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THE STABILITY OF A GENERALIZED AFFINE FUNCTIONAL EQUATION IN FUZZY NORMED SPACES

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Abstract. We obtain the general solution of the following functional equation

$$f(kx_1+x_2+\cdots+x_k)+f(x_1+kx_2+\cdots+x_k)+\cdots+f(x_1+x_2+\cdots+kx_k) + f(x_1)+f(x_2)+\cdots+f(x_k)=2kf(x_1+x_2+\cdots+x_k),\ k\geqslant 2.$$

We establish the Hyers–Ulam–Rassias stability of the above functional equation in the fuzzy normed spaces. More precisely, we show under suitable conditions that a fuzzy q-almost affine mapping can be approximated by an affine mapping. Further, we determine the stability of same functional equation by using fixed point alternative method in fuzzy normed spaces.

1. Introduction

In modelling applied problems only partial informations may be known (or) there may be a degree of uncertainty in the parameters used in the model or some measurements may be imprecise. Due to such features, we are tempted to consider the study of functional equations in the fuzzy settings. For the last 40 years, the fuzzy theory has become a very active area of research and a lot of development has been made in the theory of fuzzy sets [**1**] to find the fuzzy analogues of the classical set theory. This branch finds a wide range of applications in the field of science and engineering. Katsaras [**2**] introduced an idea of fuzzy norm on a linear space in 1984. In [**3**], the authors study the stability problems in fuzzy Banach spaces. In [**4**], Felbin introduced an alternative definition of a fuzzy norm on a linear topological structure of a fuzzy normed linear spaces. Papers [**5, 6, 7**] are good survey papers, in which results and history on stability are given.

In 1940, Ulam [**8**] raised a question concerning the stability of group homomorphism as follows: Let G_1 be a group and G_2 a metric group with the metric

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$d(.,.)$. Given $\varepsilon > 0$, does there exist any $\delta > 0$ such that, if a function $f : G_1 \to G_2$ satisfies the inequality

$$d(f(xy), f(x)f(y)) < \delta \text{ for all } x, y \in G_1,$$

then there exists a homomorphism $h : G_1 \to G_2$ with

$$d(f(x), H(x)) < \varepsilon \text{ for all } x \in G_1?$$

The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, the case of approximately additive mappings was solved by Hyers [**9**] under the assumption that G_2 is a Banach space. In 1978, a generalized version of the theorem of Hyers for approximately linear mapping was given by Rassias [**10**]. He proved that for a mapping $f : E_1 \to E_2$ such that $f(tx)$ is continuous in $t \in \mathbb{R}$ and for each fixed $x \in E_1$ assume that there exists a constant $\varepsilon > 0$ and $p \in [0, 1)$ with

$$\| f(x+y) - f(x) - f(y) \| \leqslant \varepsilon(\| x \|^p + \| y \|^p) \tag{1.1}$$

$x, y \in E_1$, then there exists a unique R-Linear mapping $T : E_1 \to E_2$ such that

$$\| f(x) - T(x) \| \leqslant \frac{2\varepsilon}{2 - 2^p} \| x \|^p \quad (x \in E_1).$$

The result of Rassias has influenced the development of what is now called the Hyers–Ulam–Rassias stability theory for functional equations. In 1994, a generalization of Rassias' theorem was obtained by Gavruta [**11**] by replacing the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. During the last decades, the stability problems of several functional equations have been extensively investigated by a number of authors (e.g. [**12**] etc.). In 1982–1989, Rassias [**13, 14**] replaced the sum which appeared on the right-hand side of equation (1.1) by the product of powers of norms.

In 2003, Radu [**15**] proposed a new method, successively developed in [**16**], to obtain the existence of the exact solutions and the error estimations, based on the fixed point alternative method. Subsequently, these results were generalized by Mihet [**17**]. Recently, Cadariu et al [**18**] studied the generalized Hyers–Ulam stability by using the direct method as well as the fixed point method for the affine type functional equation

$$f(2x + y) + f(x + 2y) + f(x) + f(y) = 4f(x + y), \text{ for all } x, y \in G.$$

We obtain the general solution of the functional equation

$$\begin{aligned} f(kx_1 + x_2 + \cdots + x_k) + f(x_1 + kx_2 + \cdots + x_k) + \cdots + f(x_1 + x_2 + \cdots + kx_k) \\ + f(x_1) + f(x_2) + \cdots + f(x_k) = 2kf(x_1 + x_2 + \cdots + x_k), \ k \geqslant 2 \end{aligned} \tag{1.2}$$

where $f : X \to Y$, X and Y are normed spaces. Then, we establish the fuzzy Hyers–Ulam–Rassias stability of the above functional equation and also we approximate a fuzzy q-almost affine mapping by an affine mapping under suitable conditions. Further, we determine the stability of functional equation (1.2) by using fixed point alternative method in fuzzy normed spaces.

2. Preliminary notes

Before we proceed to the main results, we will give some definitions and examples to illustrate the idea of fuzzy norm. Quite recently, the stability problem for the Jensen functional equation, additive functional equation, Pexiderized quadratic functional equation, cubic functional equation and mixed type additive cubic functional equations have been considered in [**19**]–[**27**].

Definition 2.1. Let X be a real linear space. A mapping $N : X \times \mathbb{R} \to [0, 1]$ (the so-called fuzzy subset) is said to be a *fuzzy norm* on X if for all $x, y \in X$ and all $s,\ t \in \mathbb{R}$,

(N_1) $N(x, t) = 0$ for $t \leqslant 0$;
(N_2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;
(N_3) $N(cx, t) = N(x, t/|c|)$ if $c \neq 0$;
(N_4) $N(x + y, t + s) \geqslant \min\{N(x, t), N(y, s)\}$;
(N_5) $N(x, .)$ is a nondecreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;
(N_6) for $x \neq 0$, $N(x, .)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a *fuzzy normed linear space*. One may regard $N(x, t)$ as the truth value of the statement that the norm of x is less than or equal to the real number t.

Example 2.1. Let $(X, \|.\|)$ be a normed linear space. One can easily verify that for each $p > 0$,

$$N_p(x, t) = \begin{cases} \frac{t}{t+p\|x\|}, & t > 0,\ \ x \in X; \\ 0, & t \leqslant 0,\ \ x \in X \end{cases}$$

is a fuzzy norm on X.

Example 2.2. Let $(X, \|.\|)$ be a normed linear space. The mapping $N : X \times \mathbb{R} \to [0, 1]$ defined by

$$N(x, t) = \begin{cases} \frac{t^2-\|x\|^2}{t^2+\|x\|^2}, & t > \|x\|; \\ 0, & t \leqslant \|x\| \end{cases}$$

is a fuzzy norm on X.

Definition 2.2. Let (X, N) be a fuzzy normed linear space. A sequence $\{x_n\}$ in X is said to be convergent if there exists an $x \in X$ such that $\lim\limits_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ and we denote it by N-$\lim_{n\to\infty} N(x_n - x, t) = x$.

Definition 2.3. Let (X, N) be a fuzzy normed linear space. A sequence $\{x_n\}$ in X is said to be Cauchy if for each $\varepsilon > 0$ and each $\delta > 0$ there exists an $n_0 \in \mathbb{N}$ such that $N(x_m - x_n, \delta) > 1 - \varepsilon$ $(m, n \geqslant n_0)$.

It is well known that every convergent sequence in a fuzzy normed linear space is a Cauchy sequence. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called fuzzy Banach space.

The remaining part of the paper is organized as follows: we discuss the general solution of functional equation (1.2) in Subection 2.1. In Subsection 2.2 we investigate the stability of functional equation (1.2) in fuzzy normed spaces and we show that, under suitable conditions, a fuzzy q-almost affine mapping can be approximated by an affine mapping. In Section 3 we prove some fuzzy stability results for functional equation (1.2) via fixed point alternative method.

Now we proceed to find the general solution of functional equation (1.2).

2.1. Solution. We begin with the following theorem.

THEOREM 2.1. *A mapping $f : X \to Y$, where X and Y are normed spaces, is a solution of functional equation* (1.2) *if and only if it is an affine mapping (i.e., it is the sum of a constant and an additive function).*

PROOF. We can easily see that any affine function f is a solution of equation (1.2). Conversely, we have two cases:

Case 1: $f(0) = 0$. If we take $x_2 = x_3 = \cdots = x_k = -x_1$ and finally replacing x_1 with x in (1.2), we obtain

$$2f(x) + (k-1)f((3-2k)x) + (k-1)f(-x) = 2kf((2-k)x), \text{ for all } x \in X. \tag{2.1}$$

Again replacing x_1 with x and putting $x_2 = x_3 = \cdots = x_k = 0$ in (1.2), we obtain

$$f(kx) = kf(x), \text{ for all } x \in X. \tag{2.2}$$

By (2.1) and (2.2), we have $f(-x) = -f(x)$, for all $x \in X$. It results that f is an odd mapping. Take $x_4 = x_5 = \cdots = x_k = 0$ in (1.2), we have

$$f(kx_1 + x_2 + x_3) + f(x_1 + kx_2 + x_3) + f(x_1 + x_2 + kx_3) + f(x_1) + f(x_2) + f(x_3) = (k+3)f(x_1 + x_2 + x_3).$$

Replace x_3 by $-x_2$ in the last equation, we get

$$f(x_1 + (k-1)x_2) + f(x_1 + (1-k)x_2) = 2f(x_1). \tag{2.3}$$

If we replace x_1 and x_2 by $\frac{u+v}{2}$ and $\frac{u-v}{2(k-1)}$, respectively, in (2.3) and using (2.2), we have $f(u+v) = f(u) + f(v)$, for all $u, v \in X$. So, f is an additive mapping.

Case 2: General case. Let us consider the function $g(x) := f(x) - f(0)$. It is clear that $g(0) = 0$ and $f(x) = g(x) + f(0)$. Replacing f in (1.2), it results

$$g(kx_1 + x_2 + \cdots + x_k) + g(x_1 + kx_2 + \cdots + x_k) + \cdots + g(x_1 + x_2 + \cdots + kx_k) + g(x_1) + g(x_2) + \cdots + g(x_k) = 2kg(x_1 + x_2 + \cdots + x_k),$$

for all $x_1, x_2, \ldots, x_k \in X$. Taking into account that $g(0) = 0$, from Case 1, we obtain that g is an additive mapping, hence $f(x) = g(x) + f(0)$ is an affine function. $\square$

2.2. Fuzzy stability. For a given mapping $f : X \to Y$, let us denote

$$Df(x_1, x_2, \ldots, x_k) = f(kx_1 + x_2 + \cdots + x_k) + f(x_1 + kx_2 + \cdots + x_k) + \cdots + f(x_1 + x_2 + \cdots + kx_k) + f(x_1) + f(x_2) + \cdots + f(x_k) - 2kf(x_1 + x_2 + \cdots + x_k).$$

2.2.1. *Fuzzy Hyers–Ulam–Rassias stability: non-uniform version.* From now on X^k will denote $X \times X \times \cdots \times X$ (k times).

Theorem 2.2. *Let X be a linear space and (Z, N') a fuzzy normed space. Let $\varphi : X^k \to Z$ be a mapping such that for some $\alpha \neq 0$ with $0 < \alpha < k$ we have $N'(\varphi(kx, 0, 0, \ldots, 0), t) \geqslant N'(\alpha\varphi(x, 0, 0, \ldots, 0), t)$ for all $x \in X$, $t > 0$ and $\lim_{n\to\infty} N'(\varphi(k^n x_1, k^n x_2, \ldots, k^n x_k), k^n t) = 1$, for all $x_1, x_2, \ldots, x_k \in X$ and all $t > 0$. Suppose that (Y, N) is a fuzzy Banach space and an odd mapping $f : X \to Y$ satisfies the inequality*

$$N(Df(x_1, x_2, \ldots, x_k), t) \geqslant N'(\varphi(x_1, x_2, \ldots, x_k), t) \tag{2.4}$$

for all $x_1, x_2, \ldots, x_k \in X$ and all $t > 0$. Then the limit $A(x) = N\text{-}\lim_{n\to\infty} \frac{f(k^n x)}{k^n}$ exists for all $x \in X$ and the mapping $A : X \to Y$ is the unique affine mapping satisfying

$$N(f(x) - A(x) - f(0), t) \geqslant N'(\varphi(x, 0, 0, \ldots, 0), (k - \alpha)t) \tag{2.5}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $x_2 = x_3 = \cdots = x_k = 0$ and replacing x_1 by x in (2.4), we get

$$N(f(kx) - kf(x) + (k-1)f(0), t) \geqslant N'(\varphi(x, 0, 0, \ldots, 0), t) \tag{2.6}$$

for all $x \in X$ and all $t > 0$. Let $g(x) := f(x) - f(0)$ for all $x \in X$. Then (2.6) implies $N(g(kx) - kg(x), t) \geqslant N'(\varphi(x, 0, 0, \ldots, 0), t)$. Replacing x by $k^n x$ in the last inequality, we obtain

$$N(g(k^{n+1}x) - kg(k^n x), t) \geqslant N'(\varphi(k^n x, 0, 0, \ldots, 0), t)$$

$$N\Big(\frac{g(k^{n+1}x)}{k^{n+1}} - \frac{g(k^n x)}{k^n}, \frac{t}{k^{n+1}}\Big) \geqslant N'\Big(\varphi(x, 0, 0, \ldots, 0), \frac{t}{\alpha^n}\Big)$$

$$N\Big(\frac{g(k^{n+1}x)}{k^{n+1}} - \frac{g(k^n x)}{k^n}, \frac{\alpha^n t}{k^{n+1}}\Big) \geqslant N'(\varphi(x, 0, 0, \ldots, 0), t) \tag{2.7}$$

for all $x \in X$ and all $t > 0$. It follows from

$$\frac{g(k^n x)}{k^n} - g(x) = \sum_{j=0}^{n-1} \frac{g(k^{j+1}x)}{k^{j+1}} - \frac{g(k^j x)}{k^j}$$

and (2.7) that

$$\begin{aligned} N\Big(\frac{g(k^n x)}{k^n} - g(x), \sum_{j=0}^{n-1} \frac{\alpha^j t}{k^{j+1}}\Big) &= N\Big(\sum_{j=0}^{n-1} \frac{g(k^{j+1}x)}{k^{j+1}} - \frac{g(k^j x)}{k^j}, \sum_{j=0}^{n-1} \frac{\alpha^j t}{k^{j+1}}\Big) \\ &\geqslant \min \bigcup_{j=0}^{n-1} \Big\{N\Big(\frac{g(k^{j+1}x)}{k^{j+1}} - \frac{g(k^j x)}{k^j}, \frac{\alpha^j t}{k^{j+1}}\Big)\Big\} \geqslant N'(\varphi(x, 0, 0, \ldots, 0), t) \end{aligned} \tag{2.8}$$

for all $x \in X$ and all $t > 0$. Replacing x by $k^m x$ in (2.8), we get

$$N\Big(\frac{g(k^{n+m}x)}{k^{n+m}} - \frac{g(k^m x)}{k^m}, \sum_{j=0}^{n-1} \frac{\alpha^j t}{k^{j+m+1}})\Big) \geqslant N'\Big(\varphi(x, 0, 0, \ldots, 0), \frac{t}{\alpha^m}\Big)$$

and so

$$N\left(\frac{g(k^{n+m}x)}{k^{n+m}}-\frac{g(k^m x)}{k^m},\sum_{j=m}^{n+m-1}\frac{\alpha^j t}{k^{j+1}}\right)\geqslant N'(\varphi(x,0,0,\dots,0),t)$$

$$N\left(\frac{g(k^{n+m}x)}{k^{n+m}}-\frac{g(k^m x)}{k^m},t\right)\geqslant N'\left(\varphi(x,0,0,\dots,0),\frac{t}{\sum_{j=m}^{n+m-1}\frac{\alpha^j}{k^{j+1}}}\right) \tag{2.9}$$

for all $x\in X$, $t>0$ and $m,n\geqslant 0$. Since $0<\alpha<k$ and $\sum_{j=0}^{\infty}(\frac{\alpha}{k})^j<\infty$, the Cauchy criterion for convergence and (N_5) imply that $\{\frac{g(k^n x)}{k^n}\}$ is a Cauchy sequence in (Y,N). Since (Y,N) is a fuzzy Banach space, this sequence converges to some point $A(x)\in Y$. Hence, we can define a mapping $A:X\to Y$ by $A(x)=N\text{-}\lim_{n\to\infty}\frac{g(k^n x)}{k^n}=N\text{-}\lim_{n\to\infty}\frac{f(k^n x)}{k^n}$ for all $x\in X$, namely. Since f is odd, A is odd. Letting $m=0$ in (2.9), we get

$$N\left(\frac{g(k^n x)}{k^n}-g(x),t\right)\geqslant N'\left(\varphi(x,0,0,\dots,0),\frac{t}{\sum_{j=0}^{n-1}\frac{\alpha^j}{k^{j+1}}}\right).$$

Taking the limit as $n\to\infty$ and using (N_6), we get

$$\begin{aligned}N(A(x)-g(x),t)&\geqslant N'\left(\varphi(x,0,0,\dots,0),\frac{t}{\sum_{j=0}^{\infty}\frac{\alpha^j}{k^{j+1}}}\right)\\&=N'(\varphi(x,0,0,\dots,0),(k-\alpha)t)\\N(f(x)-A(x)-f(0),t)&\geqslant N'(\varphi(x,0,0,\dots,0),(k-\alpha)t)\end{aligned}$$

for all $x\in X$ and all $t>0$. Now we claim that A is affine. Replacing $x_1,x_2,\dots,x_k$ by $k^n x_1,k^n x_2,\dots,k^n x_k$, respectively, in (2.4), we get

$$N\left(\frac{1}{k^n}Df(k^n x_1,k^n x_2,\dots,k^n x_k),t\right)\geqslant N'(\varphi(k^n x_1,k^n x_2,\dots,k^n x_k),k^n t)$$

for all $x_1,x_2,\dots,x_k\in X$ and all $t>0$. Since

$$\lim_{n\to\infty}N'(\varphi(k^n x_1,k^n x_2,\dots,k^n x_k),k^n t)=1,$$

A satisfies functional equation (1.2). Hence A is affine. To prove the uniqueness of A, let $A':X\to Y$ be another affine mapping satisfying (2.5). Fix $x\in X$. Clearly $A(k^n x)=k^n A(x)$ and $A'(k^n x)=k^n A'(x)$ for all $x\in X$ and all $n\in\mathbb{N}$. It follows from (2.5) that

$$\begin{aligned}N(A(x)-A'(x),t)&=N\left(\frac{A(k^n x)}{k^n}-\frac{A'(k^n x)}{k^n},t\right)\\&\geqslant\min\left\{N\left(\frac{A(k^n x)}{k^n}-\frac{g(k^n x)}{k^n},\frac{t}{2}\right),\ N\left(\frac{g(k^n x)}{k^n}-\frac{A'(k^n x)}{k^n},\frac{t}{2}\right)\right\}\\&\geqslant N'\left(\varphi(k^n x,0,0,\dots,0),\frac{k^n(k-\alpha)t}{2}\right)\geqslant N'\left(\varphi(x,0,0,\dots,0),\frac{k^n(k-\alpha)t}{2\alpha^n}\right)\end{aligned}$$

for all $x\in X$ and all $t>0$. Since $\lim\limits_{n\to\infty}\frac{k^n(k-\alpha)}{2\alpha^n}=\infty$, we obtain

$$\lim_{n\to\infty}N'\left(\varphi(x,0,0,\dots,0),\frac{k^n(k-\alpha)t}{2\alpha^n}\right)=1.$$

Thus $N(A(x)-A'(x),\ t)=1$ for all $x\in X$ and all $t>0$, and so $A(x)=A'(x)$. □

Example 2.3. Let X be a normed space and let N and N' be the fuzzy norms on X and $\mathbb{R}$, respectively, defined in Example 2.1 when $p=1$. Let $\psi:[0,\infty)\to(0,\infty)$ be a function such that $\psi(kr)\leqslant\alpha\psi(r)$ for all $r\geqslant 0$, where $0<\alpha<k$. Define

$$\begin{aligned}\varphi(x_1,x_2,\ldots,x_k)&=\psi(\|kx_1+x_2+\cdots+x_k\|)\\&\quad+\psi(\|x_1+kx_2+\cdots+x_k\|)+\cdots+\psi(\|x_1+x_2+\cdots+kx_k\|)\\&\quad+\psi(\|x_1\|)+\psi(\|x_2\|)+\cdots+\psi(\|x_k\|)-2k\psi(\|x_1+x_2+\cdots+x_k\|)\end{aligned}$$

for each $x_1,x_2,\ldots,x_k\in X$. Let $f(0)=x_0(\text{say})\in X$ be a fixed unit vector and define $f:X\to X$ by $f(x)=x+\psi(\|x\|)x_0$. Then for each $x_1,x_2,\ldots,x_k\in X$ and $t>0$, we have $N(Df(x_1,x_2,\ldots,x_k),t)\geqslant N'(\varphi(x_1,x_2,\ldots,x_k),t)$. Moreover, for each $x_1,x_2,\ldots,x_k\in X$ and $t>0$, we have

$$\begin{aligned}N'(\varphi(kx,0,\ldots,0),t)&=\frac{t}{t+\varphi(kx,0,\ldots,0)}\\&\geqslant\frac{t}{t+\alpha\varphi(x,0,\ldots,0)}=N'(\alpha\varphi(x,0,\ldots,0),t).\end{aligned}$$

Therefore, by Theorem 2.2, There exists a unique affine mapping $A:X\to X$ such that for each $x\in X$ and $t>0$,

$$N(f(x)-A(x)-f(0),t)\geqslant N'(\varphi(x,0,\ldots,0),(k-\alpha)t).$$

2.2.2. *Fuzzy Hyers–Ulam–Rassias stability: uniform version.*

Theorem 2.3. *Let X be a linear space and (Y,N) a fuzzy Banach space. Let $\varphi:\ X^k\to[0,\infty)$ be a function such that*

$$\tilde{\varphi}(x_1,x_2,\ldots,x_k)=\sum_{n=0}^{\infty}\frac{1}{k^n}\varphi(k^nx_1,k^nx_2,\ldots,k^nx_k)<\infty \tag{2.10}$$

for all $x_1,x_2,\ldots,x_k\in X$. Let $f:X\to Y$ be a uniformly approximately affine mapping with respect to φ in the sense that

$$\lim_{t\to\infty}N(Df(x_1,x_2,\ldots,x_k),t\varphi(x_1,x_2,\ldots,x_k))=1 \tag{2.11}$$

uniformly on $X\times X\times\cdots\times X$ (k-times). Then $A(x):=N\text{-}\lim_{n\to\infty}\frac{f(k^nx)}{k^n}$ for all $x\in X$ exists and defines an affine mapping $A:\ X\to Y$ such that if for some $\alpha>0,\ \delta>0$

$$N(Df(x_1,x_2,\ldots,x_k),\delta\varphi(x_1,x_2,\ldots,x_k))>\alpha \tag{2.12}$$

for all $x_1,x_2,\ldots,x_k\in X$, then $N(f(x)-A(x)-f(0),\frac{\delta}{k}\tilde{\varphi}(0,0,\ldots,0,x))>\alpha$ for all $x\in X$.

Proof. Let $\varepsilon>0$; by (2.11) we can find $t_0>0$ such that

$$N(Df(x_1,x_2,\ldots,x_k),t\varphi(x_1,x_2,\ldots,x_k))\geqslant 1-\varepsilon \tag{2.13}$$

for all $x_1, x_2, \ldots, x_k \in X$ and all $t \geqslant t_0$. Let $g(x) := f(x) - f(0)$. From (2.13), we have

$$N(Dg(x_1, x_2, \ldots, x_k), t\varphi(x_1, x_2, \ldots, x_k)) \geqslant 1 - \varepsilon \tag{2.14}$$

for all $x_1, x_2, \ldots, x_k \in X$ and all $t \geqslant t_0$. By induction on n, we will show that

$$N(g(k^n x) - k^n g(x), t\sum_{m=0}^{n-1} k^{n-m-1}\varphi(0, 0, \ldots, 0, k^m x)) \geqslant 1 - \varepsilon \tag{2.15}$$

for all $x \in X$, all $t \geqslant t_0$ and $n \in \mathbb{N}$. Putting $x_1 = x_2 = \cdots = x_{k-1} = 0$ and for our convenience replace x_k by x in (2.14), we get (2.15) for $n = 1$. Let (2.15) hold for some positive integers n. Then

$$\begin{aligned} N\Big(g(k^{n+1}x) - k^{n+1}g(x), t\sum_{m=0}^{n} k^{n-m}\varphi(0, 0, \ldots, 0, k^m x)\Big) \\ \geqslant \min\Big\{N\big(g(k^{n+1}x) - kg(k^n x), t\varphi(0, 0, \ldots, 0, k^n x)\big), \\ N(kg(k^n x) - k^{n+1}g(x), t\sum_{m=0}^{n} k^{n-m}\varphi(0, 0, \ldots, 0, k^m x))\Big\} \\ \geqslant \min\{1 - \varepsilon, 1 - \varepsilon\} = 1 - \varepsilon. \end{aligned}$$

This completes the induction argument. Let $t = t_0$ and put $n = p$. Then by replacing x with $k^n x$ in (2.15), we obtain

$$N\Big(g(k^{n+p}x) - k^p g(k^n x), t_0\sum_{m=0}^{p-1} k^{p-m-1}\varphi(0, 0, \ldots, 0, k^{n+m}x)\Big) \geqslant 1 - \varepsilon$$

$$N\Big(\frac{g(k^{n+p}x)}{k^{n+p}} - \frac{g(k^n x)}{k^n}, t_0\sum_{m=0}^{p-1} k^{-(n+m+1)}\varphi(0, 0, \ldots, 0, k^{n+m}x)\Big) \geqslant 1 - \varepsilon \tag{2.16}$$

for all integers $n \geqslant 0,\ p > 0$. The convergence of (2.10) and the equation

$$\sum_{m=0}^{p-1} k^{-(n+m+1)}\varphi(0, 0, \ldots, 0, k^{n+m}x)) = \frac{1}{k}\sum_{m=n}^{n+p-1} k^{-m}\varphi(0, 0, \ldots, 0, k^m x)$$

guarantees that for given $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{t_0}{k}\sum_{m=n}^{n+p-1} k^{-m}\varphi(0, 0, \ldots, 0, k^m x) < \delta$$

for all $n \geqslant n_0$ and $p > 0$. It follows from (2.16) that

$$\begin{aligned} N\Big(\frac{g(k^{n+p}x)}{k^{n+p}} - \frac{g(k^n x)}{k^n}, \delta\Big) \\ \geqslant \Big(\frac{g(k^{n+p}x)}{k^{n+p}} - \frac{g(k^n x)}{k^n}, t_0\sum_{m=0}^{p-1} k^{-(n+m+1)}\varphi(0, 0, \ldots, 0, k^{n+m}x)\Big) \geqslant 1 - \varepsilon \end{aligned} \tag{2.17}$$

for each $n \geqslant n_0$ and all $p > 0$. Hence $\{\frac{g(k^n x)}{k^n}\}$ is a Cauchy sequence in Y. Since Y is a fuzzy Banach space, this sequence converges to some $A(x) \in Y$. Hence we can define a mapping $A : X \to Y$ by $A(x) := N\text{-}\lim_{n\to\infty} \frac{g(k^n x)}{k^n} = N\text{-}\lim_{n\to\infty} \frac{f(k^n x)}{k^n}$, for all $x \in X$ namely. For each $t > 0$ and $x \in X$

$$\lim_{n\to\infty} N\Big(A(x) - \frac{f(k^n x)}{k^n}, t\Big) = 1.$$

Now, let $x_1, x_2, \ldots, x_k \in X$. Fix $t > 0$ and $0 < \varepsilon < 1$. Since $\lim_{n\to\infty} \frac{1}{k^n}\varphi(k^n x_1, k^n x_2, \ldots, k^n x_k) = 0$, there is some $n_1 > n_0$ such that

$$\begin{aligned}
&N(DA(x_1,x_2, \ldots, x_k), t)\\
&\quad\geqslant \min\Big\{N\Big(A(kx_1 + x_2 + \cdots + x_k) - \frac{f(k^n(kx_1 + x_2 + \cdots + x_k))}{k^n}, \frac{t}{2k+2}\Big),\\
&\qquad N\Big(A(x_1 + kx_2 + \cdots + x_k) - \frac{f(k^n(x_1 + kx_2 + \cdots + x_k))}{k^n}, \frac{t}{2k+2}\Big), \ldots,\\
&\qquad N\Big(A(x_1 + x_2 + \cdots + kx_k) - \frac{f(k^n(x_1 + x_2 + \cdots + kx_k))}{k^n}, \frac{t}{2k+2}\Big),\\
&\qquad N\Big(A(x_1) - \frac{f(k^n x_1)}{k^n}, \frac{t}{2k+2}\Big),\\
&\qquad N\Big(A(x_2) - \frac{f(k^n x_2)}{k^n}, \frac{t}{2k+2}\Big), \cdots, N\Big(A(x_k) - \frac{f(k^n x_k)}{k^n}, \frac{t}{2k+2}\Big),\\
&\qquad N\Big(A(x_1 + x_2 + \cdots + x_k) - \frac{f(k^n(x_1 + x_2 + \cdots + x_k))}{k^n}, \frac{t}{(2k+2)}\Big),\\
&\qquad N\Big(Df(k^n x_1, k^n x_2, \ldots, k^n x_k), \frac{k^n t}{2k+2}\Big)\Big\}.
\end{aligned}$$

The first $(2k+1)$ terms on the right-hand side of the above inequality tend to 1 as $n \to \infty$ and the last term is greater than $N(Df(k^n x_1, k^n x_2, \ldots, k^n x_k), t_0\varphi(k^n x_1, k^n x_2, \ldots, k^n x_k))$, i.e., by (2.13), greater than or equal to $1 - \varepsilon$. Thus $N(DA(x_1, x_2, \ldots, x_k), t) \geqslant 1 - \varepsilon$ for all $t \geqslant 0$ and $0 < \varepsilon < 1$. It follows that $N(DA(x_1, x_2, \ldots, x_k), t) = 1$ for all $t > 0$ and by (N_2), we have $DA(x_1, x_2, \ldots, x_k) = 1$, i.e.,

$$\begin{aligned}
A(kx_1 + x_2 + \cdots + x_k) + A(x_1 + kx_2 + \cdots + x_k) + \cdots + A(x_1 + x_2 + \cdots + kx_k)\\
+A(x_1) + A(x_2) + \cdots + A(x_k) = 2kA(x_1 + x_2 + \cdots + x_k).
\end{aligned}$$

To end the proof, let for some positive α and δ, (2.12) hold. Let

$$\varphi_n(x_1, x_2, \ldots, x_k) := \sum_{m=0}^{n-1} k^{-(m+1)}\varphi(k^m x_1, k^m x_2, \ldots, k^m x_k)$$

for all $x_1, x_2, \ldots, x_k \in X$. Let $x \in X$. By a similar discussion as in the begining of the proof, we can obtain from (2.12)

$$N(g(k^n x) - k^n g(x), \delta \sum_{m=0}^{n-1} k^{(n-m-1)}\varphi_n(0, 0, \ldots, 0, k^m x)) \geqslant \delta \tag{2.18}$$

for all $n \in \mathbb{N}$. Let $s > 0$. We have

$$N\big(g(x) - A(x), \delta\varphi_n(0, 0, \ldots, 0, x) + s\big)$$
$$\geqslant \min\Big\{N\Big(g(x) - \frac{g(k^n x)}{k^n}, \delta\varphi_n(0, 0, \ldots, 0, x)\Big),\ N\Big(\frac{g(k^n x)}{k^n} - A(x), s\Big)\Big\}.$$

Combining (2.17), (2.18) and the fact that

$$\lim_{n\to\infty} N\Big(\frac{g(k^n x)}{k^n} - A(x), s\Big) = \lim_{n\to\infty} N\Big(\frac{f(k^n x)}{k^n} - A(x), s\Big) = 1,$$

we obtain that $N(g(x) - A(x), \delta\varphi_n(0, 0, \ldots, 0, x) + s) \geqslant \alpha$ for n large enough. By the (upper semi) continuity of the real function $N(g(x) - A(x), .)$, we obtain that

$$N\Big(g(x) - A(x), \frac{\delta}{k}\tilde{\varphi}(0, 0, \ldots, 0, x) + s\Big) \geqslant \alpha.$$

Taking the limit as $s \to 0$, we conclude that

$$N\Big(g(x) - A(x), \frac{\delta}{k}\tilde{\varphi}(0, 0, \ldots, 0, x)\Big) \geqslant \alpha$$
$$N\Big(f(x) - A(x) - f(0), \frac{\delta}{k}\tilde{\varphi}(0, 0, \ldots, 0, x)\Big) \geqslant \alpha. \qquad \square$$

Theorem 2.4. *Let X be a linear space and (Y, N) a fuzzy Banach space. Let $\varphi : X^k \to [0, \infty)$ be a function satisfying* (2.10). *Let $f : X \to Y$ be a uniformly approximately affine mapping with respect to φ. Then there is a unique affine mapping $A : X \to Y$ such that*

$$\lim_{t\to\infty} N(f(x) - A(x) - f(0), t\tilde{\varphi}(0, 0, \ldots, 0, x)) = 1 \tag{2.19}$$

uniformly on X.

Proof. The existence of uniform limit (2.19) immediately follows from Theorem 2.3. It remains to prove the uniqueness assertion. Let A' be another affine mapping satisfying (2.19). Fix $c > 0$. Given $\varepsilon > 0$, by (2.19) for A and A', we can find some $t_0 > 0$ such that

$$N\Big(g(x) - A(x), \frac{t}{2}\tilde{\varphi}(0, 0, \ldots, 0, x)\Big) \geqslant 1 - \varepsilon,$$
$$N\Big(g(x) - A'(x), \frac{t}{2}\tilde{\varphi}(0, 0, \ldots, 0, x)\Big) \geqslant 1 - \varepsilon$$

for all $x \in X$ and all $t \geqslant t_0$. Fix some $x \in X$ and find some integer n_0 such that $t_0 \sum_{m=n}^{\infty} k^{-m}\varphi(0, 0, \ldots, 0, k^m x) < \frac{c}{2}$, for all $n \geqslant n_0$. Since

$$\sum_{m=n}^{\infty} k^{-m}\varphi(0, 0, \ldots, 0, k^m x) = \frac{1}{k^n}\sum_{m=n}^{\infty} k^{-(m-n)}\varphi(0, 0, \ldots, 0, k^{m-n}(k^n x))$$
$$= \frac{1}{k^n}\sum_{j=0}^{\infty}\frac{1}{k^j}\varphi(0, 0, \ldots, 0, k^j(k^n x)) = \frac{1}{k^n}\tilde{\varphi}(0, 0, \ldots, 0, k^n x),$$

we have

$$N(A'(x) - A(x), c) \geqslant \min\Big\{N\Big(\frac{g(k^n x)}{k^n} - A(x), \frac{c}{2}\Big),\ N\Big(A'(x) - \frac{g(k^n x)}{k^n}, \frac{c}{2}\Big)\Big\}$$

$$= \min\Big\{N\Big(\frac{g(k^nx)}{k^n} - \frac{A(k^nx)}{k^n}, \frac{c}{2}\Big),\ N\Big(\frac{A'(k^nx)}{k^n} - \frac{g(k^nx)}{k^n}, \frac{c}{2}\Big)\Big\}$$
$$= \min\Big\{N\Big(g(k^nx) - A(k^nx), \frac{k^nc}{2}\Big),\ N\Big(A'(k^nx) - g(k^nx), \frac{k^nc}{2}\Big)\Big\}$$
$$\geqslant \min\Big\{N\Big(g(k^nx) - A(k^nx), k^nt_0\sum_{m=n}^{\infty} k^{-m}\varphi(0,0,\dots,0,k^mx)\Big),$$
$$N\Big(A'(k^nx) - g(k^nx), k^nt_0\sum_{m=n}^{\infty} k^{-m}\varphi(0,0,\dots,0,k^mx)\Big)\Big\}$$
$$= \min\Big\{N\Big(g(k^nx) - A(k^nx), t_0\tilde{\varphi}(0,0,\dots,0,k^nx)\Big),$$
$$N\Big(A'(k^nx) - g(k^nx), t_0\tilde{\varphi}(0,0,\dots,0,k^mx)\Big)\Big\} \geqslant 1-\varepsilon.$$

It follows that $N(A'(x) - A(x),\ c) = 1$, for all $c > 0$. Thus $A(x) = A'(x)$ for all $x \in X$. □

Considering the control function $\varphi(x_1, x_2, \dots, x_k) = \varepsilon(\|x_1\|^p + \|x_2\|^p + \cdots + \|x_k\|^p)$ for some $\varepsilon > 0$, we obtain the following:

Corollary 2.1. *Let X be a normed linear space, (Y, N) a fuzzy Banach space, $\varepsilon \geqslant 0$, and $0 \leqslant p < 1$. Suppose that $f : X \to Y$ is a function such that*

$$\lim_{n\to\infty} N(Df(x_1, x_2, \dots, x_k),\ t\varepsilon(\|x_1\|^p + \|x_2\|^p + \cdots + \|x_k\|^p)) = 1$$

uniformly on X^k. Then there is a unique affine mapping $A : X \to Y$ such that

$$\lim_{t\to\infty} N\Big(f(x) - A(x) - f(0), \frac{\varepsilon t k^{1-p}\|x\|^p}{k^{1-p} - 1}\Big) = 1$$

uniformly on X.

Example 2.4. Let X be a Banach space, $x_0 \in X$, $0 \leqslant p < 1$ and let α and β be real numbers. Put $f(x) := \alpha x + \beta\|x\|^p x_0$ $(x \in X)$ and $\varphi(x_1, x_2, \dots, x_k) = \|x_1\|^p + \|x_2\|^p + \cdots + \|x_k\|^p$ for each $x_1, x_2, \dots, x_k \in X$. Then

$$\tilde{\varphi}(x_1, x_2, \dots, x_k) = \sum_{n=0}^{\infty} \frac{1}{k^n}\varphi(k^nx_1, k^nx_2, \dots, k^nx_k)$$
$$= \frac{k^{1-p}(\|x_1\|^p + \|x_2\|^p + \cdots + \|x_k\|^p)}{k^{1-p} - 1}.$$

Moreover, for each fuzzy norm N on X, we have

$$N\big(Df(x_1, x_2, \dots, x_k), t\varphi(x_1, x_2, \dots, x_k)\big)$$
$$= N\big(\beta x_0(\|kx_1 + x_2 + \cdots + x_k\|^p$$
$$+ \|x_1 + kx_2 + \cdots + x_k\|^p + \cdots + \|x_1 + x_2 + \cdots + kx_k\|^p$$
$$+ \|x_1\|^p + \|x_2\|^p + \cdots + \|x_k\|^p - 2k\|x_1 + x_2 + \cdots + x_k\|^p),$$
$$(\|x_1\|^p + \|x_2\|^p + \cdots + \|x_k\|^p)\big),$$

where $x_1, x_2, \dots, x_k \in X$, $t > 0$. So that

$$N(Df(x_1,x_2,\dots,x_k),t\varphi(x_1,x_2,\dots,x_k)) \geqslant N\Big(\beta x_0, \frac{t}{2k(k+2)}\Big)$$
$$(x_1,x_2,\cdots x_k \in X,\ t>0).$$

Therefore, $\lim_{t\to\infty} N(Df(x_1,x_2,\dots,x_k),t\varphi(x_1,x_2,\dots,x_k)) = 1$ uniformly on X^k. Hence the conditions of Corollary 2.1 are fulfilled.

2.2.3. *Approximation of fuzzy almost affine mapping.* Let f be a function from a fuzzy normed space (X,N') into a fuzzy Banach space (Y,N) and $q\neq 1$. Then a function f is called a *fuzzy q-almost affine function*, if

$$\begin{aligned}
N\big(&f(kx_1+x_2+\cdots+x_k)\\
&+f(x_1+kx_2+\cdots+x_k)+\cdots+f(x_1+x_2+\cdots+kx_k)\\
&+f(x_1)+f(x_2)+\cdots+f(x_k)\\
&-2kf(x_1+x_2+\cdots+x_k),t_1+t_2+\cdots+t_k\big)\\
&\qquad\qquad \geqslant \min\{N'(x_1,\ t_1^q),N'(x_2,\ t_2^q),\dots,N'(x_k,t_k^q)\}
\end{aligned}\tag{2.20}$$

for all $x_1,x_2,\dots,x_k\in X$ and all $t_1,\ t_2,\dots,\ t_k\in(0,\infty)$.
The following result gives a Hyers–Ulam–Rassias stability of the affine equation

$$\begin{aligned}
f(kx_1+x_2+\cdots+x_k)+f(x_1+kx_2+\cdots+x_k)+\cdots+f(x_1+x_2+\cdots+kx_k)\\
+f(x_1)+f(x_2)+\cdots+f(x_k)=2kf(x_1+x_2+\cdots+x_k),\ k\geqslant 2.
\end{aligned}$$

Theorem 2.5. *Let $q>1$ and f be a* fuzzy q-almost affine function *from a fuzzy normed space (X,N') into a fuzzy Banach space (Y,N). Then there is a unique affine function $A: X\to Y$ such that for each $x\in X$,*

$$N(f(x)-A(x)-f(0),t) \geqslant N'\Big(x, \frac{(k^{1-p}-1)^q}{2}t^q\Big)(x\in X, t>0)\tag{2.21}$$

where $p=\frac{1}{q}$.

Proof. Letting $x_2=x_3=\cdots=x_k=0$ and $t_1=t_2=\cdots=t_k=t$ and replacing x_1 by x, in (2.20), we get

$$N(f(kx)-kf(x)+(k-1)f(0),kt) \geqslant N'(x,\ t^q)\tag{2.22}$$

for all $x\in X$ and all $t>0$. Let $g(x):=f(x)-f(0)$. Then (2.22) implies that $N(g(kx)-kg(x),kt)\geqslant N'(x,t^q)$. Replacing x by $k^n x$ in the last inequality, we obtain $N(g(k^{n+1}x)-kg(k^n x),kt)\geqslant N'(k^n x,t^q)$. It follows that

$$N(g(k^{n+1}x)-kg(k^n x),k^{\frac{n}{q}+1}t^{\frac{1}{q}}) \geqslant N'(x,t)$$

$(x\in X,\ n\geqslant 0,\ t>0)$, whence

$$N\Big(\frac{g(k^{n+1}x)}{k}-g(k^n x),k^{np}t^p\Big) \geqslant N'(x,t)$$
$$N\Big(\frac{g(k^{n+1}x)}{k^{n+1}}-\frac{g(k^n x)}{k^n},k^{(p-1)n}t^p\Big) \geqslant N'(x,t)$$

where $p = \frac{1}{q}$. If $n > m \geqslant 0$, then

$$
\begin{aligned}
(2.23)\quad & N\Big(\frac{g(k^n x)}{k^n} - \frac{g(k^m x)}{k^m}, \sum_{j=m+1}^{n} k^{j(p-1)} t^p\Big) \\
&\geqslant N\Big(\sum_{j=m+1}^{n}\Big(\frac{g(k^j x)}{k^j} - \frac{g(k^{j-1} x)}{k^{j-1}}\Big), \sum_{j=m+1}^{n} k^{j(p-1)} t^p\Big) \\
&\geqslant \min \bigcup_{j=m+1}^{n} \Big\{N'\Big(\frac{g(k^j x)}{k^j} - \frac{g(k^{j-1} x)}{k^{j-1}}, k^{j(p-1)} t^p\Big)\Big\} \geqslant N'(x,t)
\end{aligned}
$$

$x \in X,\ t > 0$. Let $c > 0$ and ε be given. Since $\lim_{n\to\infty} N'(x,t) = 1$, there exists a $t_0 > 0$ such that $N'(x,t_0) \geqslant 1-\varepsilon$. Fix a $t > t_0$. The convergence of the series $\sum_{n=1}^{\infty} k^{n(p-1)} t^p$ guarantees that there exists an n_0 such that for each $n > m \geqslant n_0$, the inequality $\sum_{j=m+1}^{n} k^{j(p-1)} t^p < c$ holds. It follows that

$$
\begin{aligned}
N\Big(\frac{g(k^n x)}{k^n} - \frac{g(k^m x)}{k^m}, c\Big) &\geqslant N\Big(\frac{g(k^n x)}{k^n} - \frac{g(k^m x)}{k^m}, \sum_{j=m+1}^{n} k^{j(p-1)} t_0^p\Big) \\
&\geqslant N'(x,t_0) \geqslant 1-\varepsilon.
\end{aligned}
$$

Hence $\{\frac{g(k^n x)}{k^n}\}$ is a Cauchy sequence in (Y,N). Since (Y,N) is a fuzzy Banach space, this sequence converges to some point $A(x) \in Y$. Hence, we can define a mapping $A : X \to Y$ by $A(x) = N\text{-}\lim_{n\to\infty} \frac{g(k^n x)}{k^n} = N\text{-}\lim_{n\to\infty} \frac{f(k^n x)}{k^n}$ for all $x \in X$. Since f is odd, A is odd. Moreover, if we put $m = 0$ in (2.23), then we get

$$
N\Big(\frac{g(k^n x)}{k^n} - g(x), \sum_{j=1}^{n} k^{j(p-1)} t^p\Big) \geqslant N'(x,t).
$$

Therefore,

$$
N\Big(\frac{g(k^n x)}{k^n} - g(x), t^p\Big) \geqslant N'\Bigg(x, \frac{t}{\sum_{j=1}^{n} k^{j(p-1)}}\Bigg)
$$

$$
N\Big(\frac{g(k^n x)}{k^n} - g(x), t\Big) \geqslant N'\Bigg(x, \frac{t^q}{\big(\sum_{j=1}^{n} k^{j(p-1)}\big)^q}\Bigg) \tag{2.24}
$$

for all $x \in X$, $t > 0$. Next we will show that A is affine. Let $x_1, x_2, \ldots, x_k \in X$, then we have

$$
\begin{aligned}
&N(DA(x_1,x_2,\ldots,x_k),t) \\
&\geqslant \min\Big\{N\Big(A(kx_1 + x_2 + \cdots + x_k) - \frac{f(k^n(kx_1 + x_2 + \cdots + x_k))}{k^n}, \frac{t}{2k+2}\Big), \\
&\qquad N\Big(A(x_1 + kx_2 + \cdots + x_k) - \frac{f(k^n(x_1 + kx_2 + \cdots + x_k))}{k^n}, \frac{t}{2k+2}\Big), \ldots, \\
&\qquad N\Big(A(x_1 + x_2 + \cdots + kx_k) - \frac{f(k^n(x_1 + x_2 + \cdots + kx_k))}{k^n}, \frac{t}{2k+2}\Big), \ldots,
\end{aligned}
$$

$$N\Big(A(x_1)-\frac{f(k^nx_1)}{k^n},\frac{t}{2k+2}\Big),N\Big(A(x_2)-\frac{f(k^nx_2)}{k^n},\frac{t}{2k+2}\Big),$$
$$N\Big(A(x_k)-\frac{f(k^nx_k)}{k^n},\frac{t}{2k+2}\Big),$$
$$N\Big(\frac{2kf(k^n(x_1+x_2+\cdots+x_k))}{k^n}-2kA(x_1+x_2+\cdots+x_k),\frac{t}{2k+2}\Big),$$
$$N\Big(Df\Big(k^nx_1,k^nx_2,\ldots,k^nx_k,\frac{k^nt}{2k+2}\Big)\Big)\Big\}.$$

The first $(2k+1)$ terms on the right-hand side of the above inequality tend to 1 as $n\to\infty$ and the last term, by (2.20) is greater than or equal to

$$\begin{aligned}\min\Big\{N\Big(k^nx_1,\Big(\frac{k^{n-1}t}{2k+2}\Big)^q\Big),N\Big(k^nx_2,\Big(\frac{k^{n-1}t}{2k+2}\Big)^q\Big),\cdots,N\Big(k^nx_k,\Big(\frac{k^{n-1}t}{2k+2}\Big)^q\Big)\Big\}\\ =\min\Big\{N\Big(x_1,k^{n(q-1)}\Big(\frac{t}{2k(k+1)}\Big)^q\Big),N\Big(x_2,k^{n(q-1)}\Big(\frac{t}{2k(k+1)}\Big)^q\Big),\ldots,\\ N\Big(x_k,k^{n(q-1)}\Big(\frac{t}{2k(k+1)}\Big)^q\Big)\Big\}\end{aligned}$$

which tends to 1 as $n\to\infty$ (since $q>1$). Therefore

$$\begin{aligned}N(A(kx_1+x_2+\cdots+x_k)+A(x_1+kx_2+\cdots+x_k)+\cdots+A(x_1+x_2+\cdots+kx_k)\\ +A(x_1)+A(x_2)+\cdots+A(x_k)-2kA(x_1+x_2+\cdots+x_k),t)=1\end{aligned}$$

for all $x_1,x_2,\ldots,x_k\in X$ and $t>0$. This means that

$$\begin{aligned}A(kx_1+x_2+\cdots+x_k)+A(x_1+kx_2+\cdots+x_k)+\cdots+A(x_1+x_2+\cdots+kx_k)\\ +A(x_1)+A(x_2)+\cdots+A(x_k)=2kA(x_1+x_2+\cdots+x_k)\end{aligned}$$

for all $x_1,x_2,\ldots,x_k\in X$ and $t>0$. Next, we approximate the difference between f and A in a fuzzy sense. For every $x\in X$ and $t>0$, by (2.24), for large enough n, we have

$$\begin{aligned}N(A(x)-g(x),t)&\geqslant\min\Big\{N\Big(A(x)-\frac{g(k^nx)}{k^n},\frac{t}{2}\Big),N\Big(\frac{g(k^nx)}{k^n}-g(x),\frac{t}{2}\Big)\Big\}\\ &\geqslant N'\Bigg(x,\frac{t^q}{2\big(\sum_{j=1}^n k^{j(p-1)}\big)^q}\Bigg)\geqslant N'\Bigg(x,\frac{t^q}{2\big(\sum_{j=1}^\infty k^{j(p-1)}\big)^q}\Bigg).\end{aligned}$$

Therefore

$$N(A(x)-g(x),t)\geqslant N'\Big(x,\frac{(k^{1-p}-1)^q}{2}t^q\Big)$$
$$N(f(x)-A(x)-f(0),t)\geqslant N'\Big(x,\frac{(k^{1-p}-1)^q}{2}t^q\Big)$$

for all $x\in X$, $t>0$. To prove the uniqueness of A, let $A':X\to Y$ be another affine mapping satisfying (2.21). Fix $x\in X$, clearly $A(k^nx)=k^nA(x)$ and $A'(k^nx)=k^nA'(x)$ for all $x\in X$ and $n\in N$. It follows from (N_4) and (2.21) that

$$N(A(x)-A'(x),t)=N(A(k^nx)-A'(k^nx),k^nt)$$

$$\geqslant \min\Big\{N\Big(A'(k^nx)-g(k^nx),\frac{k^nt}{2}\Big),N\Big(g(k^nx)-A(k^nx),\frac{k^nt}{2}\Big)\Big\}$$
$$\geqslant N'\Big(2^nx,\frac{(k^{1-p}-1)^q}{2}\Big(\frac{k^n}{2}\Big)^qt^q\Big)=N'\Big(x,\frac{(k^{1-p}-1)^q}{2.2^n}\frac{k^{nq}}{2^q}t^q\Big)$$

for each $n\in N$. Due to $q>1$, we have $\lim_{n\to\infty}N'\big(x,\frac{(k^{1-p}-1)^q}{2.2^n}\frac{k^{nq}}{2^q}t^q\big)=1$ for each $x\in X$ and $t>0$. Therefore $A(x)=A'(x)$. □

Remark 2.1. If $N(f(x)-A(x)-f(0),.)$ is assumed to be right continuous at each point of $(0,\infty)$, then we get a better fuzzy approximation than (2.21) as follows.

$$N(f(x)-A(x)-f(0),s+t)=N(A(x)-g(x),s+t)$$
$$\geqslant \min\Big\{N\Big(A(x)-\frac{g(k^nx)}{k^n},s\Big),N\Big(\frac{g(k^nx)}{k^n}-g(x),t\Big)\Big\}.$$

Letting $s\to0$, we infer

$$N(f(x)-A(x)-f(0),t)\geqslant N\Big(\frac{g(k^nx)}{k^n}-g(x),t\Big)$$
$$\geqslant N'\Big(x,\frac{(k^{1-p}-1)^q}{2}2^qt^q\Big)=N'\Big(x,\frac{(k^{1-p}-1)^q}{2^{1-q}}t^q\Big).$$

Remark 2.2. Consider a mapping $f:X\to Y$ satisfying (2.20) for all x_1, $x_2,\dots,x_k\in X$ and a real number $q<0$. Take any $t>0$. If we choose a real number s with $0<ks<t$, then we have

$$N(Df(x_1,x_2,\dots,x_k),t)\geqslant N(Df(x_1,x_2,\dots,x_k),ks)$$
$$\geqslant \min\{N'(x_1,s^q),N'(x_2,s^q),\dots,N'(x_k,s^q)\}$$

for all $x_1,x_2,\dots,x_k\in X$. Since $q<0$, we have $\lim\limits_{s\to0^+}s^q=\infty$. This implies that

$$\lim_{s\to0^+}N'(x_1,s^q)=\lim_{s\to0^+}N'(x_2,s^q)=\cdots=\lim_{s\to0^+}N'(x_k,s^q)=1$$

and so $N(Df(x_1,x_2,\dots,x_k),t)=1$ for all $x_1,x_2,\dots,x_k\in X$ and $t>0$. By (N_2), it allows us to get $Df(x_1,x_2,\dots,x_k)=0$ for all $x_1,x_2,\dots,x_k\in X$. In other words, f is itself an affine mapping if f is a *fuzzy q-almost affine mapping* for the case $q<0$.

Remark 2.3. Using Hyer's type sequence $\{k^nf(\frac{x}{k^n})\}$ one can get 'dual' version of Theorem 2.5 when $q<1$. The case where $q=1$ remains open.

3. Fuzzy stability via fixed point approach

In this section, we deal with the stability problem via the fixed point method in fuzzy normed space. Before proceeding further, we should recall the following results related to the concept of fixed point.

Theorem 3.1 (Banach contraction principle). *Let (X,d) be a complete generalized metric space and consider a mapping $J:X\to X$ being a strictly contractive mapping, i.e., $d(Jx,Jy)\leqslant Ld(x,y)$, for all $x,y\in X$ for some (Lipschitz constant) $L<1$. Then*

(i) *The mapping has one and only one fixed point* $J(x^*) = x^*$*;*
(ii) *The fixed point* x^* *is globally attractive, that is* $\lim_{n\to\infty} J^n x = x^*$*, for any starting point* $x \in X$*;*
(iii) *One has the following estimation inequalities for all* $x \in X$ *and* $n \geqslant 0$ *:*
  (a) $d(J^n x, x^*) \leqslant L^n d(x, x^*)$
  (b) $d(J^n x, x^*) \leqslant \frac{1}{1-L} d(J^n x, J^{n+1} x)$
  (c) $d(x, x^*) \leqslant \frac{1}{1-L} d(Jx, x)$.

Theorem 3.2 (The alternative of fixed point [**28**]). *Let* (X, d) *be a complete generalized metric space and* $J\colon X \to X$ *a strictly contractive mapping with Lipschitz constant* $L < 1$*. Then for each* $x \in X$*, either* $d(J^n x,\ J^{n+1} x) = \infty$ *for all* $n \geqslant 0$*, or there exists a positive integer* n_0 *such that* $d(J^n x,\ J^{n+1} x) < \infty$*, for all* $n \geqslant n_0$*. If the second alternative holds, then*
(i) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J*;*
(ii) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X :\ d(J^{n_0} x, y) < \infty\}$*;*
(iii) $d(y, y^*) \leqslant \frac{1}{1-L} d(y, Jy)$*, for all* $y \in Y$*.*

Using the fixed point alternative, we can prove the Hyers–Ulam–Rassias stability in fuzzy normed spaces. First, we prove the following lemma which will be used in our main result.

Lemma 3.1. *Let* (Z, N') *be a fuzzy normed space and* $\varphi :\ X^k \to Z$ *a function. Let* $S = \{g' : X \to Y; g'(0) = 0\}$ *and define*

$$d(g', h) = \inf \big\{\mu \in \mathbb{R}^+ :\ N(g'(x) - h(x),\ \mu t) \geqslant N'(\varphi(x, 0, 0, \ldots, 0),\ t),\ \text{for all } x \in X \text{ and } t > 0\big\}$$

for all $h \in S$*. Then* d *is a complete generalized metric on* S*.*

Proof. Let $g', h, k \in S,\ d(g', h) < \zeta_1$ and $d(h, k) < \zeta_2$. Then

$$N(g'(x) - h(x), \zeta_1 t) \geqslant N'(\varphi(x, 0, 0, \ldots, 0), t) \text{ and}$$
$$N(h(x) - k(x), \zeta_2 t) \geqslant N'(\varphi(x, 0, 0, \ldots, 0), t)$$

for all $x \in X$ and $t > 0$. Thus

$$\begin{aligned} N(g'(x) - k(x), (\zeta_1 + \zeta_2)t) &\geqslant \min\{N(g'(x) - h(x), \zeta_1 t), N(h(x) - k(x), \zeta_2 t)\} \\ &\geqslant N'(\varphi(x, 0, 0, \ldots, 0), t) \end{aligned}$$

for each $x \in X$ and $t > 0$. By definition $g(h, k) < \zeta_1 + \zeta_2$. This proves the triangle inequality for d. The rest of the proof can be done along the same lines as in Lemma 2.1 [**29**]. □

Theorem 3.3. *Let* X *be a linear space and* (Z, N') *a fuzzy normed space. Suppose that a function* $\varphi : X^k \to Z$ *satisfies* $\varphi(kx_1, kx_2, \ldots, kx_k) = \alpha\varphi(x_1, x_2, \ldots, x_k)$ *for all* $x_1, x_2, \ldots, x_k \in X$ *and* $\alpha \neq 0$*. Suppose that* (Y, N) *is a fuzzy Banach space and* $f : X \to Y$ *is a mapping satisfying*

$$N(Df(x_1, x_2, \ldots, x_k), t) \geqslant N'(\varphi(x_1, x_2, \ldots, x_k), t) \tag{3.1}$$

for all $x_1, x_2, \ldots, x_k \in X$ *and* $t > 0$. *If for some* $0 < \alpha < k$

$$N'(\varphi(kx, 0, \ldots, 0), t) \geqslant N'(\alpha\varphi(x, 0, \ldots, 0), t), \ \forall x \in X \text{ and } t > 0 \tag{3.2}$$

and $\lim_{n\to\infty} N'(\varphi(k^n x_1, k^n x_2, \ldots, k^n x_k), k^n t) = 1$, *for all* $x_1, x_2, \ldots, x_k \in X$ *and* $t > 0$. *Then there exists a unique affine mapping* $A : X \to Y$ *such that*

$$N(f(x) - A(x) - f(0), t) \geqslant N'(\varphi(x, 0, \ldots, 0), (k - \alpha)t), \ \forall x \in X \text{ and } t > 0.$$

Proof. Put $x_2 = x_3 = \cdots = 0$ and replacing x_1 by x in (3.1), we get

$$N(f(kx) - kf(x) + (k-1)f(0), t) \geqslant N'(\varphi(x, 0, 0, \ldots, 0), t) \tag{3.3}$$

for all $x \in X$ and all $t > 0$. Let $g(x) := f(x) - f(0)$. Then (3.3) implies

$$N\Big(\frac{g(kx)}{k} - g(x), \frac{t}{k}\Big) \geqslant N'(\varphi(x, 0, 0, \ldots, 0), t).$$

Consider the set $S = \{g' : X \to Y, g'(0) = 0\}$ together with the mapping d defined on $S \times S$ by

$$d(g', h) = \inf\{\mu \in \mathbb{R}^+ : N(g'(x) - h(x), \mu t) \geqslant N'(\varphi(x, 0, 0, \ldots, 0), t), \forall x \in X,\ t > 0\}.$$

It is known that (d, S) is a complete generalized metric space by Lemma 3.1. Now, we define the linear mapping $J : S \to S$ such that $Jg'(x) = \frac{1}{k}g'(kx)$. It is easy to see that J is a strictly contractive self-mapping of S with the Lipschitz constant $\frac{\alpha}{k}$. Indeed, let $g', h \in S$ be given such that $d(g', h) = \varepsilon$. Then

$$N(g'(x) - h(x), \varepsilon t) \geqslant N'(\varphi(x, 0, \ldots, 0), t)$$

for all $x \in X$ and $t > 0$. Thus

$$\begin{aligned} N\Big(Jg'(x) - Jh(x), \frac{\alpha}{k}\varepsilon t\Big) &= N\Big(\frac{1}{k}g'(kx) - \frac{1}{k}h(kx), \frac{\alpha}{k}\varepsilon t\Big) \\ &= N(g'(kx) - h(kx), \alpha\varepsilon t) \geqslant N'(\varphi(kx, 0, \ldots, 0), \alpha t). \end{aligned}$$

It follows from (3.2) that

$$N\Big(Jg'(x) - Jh(x), \frac{\alpha}{k}\varepsilon t\Big) \geqslant N'(\alpha\varphi(x, 0, \ldots, 0), \alpha t) = N'(\varphi(x, 0, \ldots, 0), t)$$

for all $x \in X$ and $t > 0$. Therefore

$$d(g', h) = \varepsilon \Rightarrow d(Jg', Jh) \leqslant \frac{\alpha}{k}\varepsilon.$$

This means that $d(Jg', Jh) \leqslant \frac{\alpha}{k}d(g', h)$, for all $g', h \in S$. Next, from

$$N\Big(\frac{g(kx)}{k} - g(x), \frac{\alpha}{k}\Big) \geqslant N'(\varphi(x, 0, \ldots, 0),\ t),$$

we have $d(g, Jg) \leqslant \frac{1}{k}$. Using the fixed point alternative, we deduce the existence of a fixed point of J, that is, there exists a mapping $A : X \to Y$ such that A is a fixed point of J, i.e., $kA(x) = A(kx)$, for all $x \in X$. Moreover, we have $d(J^n g, A) \to 0$, which implies

$$A(x) = N\text{-}\lim_{n\to\infty} \frac{g(k^n x)}{k^n} = N\text{-}\lim_{n\to\infty} \frac{f(k^n x)}{k^n},\ \forall x \in X.$$

Also $d(g,A) \leqslant \frac{1}{1-L} d(g,Jg)$ implies $d(g,A) \leqslant \frac{1}{(k-\alpha)}$. This implies that

$$N\Big(A(x)-g(x),\frac{t}{(k-\alpha)}\Big) \geqslant N'(\varphi(x,0,\dots,0),t)$$
$$N(A(x)-g(x),t) \geqslant N'(\varphi(x,0,\dots,0),(k-\alpha)t)$$

for all $x \in X$, $t>0$. Let $x_1,x_2,\dots,x_k \in X$. Then

$$N(DA(x_1,x_2,\dots,x_k),t) \geqslant N'(\varphi(x_1,x_2,\dots,x_k),t).$$

Replacing $x_1,x_2,\dots,x_k$ by $k^n x_1,k^n x_2,\dots,k^n x_k$, respectively, in the above inequality, we obtain

$$N\Big(\frac{DA(k^n x_1,k^n x_2,\dots,k^n x_k)}{k^n},t\Big) \geqslant N'(\varphi(k^n x_1,k^n x_2,\dots,k^n x_k),k^n t)$$

for all $x_1,x_2,\dots,x_k \in X$, $t>0$. Since $\lim_{n\to\infty} N'(\varphi(k^n x_1,k^n x_2,\dots,k^n x_k),k^n t)=1$, we conclude that A fulfills (1.2).

The uniqueness of A follows from the fact that A is the unique fixed point of J with the following property that there exists $\mu \in (0,\infty)$ such that

$$N(A(x)-g(x),\mu t) \geqslant N'(\varphi(x,0,\dots,0),t)$$
$$N(f(x)-A(x)-f(0),\mu t) \geqslant N'(\varphi(x,0,\dots,0),t)$$

for all $x \in X$ and $t>0$. □

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