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ABSOLUTE AND UNIFORM CONVERGENCE OF SPECTRAL EXPANSION OF THE FUNCTION FROM THE CLASS $W_p^1(G)$, $p>1$, IN EIGENFUNCTIONS OF THIRD ORDER DIFFERENTIAL OPERATOR

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Abstract. We study an ordinary differential operator of third order and absolute and uniform convergence of spectral expansion of the function from the class $W_p^1(G)$, $G=(0,1)$, $p>1$, in eigenfunctions of the operator. Uniform convergence rate of this expansion is estimated.

1. Basic notion and formulation of results

Consider on the interval $G=(0,1)$ a formal differential operator

$$Lu = u^{(3)} + p_1(x)u^{(2)} + p_2(x)u^{(1)} + p_3(x)u$$

with summable complex valued coefficients $p_1(x)\in L_2(G)$, $p_l(x)\in L_1(G)$, $l=2,3$.

Denote by $D(G)$ a class of functions absolutely continuous together with own derivatives to second order, inclusively on the closed interval $\bar{G}=[0,1]$. Following [**1**], under the eigenfunction of the operator L responding to the eigenvalue λ, we understand any not identically equal to zero function $u(x)\in D(G)$ satisfying almost everywhere in G the equation $Lu+\lambda u=0$.

Let $\{u_n(x)\}_{n=1}^{\infty}$ be a complete orthonormalized in $L_2(G)$ system consisting of eigenfunctions of the operator L, and $\{\lambda_n\}_{n=1}^{\infty}$ be an appropriate system of eigenvalues, moreover $\operatorname{Re}\lambda_n=0$ (it is supposed that the coefficients of the operator L admit the existence of such a system $\{u_n(x)\}_{n=1}^{\infty}$). Under rather smooth coefficients the existence of such systems follows from the monograph [**2**])

Denote by μ_n the number $(\mp i\lambda_n)^{1/3}$ for $\pm\operatorname{Im}\lambda_n\geqslant 0$. We will say that the function $f(x)$ belongs to $W_p^1(G), 1\leqslant p\leqslant\infty$ if $f(x)$ is absolutely continuous on $\bar{G}$ and $f'(x)\in L_p(G)$.

Introduce a partial sum of spectral expansion of the function $f(x)\in W_p^1(G)$ in the system $\{u_n(x)\}_{n=1}^{\infty}$ by $\sigma_\nu(x,f)=\sum_{\mu_n\leqslant\nu} f_n u_n(x)$, $\nu>0$, where $f_n=$

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$(f, u_n) = \int_0^1 f(x)\overline{u_n(x)}dx$. Denote $R_\nu(x, f) = f(x) - \sigma_\nu(x, f)$. We prove the following theorems.

Theorem 1.1. *Let* $p_1(x) \equiv 0$, $p_l(x) \in L_1(G)$, $l = 2, 3$; $f(x) \in W_p^1(G)$, $p > 1$, *and the following condition be fulfilled*

$$\left|f(x)\overline{u_n^{(2)}(x)}|_0^1\right| \leqslant C(f)\mu_n^\alpha \|u_n\|_\infty, \quad 0 \leqslant \alpha < 2, \quad \mu_n \geqslant 1, \tag{1.1}$$

where $C(f) > 0$ *is a constant dependent on the function* $f(x)$. *Then the spectral expansion of the function* $f(x)$ *in the system* $\{u_n(x)\}_{n=1}^\infty$ *converges absolutely and uniformly on* $\bar{G} = [0, 1]$, *and we have the estimation*

$$\sup_{x\in\bar{G}} |R_\nu(x, f)| \leqslant \text{const}\left\{C(f)\nu^{\alpha-2} + \nu^{-\beta}\|f'\|_p + \nu^{-1}(\|f\|_\infty + \|f'\|_1)\sum_{r=2}^{3}\nu^{2-r}\|p_r\|_1\right\}, \tag{1.2}$$

where $\beta = \min\left\{\frac{1}{2}, \frac{1}{q}\right\}$, $\frac{1}{p} + \frac{1}{q} = 1$, $\nu \geqslant 2$, const *is independent of* $f(x)$, *and* $\|\cdot\|_p = \|\cdot\|_{L_p(G)}$.

Corollary 1.1. *If in Theorem* 1.1, *the function* $f(x)$ *satisfies the condition* $f(0) = f(1) = 0$, *then obviously* (1.1) *is fulfilled and ve have the estimation*

$$\sup_{x\in\bar{G}} |R_\nu(x, f)| \leqslant \text{const } \nu^{-\beta}\|f'\|_p, \quad \nu \geqslant 2;$$

if $C(f) = 0$ *or* $0 \leqslant \alpha < 2 - \beta$, *then* $\sup_{x\in\bar{G}} |R_\nu(x, f)| = o(\nu^{-\beta})$, $\nu \to +\infty$.

Theorem 1.2. *Let* $p_1(x) \in L_2(G)$, $p_l(x) \in L_1(G)$, $l = 2, 3$; $f(x) \in W_2^1(G)$ *and condition* (1.1) *be fulfilled. Then the spectral expansion of the function* $f(x)$ *in system* $\{u_n(x)\}_{n=1}^\infty$ *converges absolutely and uniformly on* $\bar{G} = [0, 1]$, *and we have the estimation*

$$\sup_{x\in\bar{G}} |R_\nu(x, f)| \leqslant \text{const}\left\{C(f)\nu^{\alpha-2} + \nu^{-\frac{1}{2}}(\|f\,\overline{p_1}\|_2 + \|f'\|_2) + \nu^{-1}\|f\|_\infty\sum_{r=2}^{3}\nu^{2-r}\|p_r\|_1\right\}, \quad \nu \geqslant 2.$$

Corollary 1.2. *If in Theorem* 1.2, $C(f) = 0$ *or* $0 \leqslant \alpha < 3/2$, *then*

$$\sup_{x\in\bar{G}} |R_\nu(x, f)| = o\left(\nu^{-\frac{1}{2}}\right), \quad \nu \to +\infty.$$

Theorem 1.3. *Let* $p_1(x) \in L_2(G), p_l(x) \in L_1(G)$, $l = 2, 3$; $f(x) \in W_p^1(G)$, $1 < p < 2$, *condition* (1.1) *be fulfilled, and the system* $\{u_n(x)\}_{n=1}^\infty$ *be uniformly bounded. Then the spectral expansion of the function* $f(x)$ *in system* $\{u_n(x)\}_{n=1}^\infty$ *converges absolutely and uniformly on* $\bar{G}$, *and we have the estimation*

$$\sup_{x\in\bar{G}} |R_\nu(x, f)| \leqslant \text{const}\left\{C(f)\nu^{\alpha-2} + \nu^{-\frac{1}{2}}\|f\overline{p_1}\|_2 + \nu^{-\frac{1}{q}}\|f'\|_p + \nu^{-1}\|f\|_\infty\sum_{r=2}^{3}\nu^{2-r}\|p_r\|_1\right\}, \quad \mu \geqslant 2,$$

where $p^{-1}+q^{-1}=1$.

Corollary 1.3. *If in Theorem* 1.3*,* $C(f)=0$ *or* $0\leqslant\alpha<2-q^{-1}$*, then*

$$\sup_{x\in\bar{G}}|R_\nu(x,f)|=o\big(\nu^{-\frac{1}{q}}\big),\quad \nu\to+\infty.$$

Note that such results for the Schrodinger operator $L_1=-\frac{d^2}{dx^2}+q(x)$ were obtained in [**3**–**7**]. In the case $f(x)\in W_1^1(G)$ the absolute and uniform convergence of spectral expansion in eigenfunctions of a third order operator was studied in [**8**].

2. Some auxiliary statements

To prove, the formulated results it is necessary to estimate the Fourier coefficients of the function $f(x)$ in the system $\{u_n(x)\}_{n=1}^{\infty}$. To this end we established the representation for the eigenfunction $u_n(x)$. Introduce the notation

$$K(z)=\sum_{j=1}^{3}\omega_j\exp(i\omega_j(\operatorname{sgn}\operatorname{Im}\lambda_n)z),$$

$$M(u_n(\xi))=\frac{1}{3}\mu_n^{-2}\sum_{r=1}^{3}p_r(\xi)u_n^{(3-r)}(\xi),$$

$$X_j^{\pm}(x)=\frac{1}{3}\mu_n^{-2}\sum_{m=0}^{2}(\pm i\mu_n)^m\omega_j^{m+1}u_n^{(2-m)}(x),$$

where $\omega_1=-1$, $\omega_2=\exp(-i\pi/3)$, $\omega_3=\exp(i\pi/3)$.

Lemma 2.1. *For the eigenfunction* $u_n(x)$*, the following representations are valid* $(\lambda_n\neq 0,\ l=\overline{0,2})$

$$u_n^{(l)}(x+t)=\sum_{j=1}^{3}(-i\omega_j\mu_n)^lX_j^{-}(x)\exp(-i\omega_j\mu_nt)-\int_x^{x+t}M(u_n(\xi))K_t^{(l)}(\xi-x-t)d\xi,\quad \textit{if}\ \operatorname{Im}\lambda_k>0; \tag{2.1}$$

$$u_n^{(l)}(x+t)=\sum_{j=1}^{3}(i\omega_j\mu_n)^lX_j^{+}(x)\exp(i\omega_j\mu_nt)-\int_x^{x+t}M(u_n(\xi))K_t^{(l)}(\xi-x-t)d\xi,\quad \textit{if}\ \operatorname{Im}\lambda_n<0. \tag{2.2}$$

Proof. For definiteness we consider the case $\operatorname{Im}\lambda_n<0$. Multiply each side of the equation $Lu_n(\xi)+\lambda_nu_n(\xi)=0$ by the function $K_t^{(l)}(\xi-x-t)$ and integrate the obtained equality with respect to ξ from x to $x+t$, where $x,x+t\in G$

$$\sum_{j=1}^{3}\omega_j(i\omega_j\mu_n)^l\int_x^{x+t}u_n^{(3)}(\xi)\exp(-i\omega_j\mu_n(\xi-x-t))d\xi \tag{2.3}$$

$$+\int_x^{x+t}\left\{p_1(\xi)u_n^{(2)}(\xi)+p_2(\xi)u_n^{(1)}(\xi)+p_3(\xi)u_n(\xi)\right\}$$
$$\times K_t^{(l)}(\xi-x-t)d\xi+\lambda_n\int_x^{x+t}u_n(\xi)K_t^{(l)}(\xi-x-t)d\xi=0.$$

Integrating by parts and using $\sum_{j=1}^{3}\omega_j^s=3\delta_{3s}$ (δ_{ks} is the Kronecker symbol), we transform the first expression in equality (2.3) in the following way

$$\sum_{j=1}^{3}\omega_j(i\omega_j\mu_n)^l\int_x^{x+t}u_n^{(3)}(\xi)\exp(-i\omega_j\mu_n(\xi-x-t))d\xi$$
$$=\sum_{j=1}^{3}\omega_j(i\omega_j\mu_n)^l\sum_{m=0}^{2}(-1)^m(-i\omega_j\mu_n)^m$$
$$\times\left[u_n^{(2-m)}(x+t)-u_n^{(2-m)}(x)\exp(i\omega_j\mu_nt)\right]$$
$$-\sum_{j=1}^{3}\omega_j(i\omega_j\mu_n)^l(-\omega_j\mu_n)^3\int_x^{x+t}u_n(\xi)\exp(-i\omega_j\mu_n(\xi-x-t))d\xi$$
$$=3\mu_n^2u_n^{(l)}(x+t)$$
$$-\sum_{j=1}^{3}\omega_j(i\omega_j\mu_n)^l\sum_{m=0}^{2}(i\omega_j\mu_n)^mu_n^{(2-m)}(x)\exp(i\omega_j\mu_nt)+(-i\mu_n)^3$$
$$\times\sum_{j=1}^{2}\omega_j(i\omega_j\mu_n)^l\int_x^{x+t}u_n(\xi)\exp(-i\omega_j\mu_n(\xi-x-t)d\xi.$$

Taking this into account in (2.3) and taking into attention the equality $(-i\mu_n)^3=-\lambda_n$, we get

$$3\mu_n^2u_n^{(l)}(x+t)-\sum_{j=1}^{3}(i\omega_j\mu_n)^l\left[\sum_{m=0}^{2}\omega_j^{m+1}(i\mu_n)^mu_n^{(2-m)}(x)\right]\exp(i\omega_j\mu_nt)$$
$$+\int_x^{x+t}\left\{\sum_{\tau=1}^{3}p_r(\xi)u_n^{(3-r)}(\xi)\right\}K_t^{(l)}(\xi-x-t)d\xi=0.$$

Hence we find $u_n^{(l)}(x+t)$ and get formula (2.2). Lemma 2.1 is proved. □

For $x=0$ we write formulas (2.1) and (2.2) in a more convenient form

$$\mu_n^{-l}u_n^{(l)}(t)=\sum_{j=1}^{2}X_j^-(0)(-i\omega_j)^l\exp(-i\omega_j\mu_nt)$$
$$-(-i\omega_3)B_3^-\exp(i\omega_3\mu_n(1-t))$$
$$-\sum_{j=1}^{2}(-i)^l\omega_j^{l+1}\int_0^tM(u_n(\xi))\exp(i\omega_j\mu_n(\xi-t))d\xi \tag{2.4}$$

$$+(-i)^l\omega_3^{l+1}\int_t^1 M(u_n(\xi))\exp(i\omega_3\mu_n(\xi-t))d\xi,$$

$\operatorname{Im}\lambda_n>0$, $l=\overline{0,2}$, where

$$B_3^-=X_3^-(0)\exp(-i\omega_3\mu_n)-\omega_3\int_0^1 M(u_n(\xi))\exp(-i\omega_3\mu_n(\xi-1))d\xi;$$

$$\begin{aligned}\mu_n^{-l}u_n^{(l)}(t)=&\sum_{j=1,j\neq 2}^{2}(i\omega_j)^l X_j^+(0)\exp(i\omega_j\mu_n t)\\&+(i\omega_2)^l B_2^+\exp(-i\omega_2\mu_n(1-t))\\&-\sum_{j=1,j\neq 2}^{3}(i)^l\omega_j^{l+1}\int_0^t M(u_n(\xi))\exp(-i\omega_j\mu_n(\xi-t))d\xi\\&+(i)^l\omega_3^{l+1}\int_t^1 M(u_n(\xi))\exp(-i\omega_2\mu_n(\xi-t))d\xi,\end{aligned}\tag{2.5}$$

where $\operatorname{Im}\lambda_n<0$, $l=\overline{0,2}$

$$B_2^+=X_2^+(0)\exp(i\omega_2\mu_n)-\omega_2\int_0^1 M(u_n(\xi))\exp(-i\omega_2\mu_n(\xi-t))d\xi.$$

For the coefficients in formulas (2.4) and (2.5) we note that if the system $\{u_n(x)\}_{n=1}^\infty$ is a Bessel system in $L_2(G)$, then for, them the following estimations are fulfilled (see [**9**, **10**])

$$\begin{aligned}|X_1^\pm(0)|\leqslant\|u_n\|_2,\qquad |X_2^\pm(0)|\leqslant C\|u_n\|_\infty\\ |B_2^+|\leqslant C\|u_n\|_\infty,\qquad |B_3^-|\leqslant C\|u_n\|_\infty.\end{aligned}\tag{2.6}$$

Now, using formulas (2.4) and (2.5), estimate the Fourier coefficients of the function $f(x)\in W_p^1(G), p>1$, in system $\{u_n(x)\}_{n=1}^\infty$.

Lemma 2.2. *Let the function $f(x)\in W_p^1(G), p>1$, and the system $\{u_n(x)\}_{n=1}^\infty$ satisfy condition* (1.1). *Then for the Fourier coefficients f_n the estimations* ($\mu_n\geqslant 1$) *are valid*

$$\begin{aligned}|f_n|\leqslant\text{const}\Big\{&C(f)\mu_n^{\alpha-3}\|u_n\|_\infty+\mu_n^{-1}|(f\,\overline{p_1},\mu_n^{-2}u_n^{(2)})|\\&+\mu_n^{-1}\big|\big(f',\mu_n^{-2}u_n^{(2)}\big)\big|+\mu_n^{-2}\Big(\sum_{r=2}^{3}\mu_n^{2-r}\|p_r\|_1\Big)\|f\|_\infty\|u\|_\infty\Big\};\end{aligned}\tag{2.7}$$

$$\begin{aligned}|f_n|\leqslant\text{const}\Big\{\Big[&C(f)\mu_n^{\alpha-3}+\mu_n^{-1}|(f',\exp(i\omega_3\mu_n t))|\\&+\mu_n^{-1}|(f',\exp(-i\omega_2\mu_n(1-t)))|+(\|f\|_\infty+\|f'\|_1)\mu_n^{-2}\sum_{r=2}^{3}\mu_n^{2-r}\|p_r\|_1\Big]\\&\times\|u_n\|_\infty+\mu_n^{-1}|(f',\exp(-i\mu_n t))|\Big\},\end{aligned}\tag{2.8}$$

if $p_1(x) \equiv 0, \operatorname{Im}\lambda_n < 0$;

$$(2.9)\quad |f_n| \leqslant \mathrm{const}\Big\{\Big[C(f)\mu_n^{\alpha-3} + \mu_n^{-1}|(f', \exp(-i\omega_2\mu_n t))| + \mu_n^{-1}|(f', \exp(i\omega_3\mu_n(1-t)))| + (\|f\|_\infty + \|f'\|_1)\mu_n^{-2}\sum_{r=2}^{3}\mu_n^{2-r}\|p_r\|_1\Big] \times \|u_n\|_\infty + \mu_n^{-1}|(f', \exp(i\mu_n t))|\Big\},$$

if $p_1(x) \equiv 0, \operatorname{Im}\lambda_n > 0$.

Proof. By definition of the eigenfunction $u_n(x)$, the Fourier coefficients f_n for $\mu_n \geqslant 1$ are calculated by the formula

$$(2.10)\qquad f_n = (f, u_n) = -\frac{1}{\lambda_n}(f, Lu_n) = -\frac{1}{\lambda_n}(f, u^{(3)}) - \frac{1}{\lambda_n}\big(f, p_1u_n^{(2)}\big) - \frac{1}{\lambda_n}\big(f, p_2u_n^{(1)}\big) - \frac{1}{\lambda_n}(f, p_3u_n).$$

Applying the estimation (see [**10**])

$$(2.11)\qquad \big\|u_n^{(s)}\big\|_\infty \leqslant \mathrm{const}(1+\mu_m)^{s+\frac{1}{p}}\|u_n\|_p, \quad p \geqslant 1, s = \overline{0,2}$$

we find

$$(2.12)\qquad \frac{1}{|\lambda_n|}\big|\big(f, p_2u_n^{(1)}\big)\big| + \frac{1}{|\lambda_n|}|(f, p_3u_n)| \leqslant \mathrm{const}\,\|f\|_\infty\mu_n^{-2}\bigg(\sum_{r=2}^{3}\mu_n^{2-r}\|p_r\|_1\bigg)\|u_n\|_\infty.$$

Making integration in parts in the first summand on the right-hand side of equality (2.10) and taking into account condition (1.1), we get

$$(2.13)\qquad \frac{1}{|\lambda_n|}\big|\big(f, u_n^{(3)}\big)\big| \leqslant C(f)\mu_n^{\alpha-3}\|u_n\|_\infty + \mu_n^{-3}\big|\big(f', u_n^{(2)}\big)\big|.$$

From (2.10), (2.12) and (2.13) it follows estimation (2.7).

Estimate the expression $\mu_n^{-3}\big|\big(f', u_n^{(2)}\big)\big|$ in the case $p_1(x) \equiv 0$. For that we use formulas (2.4) and (2.5) depending on the sign of $\operatorname{Im}\lambda_n$. For definiteness we consider the case $\operatorname{Im}\lambda_n < 0$ and apply formula (2.5) for $l = 2$. Then by estimations (2.6), (2.11) and

$$|M(u_n(\xi))| \leqslant \frac{1}{3}\mu_n^{-2}\sum_{r=2}^{3}|p_r(\xi)|\big|u_n^{(3-r)}(\xi)\big| \leqslant \mathrm{const}\,\mu_n^{-1}\bigg(\sum_{r=2}^{3}|p_r(\xi)|\mu_n^{2-r}\bigg)\|u_n\|_\infty$$

we get that

$$\mu_n^{-3}\big|\big(f', u_n^{(2)}\big)\big| = \mu_n^{-1}\big|\big(f', \mu_n^{-2}u_n^{(2)}\big)\big|$$

$$
\begin{aligned}
&\leqslant \mu_n^{-1} \sum_{j=1, j\neq 2}^{3} |X_j^{+}(0)||(f', \exp(i\omega_j \mu_n t))| \\
&\quad + \mu_n^{-1} |B_2^{+}||(f', \exp(-i\omega_2 \mu_n (1-t)))| \\
&\quad + \mu_n^{-1} \sum_{j=1, j\neq 2}^{3} \left| \left(f', \int_0^t M(u_n(\xi)) \exp(-i\omega_j \mu_n (\xi - t) d\xi) \right) \right| \\
&\quad + \mu_n^{-1} \left| \left(f', \int_t^1 M(u_n(\xi)) \exp(-i\omega_2 \mu_n (\xi - t) d\xi) \right) \right| \\
&\leqslant \operatorname{const} \mu_n^{-1} \big\{ |(f', \exp(-i\mu_n t))| + \big[|(f', \exp(i\omega_3 \mu_n t))| \\
&\qquad\qquad + |(f', \exp(-i\omega_2 \mu_n (1-t)))| \big] \|u_n\|_\infty \big\} \\
&\quad + \operatorname{const} \mu_n^{-2} \|f'\|_1 \left(\sum_{r=2}^{3} \mu_n^{2-r} \|p_r\|_1 \right) \|u_n\|_\infty .
\end{aligned}
$$

Thus, in the case $p_1(x) \equiv 0$, $\operatorname{Im} \lambda_n < 0$ the following estimation is fulfilled

$$
\begin{aligned}
(2.14) \quad \mu_n^{-3} \big| (f', u_n^{(3)}) \big| &\leqslant \operatorname{const} \mu_n^{-1} \Bigg\{ |(f', \exp(-i\mu_n t))| + \Bigg[|(f', \exp(i\omega_3 \mu_n t))| \\
&+ |(f', \exp(i\omega_2 \mu_n (t-1)))| + \|f'\|_1 \mu_n^{-1} \sum_{r=2}^{3} \mu_n^{2-r} \|p_r\|_1 \Bigg] \|u_n\|_\infty \Bigg\}.
\end{aligned}
$$

Consequently, estimation (2.8) follows from (2.10), (2.12)–(2.14). Estimation (2.9) for $p_1(x) \equiv 0$, $\operatorname{Im} \lambda_n > 0$ is proved in the same way. The Lemma 2.2 is proved. □

Lemma 2.3 (see [**10**]). *Let* $p_1(x) \in L_2(G)$, $p_l(x) \in L_1(G)$, $l = 2, 3$ *and* $\{u_n(x)\}$ *be an orthonormalized in* $L_2(G)$ *system of eigenfunctions of the operator* L. *Then the following estimations are fulfilled:*

$$
(2.15) \qquad \sum_{\tau \leqslant \mu_n \leqslant tau+1} 1 \leqslant \operatorname{const}, \quad \textit{for all } \tau \geqslant 0,
$$

$$
(2.16) \qquad \sum_{0 \leqslant \mu_n \leqslant \tau} \|u_n\|_\infty^2 \leqslant \operatorname{const}(1+\tau), \quad \textit{for all } \tau > 0.
$$

Lemma 2.4 (see [**9**]). *If conditions of Lemma* 2.3 *are fulfilled, then the system* $\{\mu_n^{-2} u_n^{(2)}(x)\}$, $\mu_n \geqslant 1$, *is a Bessel system, i.e., for any* $g(x) \in L_2(G)$ *the following inequality is fulfilled*

$$
\sum_{\mu_n \geqslant 1} \left| \left(f, \mu_n^{-2} u_n^{(2)}(x) \right) \right|^2 \leqslant \operatorname{const} \|g\|_2^2.
$$

Lemma 2.5. *Under conditions of Lemma* 2.3, *the system* $\{\exp(-i\mu_n t)\}$ *for* $\operatorname{Im} \lambda_n < 0$ *and the system* $\{\exp(i\mu_n t)\}$ *for* $\operatorname{Im} \lambda_n > 0$ *satisfies the Riesz inequality for* $1 < p \leqslant 2$.

Proof. As these system satisfy the Bessel inequality in $L_2(G)$ (see [**11**]) subject to condition (2.15), furthermore, for any $g(x) \in L_1(G)$ it is valid

$$\left| \int_0^1 g(x)\overline{\varphi_n(x)}dx \right| \leqslant \text{const}\, \|g\|_1$$

where $\{\varphi_n(x)\}$ is any from the above mentioned systems, then by the Riesz–Torin theorem (see [**12**]), for these systems the Riesz inequality is valid, i.e.,

$$\sum_n \left| \int_0^1 g(x)\overline{\varphi_n(x)}dx \right|^q \leqslant \text{const}\, \|g\|_p^q$$

for any $g(x) \in L_p(G)$, $1 < p \leqslant 2$. Lemma 2.5 is proved. □

Lemma 2.6. *Let the conditions of Lemma* 2.3 *be fulfilled. Then*

$$\sum_{\mu_n \geqslant \mu} \frac{\|u_n\|_\infty^2}{\mu_n^{\theta+1}} \leqslant \text{const}\, \mu^{-\theta}, \quad \theta > 0, \quad \textit{for all } \mu \geqslant 2. \tag{2.17}$$

Proof. By estimations (2.15), (2.16) and Abel transformation for any $l \in N$

$$\begin{aligned}
\sum_{\mu \leqslant \mu_n \leqslant [\mu]+l} \frac{\|u_n\|_\infty^2}{\mu_n^{1+\theta}} &\leqslant \sum_{k=[\mu]}^{[\mu]+l} \frac{1}{k^{1+\theta}} \Big(\sum_{k \leqslant \mu_n < k+1} \|u_n\|_\infty^2 \Big) \\
&\leqslant \sum_{k=[\mu]}^{[\mu]+l-1} \Big(\sum_{1 \leqslant \mu_n \leqslant k+1} \|u_n\|_\infty^2 \Big) \Big(\frac{1}{k^{1+\theta}} - \frac{1}{(k+1)^{1+\theta}} \Big) \\
&\quad + \Big(\sum_{1 \leqslant \mu_n \leqslant [\mu]+l} \|u_n\|_\infty^2 \Big) ([\mu]+l)^{-(1+\theta)} + \Big(\sum_{1 \leqslant \mu_n \leqslant [\mu]-1} \|u_n\|_\infty^2 \Big) [\mu]^{-(1+\theta)} \\
&\leqslant \text{const} \sum_{k=[\mu]}^{[\mu]+l-1} (k+1) \frac{(k+1)^\theta (1+\theta)}{(k(k+1))^{1+\theta}} \\
&\quad + \text{const}([\mu]+l)^{-\theta} + \text{const}[\mu]^{-\theta} \leqslant \text{const} \Big(\sum_{k=[\mu]}^{\infty} \frac{1+\theta}{k^{1+\theta}} + [\mu]^{-\theta} \Big).
\end{aligned}$$

Hence, by the arbitrariness of the natural number l we get estimation (2.17). □

Lemma 2.7. *Let the conditions of Lemma* 2.3 *be fulfilled. Then*

$$\sum_{\mu_n \geqslant \mu} \frac{\|u_n\|_\infty^p}{\mu_n^p} \leqslant \text{const}\, \mu^{1-p}, \quad 1 < p \leqslant 2, \quad \textit{for all } \mu \geqslant 2. \tag{2.18}$$

Proof. For $p = 2$ estimation (2.18) follows from (2.17) for $\theta = 1$. Consider the case $p \neq 2$ and apply the Holder inequality for $p' = \frac{2}{p}$, $q' = \frac{2}{2-p}$:

$$\sum_{\mu_n \geqslant \mu} \frac{\|u_n\|_\infty^p}{\mu_n^p} = \sum_{\mu_n \geqslant \mu} \frac{\|u_n\|_\infty^p}{\mu_n^{p-\frac{1}{2}}} \frac{1}{\mu_n^{\frac{1}{2}}} \leqslant \Big(\sum_{\mu_n \geqslant \mu} \frac{\|u_n\|_\infty^2}{\mu_n^{2-\frac{1}{p}}} \Big)^{p/2} \Big(\sum_{\mu_n \geqslant \mu} \frac{1}{\mu_n^{\frac{1}{2-p}}} \Big)^{\frac{2-p}{2}}$$

$$\leqslant \Big(\sum_{\mu_n\geqslant\mu}\frac{\|u_n\|_\infty^2}{\mu_n^{2-\frac{1}{p}}}\Big)^{p/2}\Big(\sum_{k=[\mu]}^{\infty}\frac{1}{k^{\frac{1}{2-p}}}\Big(\sum_{k\leqslant\mu_n\leqslant k+1}1\Big)\Big)^{\frac{2-p}{2}}.$$

Having applied here Lemma 2.6 for $\theta = 1-\frac{1}{p}$ and estimation (2.15), we get

$$\sum_{\mu_n\geqslant\mu}\frac{\|u_n\|_\infty^2}{\mu_n^p}\leqslant \mathrm{const}\,\big(\mu^{(\frac{1}{p}-1)}\big)^{p/2}[\mu]^{\frac{1-p}{2}}\leqslant \mathrm{const}\,\mu^{1-p}.$$

Lemma 2.7 is proved. □

Lemma 2.8 (see [**9**]). *Let $\{\alpha_m\}_{m=0}^{\infty}$ be a numerical sequence with the elements $\alpha_m\geqslant 0, \beta$ be a complex number for which* $\mathrm{Re}\beta>0$. *Then for the inequality*

$$\Big(\sum_{m=0}^{\infty}\alpha_m\Big|\int_G f(x)\exp(-m\beta x)dx\Big|^q\Big)^{\frac{1}{q}}\leqslant M_p\|f\|_p$$

to be fulfilled for any function $f(x)\in L_p(G)$, $1<p\leqslant 2$, $q=\frac{p}{p-1}$, it is necessary and sufficient the existence of a constant K such that for all $N=1,2,\dots$ the following estimation is valid $\sum_{m=0}^{N}\alpha_m\leqslant KN$.

Lemma 2.9. *If under the conditions of Lemma* 2.3, *we have that for each system $\big\{\|u_n\|_\infty^{\frac{2}{q}}e^{i\omega_3\mu_n t}\big\}_{n=1}^{\infty}$ and $\big\{\|u_n\|_\infty^{\frac{2}{q}}e^{-i\omega_2\mu_n(1-t)}\big\}_{n=1}^{\infty}$ for* $\mathrm{Im}\,\lambda_n<0$, *for each system $\big\{\|u_n\|_\infty^{\frac{2}{q}}e^{-i\omega_2\mu_n t}\big\}_{n=1}^{\infty}$ and $\big\{\|u_n\|_\infty^{\frac{2}{q}}e^{i\omega_3\mu_n(1-t)}\big\}_{n=1}^{\infty}$ for* $\mathrm{Im}\,\lambda_n<0$ *the Riesz inequality is fulfilled for $1<p\leqslant 2$, where $p^{-1}+q^{+1}=1$.*

Proof. Let us consider the first one of these systems and prove for it the Riesz inequality (the remaining systems are considered similarly). Since $\mu_n\in[0,+\infty)$ and $\omega_3=i\frac{\sqrt{3}}{2}+\frac{1}{2}$, then $i\omega_3\mu_n t=\big(i\frac{1}{2}-\frac{\sqrt{3}}{2}\big)\mu_n t$ and $|e^{i\omega_3\mu_n t}|=e^{-\frac{\sqrt{3}}{2}\mu_n t}$. Taking this into account, we get that for any $f(x)\in L_p(G)$

$$\begin{aligned}\sum_{\mathrm{Im}\,\lambda_n<0}\|u_n\|_\infty^2\Big|\int_0^1\overline{f(t)}e^{i\omega_3\mu_n t}dt\Big|^q &\leqslant \sum_{\mathrm{Im}\,\lambda_n<0}\|u_n\|_\infty^2\Big|\int_0^1\overline{f(t)}e^{\left(i\frac{1}{2}-\frac{\sqrt{3}}{2}\right)\mu_n t}dt\Big|^q\\ &\leqslant\sum_{n=1}^{\infty}\|u_n\|_\infty^2\Big(\int_0^1|f(t)|e^{-\frac{1}{2}\mu_n t}dt\Big)^q\\ &\leqslant\sum_{k=0}^{\infty}\sum_{k\leqslant\mu_n<k+1}\|u_n\|_\infty^2\Big(\int_0^1|f(t)|e^{-\frac{1}{2}\mu_n t}dt\Big)^q\\ &\leqslant\sum_{k=0}^{\infty}\Big(\sum_{k\leqslant\mu_n<k+1}\|u_n\|_\infty^2\Big)\Big(\int_0^1|f(t)|e^{-\frac{1}{2}kt}dt\Big)^q\\ &=\sum_{k=0}^{\infty}\alpha_k\Big(\int_0^1|f(t)|e^{-\frac{1}{2}kt}dt\Big)^q,\end{aligned}$$

where $\alpha_k=\sum_{k\leqslant\mu_n<k+1}\|u_n\|_\infty^2$.

By inequality (2.16), for any natural number N it is fulfilled:

$$\sum_{k=0}^{N}\alpha_k=\sum_{k=0}^{N}\bigg(\sum_{k\leqslant\mu_n<k+1}\|u_n\|_\infty^2\bigg)=\sum_{0\leqslant\mu_n<N+1}\|u_n\|_\infty^2\leqslant\text{const}\cdot N.$$

Consequently, the condition of Lemma 2.8 is fulfilled. Therefore, the following inequality is valid

$$\bigg\{\sum_{k=1}^{\infty}\alpha_k\bigg(\int_0^1|f(t)|e^{-\frac{1}{2}kt}dt\bigg)^q\bigg\}^{1/q}\leqslant M(p)\|f\|_p.$$

Lemma 2.9 is proved. □

3. Proof of the basic results

Proof of Theorem 1.1. If suffices to consider the case $1<p\leqslant 2$. Prove the uniform convergence of the series $\sum_{n=1}^{\infty}|f_n||u_n(x)|$ on $\bar{G}$. To this end we partition this series in two sums: $\sum_{0\leqslant\mu_n\leqslant 2}|f_n||u_n(x)|$ and $\sum_{\mu_n>2}|f_n||u_n(x)|$. The first sum by inequality (2.16) doesn't exceed the quantity $\text{const}\,\|f\|_1$. For investigating the second series we apply Lemma 2.2, i.e., estimations (2.8) and (2.9) depending on the sign of $\operatorname{Im}\lambda_n$. For that we represent the given series in the form

$$\sum_{\mu_n>2}|f_n||u_n(x)|=\sum_{n\in J_1}|f_n||u_n(x)|+\sum_{n\in J_2}|f_n||u_n(x)|=I_1+I_2,$$

where $J_1=\{n:\mu_n>2,\operatorname{Im}\lambda_n<0\}$, $J_2=\{n:\mu_n>2,\operatorname{Im}\lambda_n>0\}$. By estimation (2.8)

$$\begin{aligned}
I_1=\sum_{n\in J_1}|f_n||u_n(x)|&\leqslant\text{const}\sum_{n\in J_1}C(f)\mu_n^{\alpha-3}\|u_n\|_\infty^2\\
&+\text{const}\sum_{n\in J_1}\mu_n^{-1}|(f',e^{i\omega_3\mu_n t})|\|u_n\|_\infty^2\\
&+\text{const}\sum_{n\in J_1}\mu_n^{-1}|(f',e^{-i\omega_2\mu_n(1-t)})|\|u_n\|_\infty^2\\
&+\text{const}(\|f\|_\infty+\|f'\|_1)\sum_{n\in J_1}\mu_n^{-2}\bigg(\sum_{r=2}^{3}\mu_n^{2-r}\|p_r\|_1\bigg)\|u_n\|_\infty^2\\
&+\text{const}\sum_{n\in J_1}\mu_n^{-1}|(f,e^{-i\mu_n t})|\|u_n\|_\infty\\
&=\text{const}(I_1^1+I_1^2+I_1^3+I_1^4+I_1^5).
\end{aligned}$$

Estimate the series $I_1^j, j=\overline{1,5}$. By Lemma 2.6 and condition $0\leqslant\alpha<2$ we find

(3.1)
$$\begin{aligned}
I_1^1=C(f)\sum_{n\in J_1}\mu_n^{\alpha-3}\|u_n\|_\infty^2&\leqslant C(f)\sum_{\mu_n\geqslant 2}\frac{\|u_n\|_\infty^2}{\mu_n^{1+(2-\alpha)}}\\
&\leqslant\text{const}\,C(f)2^{\alpha-2}<\infty.
\end{aligned}$$

For estimating the series I_1^2 we apply at first the Holder inequality for the sum, and then Lemmas 2.6 and 2.9:

$$\begin{aligned}
I_1^2 &= \sum_{n\in J_1} \|u_n\|_\infty^{2/p}\mu_n^{-1}\big(\|u_n\|_\infty^{2/q}|(f', e^{i\omega_3\mu_n t})|\big) \\
&\leqslant \Big\{\sum_{n\in J_1}\frac{\|u_n\|^2}{\mu_n^p}\Big\}^{1/p}\Big\{\sum_{n\in J_1}\|u_n\|_\infty^2|(f', e^{i\omega_3\mu_n t})|^q\Big\}^{1/q} \\
&\leqslant \Big\{\sum_{\mu_n\geqslant 2}\frac{\|u_n\|^2}{\mu_n^p}\Big\}^{1/p}\Big\{\sum_{n\in J_1}\|u_n\|_\infty^2|(f', e^{i\omega_3\mu_n t})|^q\Big\}^{1/q} \\
&\leqslant \mathrm{const}\, 2^{-1/q}M(p)\|f'\|_p < \infty.
\end{aligned}$$

The series I_1^3 is estimated in the same way as the series I_1^2. For estimating the series I_1^4 we apply Lemma 2.6.

$$\begin{aligned}
I_1^4 &= (\|f\|_\infty + \|f'\|_1)\sum_{n\in J_1}\mu_n^{-2}\bigg(\sum_{r=2}^{3}\mu_n^{2-r}\|p_r\|_1\bigg)\|u\|_\infty^2 \\
&\leqslant \mathrm{const}(\|f\|_\infty + \|f'\|_1)\bigg(\sum_{r=2}^{3}\|p_r\|_1 2^{1-r}\bigg) < \infty.
\end{aligned} \tag{3.2}$$

Now estimate the series I_1^5. For that we apply the Holder inequality, and then Lemmas 2.5 and 2.7:

$$\begin{aligned}
I_1^5 &= \sum_{n\in J_1}\frac{\|u_n\|_\infty}{\mu_n}|(f', e^{-i\mu_n t})| \\
&\leqslant \bigg(\sum_{n\in J_1}\frac{\|u_n\|_\infty^p}{\mu_n^p}\bigg)^{1/p}\bigg(\sum_{n\in J_1}|(f', e^{-i\mu_n t})|^q\bigg)^{\frac{1}{q}} \leqslant \mathrm{const}\,\|f'\|_p 2^{-\frac{1}{q}} < \infty.
\end{aligned}$$

Thus, the series I_1 uniformly converges on $\bar{G}$. Applying estimation (2.9) for the coefficients f_n in the same way we prove the uniform convergence of the series I_2 on $\bar{G}$. Consequently, the series $\sum_{n=1}^{\infty}|f_n|\,|u_n(x)|$ uniformly converges on $\bar{G}$. By the completeness of the system $\{u_n(x)\}_{n=1}^{\infty}$ in $L_2(G)$ and continuity of the function $f(x)$ on $\bar{G}$ the series $\sum_{n=1}^{\infty} f_n u_n(x)$ uniformly converges to $f(x)$, i.e. it holds the equality

$$f(x) = \sum_{n=1}^{\infty} f_n u_n(x), \quad x \in \bar{G} \tag{3.3}$$

Now establish estimate (1.2). By equality (3.3)

$$|R_\nu(x,f)| = |f(x) - \sigma_\nu(x,f)| = \bigg|\sum_{\mu_n>\nu} f_n u_n(x)\bigg|$$

$$\leqslant \sum_{\mu_n>\nu} |f_n||u_n(x)| = \sum_{n\in A_1(\nu)} + \sum_{n\in A_2(\nu)} = T_1(\nu)+T_2(\nu),$$

where $A_1(\nu) = \{n : \mu_n \geqslant \nu, \operatorname{Im}\lambda_n < 0\}$, $A_2(\nu) = \{n : \mu_n \geqslant \nu, \operatorname{Im}\lambda_n < 0\}$. The series $T_1(\nu), T_2(\nu)$ are estimated by the scheme demonstrated by estimating the series I_1. As a result we find

$$T_j(\nu) \leqslant \text{const}\left\{C(f)\nu^{\alpha-2} + \nu^{-\frac{1}{q}}\|f'\|_p + \nu^{-1}(\|f\|_\infty + \|f'\|_1)\left(\sum_{r=2}^{3}\|p_r\|_1\nu^{2-r}\right)\right\},$$
$$j=1,2, \quad x\in\bar{G}.$$

Consequently, estimation (1.2) is valid for $1 < p \leqslant 2$. For $p > 2$ the validity of estimation (1.2) follows from the embedding $L_p(G) \subset L_2(G)$.

Theorem 1.1 is proved. □

Corollary 1.1 follows from Theorem 1.1 with regard to the inequality $\|f\|_\infty \leqslant \|f'\|_1$ that is fulfilled for the function$f(x) \in W_p^1(G)$, $f(0) = f(1) = 0$.

Proof of Theorem 1.2. In the present case it is necessary to prove the uniform convergence of the series $\sum_{\mu_n\geqslant 2}|f_n||u_n(x)|$ on $\bar{G}$. By estimation (2.7) we find

$$\begin{aligned}\sum_{\mu_n\geqslant 2}|f_n||u_n(x)| \leqslant \text{const}\Bigg\{&C(f)\sum_{\mu_n\geqslant 2}\mu^{\alpha-3}\|u_n\|_\infty^2\\
&+\sum_{\mu_n\geqslant 2}\mu_n^{-1}\left|\left(f\,\overline{p_1}, \frac{u_n^{(2)}}{\mu_n^2}\right)\right|\|u_n\|_\infty^2 + \sum_{\mu_n\geqslant 2}\mu_n^{-1}\|u_n\|_\infty\left|\left(f', \frac{u_n^{(2)}}{\mu_n^2}\right)\right|\\
&+\|f\|_\infty\sum_{\mu_n\geqslant 2}\mu_n^{-2}\|u_n\|_\infty^2\left(\sum_{r=2}^{3}\mu_n^{2-r}\|p_r\|_1\right)\Bigg\}\\
&\leqslant \text{const}\{V_1+V_2+V_3+V_4\}.\end{aligned}$$

The series V_1 and V_4 are estimated in the same way as the series I_1^1 and I_1^4. For V_1 estimation (3.1) is fulfilled, and for V_4 estimation (3.2) is fulfilled by changing the multiplier $(\|f\|_\infty + \|f'\|_1)$ by the multiplier $\|f\|_\infty$.

By estimating the series V_2 and V_3, we apply the Bessel inequality for the system $\{u_n^{(2)}(x)/\mu_n^2\}$, $\mu_n \geqslant 2$, whose validity is established in the paper [**9**], and also Lemma 2.7 for $p = 2$.

As a result, we find

$$\begin{aligned}V_2 = \sum_{\mu_n\geqslant 2}\mu_n^{-1}\|u_n\|_\infty\left(f\,\overline{P_1}, \frac{u_n^{(2)}}{\mu_n^2}\right) &\leqslant \left(\sum_{\mu_n\geqslant 2}\frac{\|u_n\|_\infty^2}{\mu_n^2}\right)^{1/2}\\
\times\left(\sum_{\mu_n\geqslant 2}\left|\left(f\,\overline{p_1}, \frac{u_n^{(2)}}{\mu_n^2}\right)\right|^2\right)^{1/2} &\leqslant \text{const}\,2^{-\frac{1}{2}}\|f\,\overline{p_1}\|_2;\end{aligned}$$

$$V_3 \leqslant \Bigg(\sum_{\mu_n \geqslant 2} \frac{\|u_n\|_\infty^2}{\mu_n^2}\Bigg)^{1/2} \Bigg(\sum_{\mu_n \geqslant 2} \bigg|\bigg(f', \frac{u_n^{(2)}}{\mu_n^2}\bigg)\bigg|^2\Bigg)^{1/2} \leqslant \operatorname{const} 2^{-\frac{1}{2}} \|f'\|_2.$$

Consequently, the series $\sum_{n=1}^\infty f_n u_n(x)$ converges absolutely and uniformly on $\bar{G}$, and the following equality $f(x) = \sum_{n=1}^\infty f_n u_n(x)$, $x \in G$, is valid. It is easy to see that for the remainder $R_\nu(x, f)$ of this series the following estimation will be valid (in the remainder the summation is conducted according to numbers n, for which $\mu_n > \nu$)

$$\sup |R_\nu(x,f)| \leqslant \operatorname{const} \Big\{ C(f)\nu^{\alpha-2} + \nu^{-\frac{1}{2}}(\|f\,\overline{p_1}\|_2 + \|f'\|_2) + \nu^{-1}\sum_{r=2}^{3} \nu^{2-r}\|p_r\|_1 \Big\},$$
$$\nu \geqslant 2.$$

Theorem 1.2 is proved. □

For justification of Corollary 1.2, it suffices to take into account that the sequence of remainders of the converging series tends to zero, more exactly,

$$\sum_{\mu_n \geqslant \nu} \bigg|\bigg(f\,\overline{p_1}, \frac{u_n^{(2)}}{\mu_n^2}\bigg)\bigg|^2 = o(1), \quad \nu \to +\infty;$$

$$\sum_{\mu_n \geqslant \nu} \bigg|\bigg(f', \frac{u_n^{(2)}}{\mu_n^2}\bigg)\bigg|^2 = o(1), \quad \nu \to +\infty.$$

Proof of Theorem 1.3. By orthonormality of the system $\{u_n(x)\}_{n=1}^\infty$ in $L_2(G)$, condition (2.15) is fulfilled. On the other hand,

$$1 = |(u_n, u_n)| \leqslant \|u_n\|_p \|u_n\|_q \leqslant \|u_n\|_\infty \|u_n\|_q.$$

Hence we find $\|u_n\|_q^{-q} \leqslant \|u_n\|_\infty^q$. Therefore, by inequality (2.15) and uniform boundedness of the system $\{u_n(x)\}_{n=1}^\infty$ we find

$$\sum_{0 \leqslant \mu_n \leqslant \tau} \|u_n\|_\infty^q \|u_n\|_q^{-q} \leqslant \sum_{0 \leqslant \mu_n \leqslant \tau} \|u_n\|_\infty^{2q} \leqslant C \sum_{0 \leqslant \mu_n \leqslant \tau} 1 \leqslant \operatorname{const} \tau, \quad \text{for all } \tau > 0.$$

Thus, for the system $\{u_n(x)\}_{n=1}^\infty$, all the conditions of the sufficiency part of [**9**, Theorem 3] are fulfilled. Therefore, for the system $\{\frac{u_n(x)}{\mu_n^2}\}$, $\mu_n \geqslant 1$ the Riesz inequality is valid.

For proving Theorem 1.3, it suffices to estimate the series $\mathbf{V}_3$ (all remaining series V_1, V_2, V_4 were estimated in Theorem 1.2 without requirement of uniform boundedness of the system $\{u_n(x)\}_{n=1}^\infty$). At first, apply the Hölder inequality, and then the Riesz inequality and Lemma 2.7. As a result, for V_3 and its remainder we get

$$\begin{aligned} V_3 &= \sum_{\mu_n \geqslant 2} \mu_n^{-1} \|u_n\|_\infty \bigg|\bigg(f', \frac{u_n^{(2)}}{\mu_n^2}\bigg)\bigg| \\ &\leqslant \Bigg(\sum_{\mu_n \geqslant 2} \frac{\|u_n\|_\infty^p}{\mu_n^p}\Bigg)^{1/p} \Bigg(\sum_{\mu_n \geqslant 2} \bigg|\bigg(f', \frac{u_n^{(2)}}{\mu_n^2}\bigg)\bigg|^q\Bigg)^{1/q} \leqslant \operatorname{const} 2^{-\frac{1}{q}} \|f'\|_2; \end{aligned}$$

$$\sum_{\mu_n\geqslant\nu}\frac{\|u_n\|_\infty}{\mu_n}\left|\left(f,\frac{u_n^{(2)}}{\mu_n^2}\right)\right|$$
$$\leqslant\left(\sum_{\mu_n\geqslant\nu}\frac{\|u_n\|_\infty^p}{\mu_n^p}\right)^{1/p}\left(\sum_{\mu_n\geqslant\nu}\left|\left(f',\frac{u_n^{(2)}}{\mu_n}\right)\right|^q\right)^{1/q}\leqslant \operatorname{const}\nu^{-\frac{1}{q}}\|f'\|_p.$$

Theorem 1.3 is proved. □

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