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REPRESENTATION WITH MAJORANT OF THE SCHWARZ LEMMA AT THE BOUNDARY

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Abstract. Let f be a holomorphic function in the unit disc and $|f(z)-1|<1$ for $|z|<1$. We generalize the uniqueness portion of Schwarz's lemma and provide sufficient conditions on the local behavior of f near a finite set of boundary points that needed for f to be a finite Blaschke product.

1. Introduction

Let f be a holomorphic function in the unit disc $D=\{z:|z|<1\}$, $f(0)=0$ and $|f(z)|<1$ for $|z|<1$. In accordance with the classical Schwarz lemma, for any point z in the disc D, we have $|f(z)|\leqslant|z|$ and $|f'(0)|\leqslant 1$. Equality in these inequalities (in the first one, for $z\neq 0$) occurs only if $f(z)=\lambda z$, $|\lambda|=1$ [**6**, p. 329].

In recent years, a boundary version of Schwarz lemma was investigated in Burns and Krantz [**4**], Chelts [**5**] and also in papers of a few other authors. They studied the uniqueness portion of the Schwarz lemma.

D. Burns and S. G. Krantz proved the following interesting theorem related to the uniqueness part of the classical Schwarz lemma for single variable functions [**6**].

Theorem 1.1. *Let $f:D\to D$ be a holomorphic function such that*

$$f(z)=z+O((z-1)^4) \tag{1.1}$$

as $z\to 1$. Then $f(z)=z$ on the disc.

Chelst [**5**] takes the Burns–Krantz theorem much further and takes the Blaschke product instead of the model function $f(z)=z$.

Theorem 1.2. *Let $f:D\to D$ be a holomorphic function and $\phi:D\to D$ a finite Blaschke product which equals $\tau\in\partial D$ on a finite set $A_f\subset\partial D$. If* (i) *for a given $\gamma_0\in A_f$, $f(z)=\phi(z)+O((z-1)^4)$, as $z\to\gamma_0$, and* (ii) *for all $\gamma\in A_f-\{\gamma_0\}$, $f(z)=\phi(z)+O((z-1)^{k_\gamma})$, for some $k_\gamma\geqslant 2$ as $z\to\gamma$, then $f(z)=\phi(z)$ on the disc.*

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Let $\mathfrak{N}$ be a class of functions $\mu : (0, +\infty) \to (0, +\infty)$ for each of which $\log \mu(x)$ is concave with respect to $\log x$. For each function $\mu \in \mathfrak{N}$ the limit

$$\mu_0 = \lim_{x\to 0} \frac{\log \mu(x)}{\log x},$$

exists, and $-\infty < \mu_0 \leqslant +\infty$. Here, the function $\mu \in \mathfrak{N}$ is called bilogaritmic concave majorant [**12**].Obviously $x^\alpha \in \mathfrak{N}$ for any $\alpha > 0$.

In this paper, more general majorants will be taken instead of power majorants in conditions (i), (ii) and (1.1). Also, the approach to the boundary of z in conditions (i), (ii) and (1.1) are inside of the unit disc. Instead of these conditions, the behavior of f at the boundary will be considered. In other words, the approach of z from ∂D to points γ will be considered as a condition. Such results were first announced in [**2**].

Let $\mathbb{U}(z, r)$ be an open disc with centre z and radius r.

We propose the following assertion for the proofs of our theorems:

(A) Let $u = u(z)$ be a positive harmonic function on the open disk $\mathbb{U}(z, r_0)$, $r_0 > 0$. Suppose that for $\theta_0 \in [0, 2\pi)$, $\lim_{r\to r_0} u(re^{i\theta_0}) = 0$ is satisfied. Then

$$\liminf_{r\to r_0} \frac{u(re^{i\theta_0})}{r_0 - r} > 0.$$

This assertion follows from the Harnack inequality. For more general results and related estimates, see [**8**, Theorem 1.1], [**9**], [**10**].

In addition, after having submitted the present paper, [**11**] was posed on ResearchGate in which further results are discussed.

2. Main Results

Let $d(z, G)$ be a distance of the point z from the set G .

Theorem 2.1. *Let $\mu \in \mathfrak{N}$, $\mu_0 > 3$, f be a holomorphic function in the unit disc that is continuous on $\overline{D} \cap \mathbb{U}\ (1, \delta_0)$ for some $\delta_0 > 0$ and $|f(z) - 1| < 1$ for $|z| < 1$ such that*

$$f(z) = 1 + z + O(\mu(|z - 1|)), \quad z \in \partial D, \quad z \to 1.$$

Then $f(z) = 1 + z$.

Proof. Consider the function

$$\varphi(z) = f(z) - 1 = 1 + z + O(\mu(|z - 1|)) - 1 = z + O(\mu(|z - 1|)).$$

There exists a number $c_1 > 0$ such that

$$|\varphi(z) - z| \leqslant c_1 \mu(|z - 1|), \quad \forall z \in \partial D \cap \mathbb{U}(1, \delta_0).$$

Let us denote k and c_2 as follows

$$k = \sup_{\substack{|z-1|=\delta_0 \\ z\in D}} |\varphi(z) - z|, \quad c_2 = \max\left\{\frac{k}{\mu(\delta_0)}, c_1\right\}.$$

It can be easily seen that for all boundary points of the set $D \cap \mathbb{U}(1,\delta_0)$, the inequality $|\varphi(z) - z| \leqslant c_2\mu(|z-1|)$ is satisfied. Applying Theorem 3 in [**12**] (see also [**3**], [**1**]) to the set $D \cap \mathbb{U}(1,\delta_0)$ and the function $\varphi(z) - z$, one receives

$$|\varphi(z) - z| \leqslant c_2\mu(|z-1|), \quad \forall z \in D \cap \mathbb{U}(1,\delta_0). \tag{2.1}$$

From $\mu_0 > 3$, we take that there exist some positive constants $\varepsilon > 0$ and $\sigma < \min(\delta_0, 1)$ such that

$$\frac{\log \mu(x)}{\log x} \geqslant 3 + \varepsilon, \quad \forall x \in (0,\sigma).$$

Otherwise

$$\mu(x) \leqslant x^{3+\varepsilon}, \quad \forall x \in (0,\sigma). \tag{2.2}$$

By (2.1) and (2.2), we obtain

$$|\varphi(z) - z| \leqslant c_2|z-1|^{3+\varepsilon} \quad \forall z \in D \cap \mathbb{U}(1,\sigma). \tag{2.3}$$

Consider the harmonic function h defined as

$$h(z) = \operatorname{Re}\Big(\frac{1+\varphi(z)}{1-\varphi(z)}\Big) - \operatorname{Re}\Big(\frac{1+z}{1-z}\Big).$$

The function

$$\frac{1+\varphi(z)}{1-\varphi(z)}$$

maps the disc D to the right half plane and hence, the first term of $h(z)$ is nonnegative, the second term is zero on $\partial D \setminus \{1\}$. Therefore,

$$\liminf_{z\to\varsigma, z\in D} h(z) \geqslant 0, \quad \forall \varsigma \in \partial D \setminus \{1\}. \tag{2.4}$$

With the simple calculations, we take

$$h(z) = \operatorname{Re}\Big(\frac{2(\varphi(z) - z)}{(1-\varphi(z))(1-z)}\Big).$$

From (2.3),

$$\lim_{z\to 1, z\in D} \frac{1-\varphi(z)}{1-z} = 1$$

and, so there exists $\delta_1 \in (0,\sigma)$ such that

$$|1-\varphi(z)| \geqslant \frac{1}{2}|z-1| \quad \forall z \in D \cap \mathbb{U}(1,\delta_1).$$

Therefore,

$$|(1-\varphi(z))(1-z)| \geqslant \frac{1}{2}|z-1|^2 \quad \forall z \in D \cap \mathbb{U}(1,\delta_1)$$

and

$$\left|\frac{2(\varphi(z) - z)}{(1-\varphi(z))(1-z)}\right| \leqslant \frac{c_2|z-1|^{3+\varepsilon}}{\frac{1}{2}|z-1|^2} = 2c_2|z-1|^{3+\varepsilon} \quad \forall z \in D \cap \mathbb{U}(1,\delta_1). \tag{2.5}$$

Consequently,

$$\lim_{z\to 1, z\in D} h(z) = 0. \tag{2.6}$$

Applying the maximum principle [**7**, p. 48] to the harmonic function $h(z)$, from (2.4) and (2.6), we conclude either $h(z) > 0$, $\forall z \in D$ or $h \equiv 0$. If h is not a constant, taking $z = r$ in (2.5) gives us

$$\lim_{r\to 1} \frac{h(r)}{1-r} = 0. \tag{2.7}$$

(2.6) and (2.7) contradict with assertion (A) statement. Consequently, $h \equiv 0$. This implies that $\varphi(z) = z$ and $f(z) = (1 + z)$. □

THEOREM 2.2. *Let $\phi : D \to D$ be a finite Blaschke product which equals $\tau \in \partial D$ on a finite set $A_f \subset \partial D$. Let f be a holomorphic function in the unit disc that is continuous on $\overline{D} \cap \{z : d(z, A_f) < \delta_0\}$ for some $\delta_0 > 0$ and $|f(z)-1| < 1$ for $|z| < 1$. Assume that $\mu^1, \mu^2 \in \mathfrak{N}$, $\mu_0^1 > 3$, $\mu_0^2 > 2$. Suppose that the following conditions are satisfied* (i) *for a given $\gamma_0 \in A_f$*

$$f(z) = 1 + \phi(z) + O(\mu^1(|z - \gamma_0|)), \quad z \in \partial D, \quad z \to \gamma_0, \tag{2.8}$$

(ii) *for all $\gamma \in A_f - \{\gamma_0\}$*

$$f(z) = 1 + \phi(z) + O(\mu^2(|z - \gamma|)), \quad z \in \partial D, \quad z \to \gamma. \tag{2.9}$$

Then $f(z) = 1 + \phi(z)$.

PROOF. Let $\Phi(z) = f(z) - 1$. By using (2.8) and (2.9), we obtain for a given $\gamma_0 \in A_f$

$$\Phi(z) = \phi(z) + O(\mu^1(|z - \gamma_0|)), \quad z \in \partial D, \quad z \to \gamma_0 \tag{2.10}$$

and for all $\forall \gamma \in A_f - \{\gamma_0\}$

$$\Phi(z) = \phi(z) + O(\mu^2(|z - \gamma|)), \quad z \in \partial D, \quad z \to \gamma.$$

Without loss of generality, we may assume that $\tau = 1$ and $\gamma_0 = 1$. Due to (2.10), there exist numbers $c_3 > 0$, $\delta_0 \in (0, 1)$ such that

$$|\Phi(z) - \phi(z)| \leqslant c_3 \mu^1(|\varsigma - 1|), \quad \forall \varsigma \in \partial D, \quad |\varsigma - 1| < \delta_0.$$

Let us denote p and c_4 as follows;

$$p = \sup_{\substack{|z-1|=\delta_0 \\ z\in D}} |\Phi(z) - \phi(z)|, \quad c_4 = \max\left\{\frac{p}{\mu^1(\delta_0)}, c_3\right\}.$$

It can be easily seen that for all boundary points of the set $D \cap \mathbb{U}(1, \delta_0)$, the inequality

$$\limsup_{z\to\varsigma, z\in D} |\Phi(z) - \phi(z)| \leqslant c_4 \mu^1(|\varsigma - 1|)$$

is satisfied. Applying Theorem 3 of [**2**] to the set $D \cap \mathbb{U}(1, \delta_0)$ and the function $\Phi(z) - \phi(z)$, one yields

$$|\Phi(z) - \phi(z)| \leqslant c_4 \mu^1(|z - 1|), \quad \forall z \in D \cap \mathbb{U}(1, \delta_0). \tag{2.11}$$

From $\mu_0 > 3$, there are some positive constants ε and $\sigma < \min(\delta_0, 1)$ such that inequality (2.2) is satisfied. Combining (2.2) and (2.11), we obtain

$$|\Phi(z) - \phi(z)| \leqslant c_4(|z - 1|)^{3+\varepsilon}, \quad \forall z \in D \cap \mathbb{U}(1, \sigma). \tag{2.12}$$

Analogously, for any point $\gamma \in A_f - \{1\}$, from conditions $\mu_0^2 > 2$ and (2.9), we have

$$|\Phi(z) - \phi(z)| \leqslant c_5(|z-\gamma|)^{2+\varepsilon}, \quad \forall z \in D \cap \mathbb{U}(\gamma, \sigma_1) \tag{2.13}$$

with some positive constants c_5 and σ_1

We introduce the harmonic function

$$\psi(z) = \operatorname{Re}\Big(\frac{1+\Phi(z)}{1-\Phi(z)}\Big) - \operatorname{Re}\Big(\frac{1+\phi(z)}{1-\phi(z)}\Big).$$

Since a finite Blaschke product ϕ is a holomorphic function throughout $\overline{D}$ and $|\phi| = 1$ on ∂D, we have that the second term of ψ is zero on $\partial D - A_f$. The first term of ψ is nonnegative. Consequently,

$$\liminf_{z \to \varsigma, z \in D} \psi(z) \geqslant 0, \quad \forall \varsigma \in \partial D - A_f \tag{2.14}$$

Now, let us examine the behavior of the function ψ at points of set A_f. After simple calculations, we obtain

$$\psi(z) = \operatorname{Re}\Big(\frac{2(\Phi(z) - \phi(z))}{(1-\Phi(z))(1-\phi(z))}\Big).$$

Now, let's take any point $\gamma \in A_f - \{1\}$. For the finite Blaschke product ϕ, obviously $|\phi'(z)| > 0$ for any z, $|z| = 1$. If $|\phi'(\gamma)| = c_\gamma$, then there exists a constant $\sigma_\gamma \in (0, \sigma_1)$ such that

$$|1 - \phi(z)| \geqslant \frac{c_\gamma}{2}|\gamma - z| \quad \forall z \in D \cap \mathbb{U}(\gamma, \sigma_\gamma). \tag{2.15}$$

From (2.13), we get $\lim_{z \to \gamma} \frac{1-\Phi(z)}{1-z} = c_\gamma$ and there exists $\sigma'_\gamma \in (0, \sigma_\gamma)$ such that

$$|1 - \Phi(z)| \geqslant \frac{c_\gamma}{2}|\gamma - z| \quad \forall z \in D \cap \mathbb{U}(\gamma, \sigma'_\gamma). \tag{2.16}$$

Then, from (2.13), (2.15) and (2.16),

$$\Big|\frac{2(\Phi(z) - \phi(z))}{(1-\Phi(z))(1-\phi(z))}\Big| \leqslant \frac{8c_5}{c_\gamma^2} \quad \forall z \in D \cap \mathbb{U}(\gamma, \sigma'_\gamma).$$

Thus, the function $\psi(z)$ is bounded below (even bounded) at the some neighborhood of every point $\gamma \in A_f - \{1\}$.

Using (2.12) (instead of (2.13), analogously we may conclude

$$|\psi(z)| \leqslant c_6|z-1|^{1+\varepsilon} \quad \forall z \in D \cap \mathbb{U}(\gamma, \sigma'_\gamma) \tag{2.17}$$

for some positive constants c_6 and σ', from which in particular

$$\lim_{z \to 1, z \in D} \psi(z) = 0. \tag{2.18}$$

So, the harmonic function $\psi(z)$ satisfied condition (2.14) and is bounded below at the some neighborhood of every point of finite set A_f. Then, from the Phragmen–Lindelöf principle, [**7**] we obtain either $\psi(z) > 0$, $\forall z \in D$ or $\psi \equiv 0$. Taking $z = r$ in (2.17) give us

$$\lim_{z \to 1} \frac{\psi(r)}{1-r} = 0. \tag{2.19}$$

If ψ is not constant, (2.18) and (2.19) contradict with assertion (A) statement. Consequently, $\psi \equiv 0$. This implies that $\Phi(z) = \phi(z)$ and $f(z) = (1 + \phi(z))$. $\square$

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