2.3) and by the monotonicity of $g(r)$, we see that there exists $\lambda\in(0,1)$ such that $f_{u,\alpha}(r)$ is strictly increasing in $(0,\lambda]$ and strictly decreasing in $[\lambda,1)$. Therefore, $f_{u,\alpha}(r)>0$ for $r\in(0,1)$ follows from (2.1) and (2.5) together with the piecewise monotonicity of $f_{u,\alpha}(r)$. □

3. Main results

Now we are in a position to state and prove our main results.

THEOREM 3.1. *If* $\alpha\in(0,1)$ *and* $\lambda,\mu\in\left(\frac{1}{2},1\right)$, *then the double inequality*

$$\begin{aligned}C(\lambda a+(1-\lambda)b,\lambda b+(1-\lambda)a)&<\alpha A(a,b)+(1-\alpha)T(a,b)\\&<C(\mu a+(1-\mu)b,\mu b+(1-\mu)a)\end{aligned}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if

$$\lambda \leqslant \frac{1}{2} + \frac{\sqrt{1-\alpha}}{4} \quad \textit{and} \quad \mu \geqslant \frac{1}{2}\Big[1 + \sqrt{(1-\alpha)(4/\pi - 1)}\Big].$$

PROOF. Since $A(a,b)$, $T(a,b)$, and $C(a,b)$ are symmetric and homogeneous of degree one, without loss of generality, assume that $a > b$. Let $p \in \left(\frac{1}{2}, 1\right)$, $t = \frac{b}{a} \in (0,1)$, and $r = \frac{1-t}{1+t}$. Then

$$\begin{aligned}
&C\big(pa + (1-p)b, pb + (1-p)a\big) - \alpha A(a,b) - (1-\alpha)T(a,b) \\
&\quad = a\frac{[p + (1-p)b/a]^2 + (pb/a + 1 - p)^2}{1 + b/a} - \alpha a \frac{1 + b/a}{2} \\
&\quad\quad - (1-\alpha)\frac{2a}{\pi}\mathcal{E}\Big(\sqrt{1 - (b/a)^2}\Big) \\
&\quad = a\bigg\{\frac{[p + (1-p)t]^2 + (pt + 1 - p)^2}{1+t} - \alpha\frac{1+t}{2} - (1-\alpha)\frac{2}{\pi}\mathcal{E}\Big(\sqrt{1-t^2}\Big)\bigg\} \\
&\quad = a\bigg\{\frac{(1-2p)^2 r^2 + 1}{1+r} - \alpha\frac{1}{1+r} - (1-\alpha)\frac{2}{\pi}\frac{2\mathcal{E} - (r')^2\mathcal{K}}{1+r}\bigg\} \\
&\quad = \frac{a}{1+r}\bigg[(1-2p)^2 r^2 + 1 - \alpha - (1-\alpha)\frac{2}{\pi}\big(2\mathcal{E} - (r')^2\mathcal{K}\big)\bigg].
\end{aligned}$$

From this and Lemma 2.2, Theorem 3.1 follows. □

COROLLARY 3.1. *For $r \in (0,1)$ and $r' = \sqrt{1-r^2}$, we have*

$$\frac{\pi}{2}\left[\frac{17 + 30r' + 17(r')^2}{8(1+r')} - \frac{3(1+r')}{2}\right] < \mathcal{E}(r) < \pi\left[\frac{r' + 2(1-r')^2/\pi}{1+r'}\right]. \tag{3.1}$$

PROOF. This follows from letting $\alpha = \frac{3}{4}$, $\lambda = \frac{5}{8}$, and $\mu = \frac{1}{2}\Big(1 + \frac{\sqrt{4/\pi - 1}}{2}\Big)$ in Theorem 3.1. □

4. Remarks

REMARK 4.1. Recently, the complete elliptic integrals have attracted attention of numerous mathematicians. In [**9**], it was established that

$$\begin{aligned}
\frac{\pi}{2}\left[\frac{1}{2}\sqrt{\frac{1+(r')^2}{2}} + \frac{1+r'}{4}\right] &< \mathcal{E}(r) \\
&< \frac{\pi}{2}\left[\frac{4-\pi}{(\sqrt{2}-1)\pi}\sqrt{\frac{1+(r')^2}{2}} + \frac{(\sqrt{2}\,\pi - 4)(1+r')}{2(\sqrt{2}-1)\pi}\right],
\end{aligned} \tag{4.1}$$

for all $r \in (0,1)$. In [**11**] it was proved that

$$\frac{\pi}{2} - \frac{1}{2}\log\frac{(1+r)^{1-r}}{(1-r)^{1+r}} < \mathcal{E}(r) < \frac{\pi - 1}{2} + \frac{1-r^2}{4r}\log\frac{1+r}{1-r}, \tag{4.2}$$

for all $r \in (0,1)$. In [**19**] it was presented that

$$\frac{\pi}{2}\frac{\sqrt{6 + 2\sqrt{1-r^2} - 3r^2}}{2\sqrt{2}} \leqslant \mathcal{E}(r) \leqslant \frac{\pi}{2}\frac{\sqrt{10 - 2\sqrt{1-r^2} - 5r^2}}{2\sqrt{2}} \tag{4.3}$$

for all $r \in (0,1)$. In [**9**] it was pointed out that the bounds in (4.1) for $\mathcal{E}(r)$ are better than the bounds in (4.2) for some $r \in (0,1)$.

Remark 4.2. The lower bound in (3.1) for $\mathcal{E}(r)$ is better than the lower bound in (4.1). Indeed,

$$\frac{17+30x+17x^2}{8(1+x)} - \frac{3(1+x)}{2} - \left[\frac{1}{2}\sqrt{\frac{1+x^2}{2}} + \frac{1+x}{4}\right] = \frac{3x^2+2x+3-2\sqrt{2(1+x^2)}\,(1+x)}{8(1+x)}$$

and

$$(3x^2+2x+3)^2 - \left[2\sqrt{2(1+x^2)}\,(1+x)\right]^2 = (1-x)^4 > 0$$

for all $x \in (0,1)$.

Remark 4.3. The following equivalence relations show that the lower bound in (3.1) for $\mathcal{E}(r)$ is better than the lower bound in (4.3):

$$\frac{17+30x+17x^2}{8(1+x)} - \frac{3(1+x)}{2} > \frac{\sqrt{6+2x-3(1-x^2)}}{2\sqrt{2}}$$
$$\Leftrightarrow (5x^2+6x+5)^2 > 8(x+1)^2(3x^2+2x+3) \Leftrightarrow (x-1)^4 > 0,$$

where $x \in (0,1)$.

Remark 4.4. This paper is a slightly revised version of the preprint [**12**].

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(Received 05 06 2014)

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