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APPROXIMATION BY GENERALIZED SRIVASTAVA–GUPTA OPERATORS BASED ON CERTAIN PARAMETER

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Abstract. We establish some direct results in simultaneous approximation for a generalization of the Srivastava–Gupta operators. We establish pointwise convergence, Voronovskaja type asymptotic formula and an error estimate in terms of modulus of continuity of the function.

1. Introduction

In the year 2003, Srivastava and Gupta [**18**] introduced a general family of summation-integral type operators which include some well-known operators as special cases. They estimated the rate of convergence for functions of bounded variation. After that several researchers studied different approximation properties of these important general form of operators, we mention some of them as [**10**, **19**, **22**], etc. Also, very recently in order to modify the Phillips operators, Păltănea [**15**] proposed a generalization of the well known Phillips operators [**17**] (see also [**5**]) based on the parameter $\rho > 0$, which provide a link with classical Szász operators as $\rho \to \infty$. As the operators introduced in [**18**] include Phillips operators as special case, this motivated us to study in this direction and here we consider the generalization of the Srivastava–Gupta operators.

For $f \in C_\gamma[0,\infty) = \{f \in C[0,\infty) : f(t) = O(t^\gamma), \gamma > 0\}$, we define the following generalization of Srivastava–Gupta operators based on certain parameter $\rho > 0$ in the following way:

$$L_{n,\rho}(f;x) = \sum_{k=1}^{\infty} p_{n,k}(x,c) \int_0^\infty \Theta_{n,k}^{\rho}(t,c) f(t)\,dt + p_{n,0}(x,c) f(0), \tag{1.1}$$

$$p_{n,k}(x,c) = \frac{(-x)^k}{k!}\phi_{n,c}^{(k)}(x), \quad \Theta_{n,k}^{\rho}(t,c) = \begin{cases} \frac{n\rho}{\Gamma(k\rho)} e^{-n\rho t}(n\rho t)^{k\rho-1}, & c = 0 \\ \frac{\Gamma(n\rho/c+k\rho)}{\Gamma(k\rho)\Gamma(n\rho/c)} \frac{c^{k\rho}t^{k\rho-1}}{(1+ct)^{n\rho/c+k\rho}}, & c = 1,2,3,\dots \end{cases}$$

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$$\phi_{n,c}(x)=\begin{cases}e^{-nx}, & c=0,\\ (1+cx)^{-n/c}, & c=1,2,3,\dots\end{cases}$$

Here $\{\phi_{n,c}(x)\}_{n=1}^{\infty}$ is a sequence of functions defined on the closed interval $[0,b]$, $b>0$, satisfying the following properties. For each $n\in\mathbb{N}$ and $k\in\mathbb{N}^0:=\mathbb{N}\cup\{0\}$:

(i) $\phi_{n,c}\in C^{\infty}[a,b]$, $b>a\geqslant 0$,

(ii) $\phi_{n,c}(0)=1$,

(iii) $\phi_{n,c}$ is completely monotone so that $(-1)^k\phi_{n,c}^{(k)}(x)\geqslant 0$, $x\in[0,b]$,

(iv) there exists an integer c such that

$$\phi_{n,c}^{(k+1)}(x)=-n\phi_{n+c,c}^{(k)}(x),\quad n>\max\{0,-c\};\ x\in[0,b],$$

(see [**18**,**19**]). It can be easily verified that the operators $L_{n,\rho}(f,x)$ are well defined for $f\in C_{\gamma}[0,\infty)$.

For $\rho=1$, operators (1.1) reduce to the Srivastava–Gupta operators [**18**]. When $c=0$, $\rho>0$, we immediately get the generalized operators due to Păltănea [**15**]. In the case $c=0$, he studied some approximation properties in simultaneous of these operators in [**16**]. Also, for $c=0$ and $\rho=1$ operators (1.1) reduce to Phillips operators [**5**,**17**].

Remark 1.1. It is observed that the general form of operators (1.1) provide link with the classical Szász and Baskakov operators. For $f\in\bar{\Pi}$, where $\bar{\Pi}$ be the closure of the space of polynomials, we have

$$\lim_{\rho\to\infty}L_{n,\rho}(t^r,x)=\sum_{k=0}^{\infty}p_{n,k}(x,c)\Big(\frac{k}{n}\Big)^r,\quad \text{uniformly for all } x\in[0,\infty).$$

Obviously,

$$\int_0^{\infty}\Theta_{n,k}^{\rho}(t,c)t^r dt=\frac{\Gamma(k\rho+r)}{\Gamma(k\rho)}\frac{1}{\prod_{i=1}^{r}(n\rho-ic)},$$

$$\lim_{\rho\to\infty}\frac{\Gamma(k\rho+r)}{\Gamma(k\rho)}\frac{1}{\prod_{i=1}^{r}(n\rho-ic)}=\Big(\frac{k}{n}\Big)^r.$$

From this the result follows immediately. For $c=0$ the link with the classical Szász operators was proved in [**15**] in more general conditions.

Thus in case $\rho\to\infty$ we have the following links with the generalized Srivastava–Gupta operators: For $c=0$ the operator $L_{n,\rho}$ provides the link with the classical Szász operators (see [**15**]) and for $c=1$ the operators $L_{n,\rho}(f,x)$ reduce to the well known Baskakov operators.

The study in simultaneous approximation began with a remarkable result for the Bernstein polynomials $B_n(f)$ owing to Wigert [**21**] after that Lorentz [**12**], who proved that $B_n^{(r)}(f)\to f(r)$, whenever the latter exists at the particular point $x\in[0,1]$, $r=1,2,3,\dots$ being arbitrary. For further study in this direction, we refer the reader to [**2**–**4**,**6**,**8**,**9**,**11**,**13**,**14**].

The aim of this paper is to study some results in simultaneous approximation by the operators $L_{n,\rho}$. First, we establish the basic pointwise convergence theorem, Voronovskaja type asymptotic formula and then proceed to study the degree of this approximation.

2. Preliminaries

LEMMA 2.1. [**7**] *Let $m \in \mathbb{N} \cup 0$. If the m-th order is defined as*

$$T_{n,m}(x) = \sum_{k=0}^{\infty} p_{n,k}(x,c)\Big(\frac{k}{n} - x\Big)^m,$$

then $T_{n,0}(x) = 1$, $T_{n,1}(x) = 0$ and also there holds the recurrence relation:

$$nT_{n,m+1}(x) = x(1+cx)[T'_{n,m}(x) + mT_{n,m-1}(x)].$$

Consequently, we have $T_{n,m}(x) = O(n^{-[(m+1)/2]})$.

REMARK 2.1. For $\Theta^{\rho}_{n,k}(t,c)$ the following identity holds:

$$(1+ct)[t\Theta^{\rho}_{n,k}(t,c)]' = \rho(k-nt)\Theta^{\rho}_{n,k}(t).$$

Taking the derivative of $[t\Theta^{\rho}_{n,k}(t,c)]$ with respect to t, we get

$$[t\Theta^{\rho}_{n,k}(t)]' = \frac{\Gamma\big(\frac{n\rho}{c} + k\rho\big)}{\Gamma(k\rho)\Gamma\big(\frac{n\rho}{c}\big)} c^{k\rho}\Bigg[\frac{k\rho t^{k\rho-1}}{(1+ct)^{\frac{n\rho}{c}+k\rho}} - \frac{(\frac{n\rho}{c}+k\rho)t^{k\rho}}{(1+ct)^{\frac{n\rho}{c}+k\rho+1}}c\Bigg],$$

simplifying the expression, identity is immediate.

LEMMA 2.2. *If we define the central moments as*

$$\begin{aligned}\mu_{n,m}(x) &= L_{n,\rho}((t-x)^m, x)\\ &= \sum_{k=1}^{\infty} p_{n,k}(x,c)\int_0^{\infty} \Theta^{\rho}_{n,k}(t,c)(t-x)^m dt + p_{n,0}(x,c)(-x)^m,\ m \in \mathbb{N}\end{aligned}$$

Then, $\mu_{n,0}(x) = 1$, $\mu_{n,1}(x) = \frac{cx}{n\rho - c}$ and for $n\rho > (m+1)c$ we have the following recurrence relation:

$$\begin{aligned}(n\rho - (m+1)c)\mu_{n,m+1}(x) = \rho x(1+cx)\mu'_{n,m}(x) &+ [cx + m(1+2cx)]\mu_{n,m}(x)\\ &+ mx(1+cx)(1+\rho)\mu_{n,m-1}(x).\end{aligned}$$

PROOF. Taking the derivative of $\mu_{n,m}$, we have

$$\begin{aligned}\mu'_{n,m}(x) =& - m\sum_{k=1}^{\infty} p_{n,k}(x,c)\int_0^{\infty} \Theta^{\rho}_{n,k}(t,c)(t-x)^{m-1}dt - mp_{n,0}(x,c)(-x)^{m-1}\\ &+ \sum_{k=1}^{\infty} p'_{n,k}(x,c)\int_0^{\infty} \Theta^{\rho}_{n,k}(t,c)(t-x)^m dt + p'_{n,0}(x,c)f(-x)^m\\ =& - m\mu_{n,m-1}(x) + \sum_{k=1}^{\infty} p'_{n,k}(x,c)\int_0^{\infty} \Theta^{\rho}_{n,k}(t,c)(t-x)^m dt\\ &+ p'_{n,0}(x,c)(-x)^m.\end{aligned}$$

$$\begin{aligned}&x(1+cx)[\mu'_{n,m}(x) + m\mu_{n,m-1}(x)]\\ &\quad= \sum_{k=1}^{\infty} x(1+cx)p'_{n,k}(x,c)\int_0^{\infty} \Theta^{\rho}_{n,k}(t,c)(t-x)^m dt + x(1+cx)p'_{n,0}(x,c)(-x)^m\end{aligned}$$

using $x(1+cx)p'_{n,k}(x,c)=(k-nx)p_{n,k}(x,c)$, we get

$$
\begin{aligned}
(2.1)\quad & x(1+cx)[\mu'_{n,m}(x)+m\mu_{n,m-1}(x)]\\
&=\sum_{k=1}^{\infty}(k-nx)p_{n,k}(x,c)\int_0^{\infty}\Theta^{\rho}_{n,k}(t,c)(t-x)^m dt-nxp_{n,0}(x,c)(x)(-x)^m\\
&=\sum_{k=1}^{\infty}kp_{n,k}(x,c)\int_0^{\infty}\Theta^{\rho}_{n,k}(t,c)(t-x)^m dt-nx\mu_{n,m}(x)\\
&=I-nx\mu_{n,m}(x).
\end{aligned}
$$

We can write I as

$$
\begin{aligned}
I&=\Big[\sum_{k=1}^{\infty}p_{n,k}(x,c)\int_0^{\infty}(k-nt)\Theta^{\rho}_{n,k}(t,c)(t-x)^m dt\\
&\quad+\Big[n\sum_{k=1}^{\infty}p_{n,k}(x,c)\int_0^{\infty}\Theta^{\rho}_{n,k}(t,c)\,t(t-x)^m dt\Big]\\
&=I_1+I_2,\quad \text{(say)}.
\end{aligned}\tag{2.2}
$$

To estimate I_2 using $t=[(t-x)+x]$, we have

$$
\begin{aligned}
I_2&=n\Big[\sum_{k=1}^{\infty}p_{n,k}(x,c)\int_0^{\infty}\Theta^{\rho}_{n,k}(t,c)(t-x)^{m+1}dt\\
&\quad+x\sum_{k=1}^{\infty}p_{n,k}(x)\int_0^{\infty}\Theta^{\rho}_{n,k}(t,c)(t-x)^m dt\Big]\\
&=n\Big[\sum_{k=1}^{\infty}p_{n,k}(x,c)\int_0^{\infty}\Theta^{\rho}_{n,k}(t,c)(t-x)^{m+1}dt+p_{n,0}(x,c)(-x)^{m+1}\\
&\quad+x\sum_{k=1}^{\infty}p_{n,k}(x,c)\int_0^{\infty}\Theta^{\rho}_{n,k}(t,c)(t-x)^m dt+xp_{n,0}(x,c)(-x)^m\Big]\\
&=n\mu_{n,m+1}(x)+nx\mu_{n,m}(x).
\end{aligned}\tag{2.3}
$$

Next, to estimate I_1, using the equality, $\rho(k-nt)\Theta^{\rho}_{n,k}(t,c)=(1+ct)[t\Theta^{\rho}_{n,k}(t,c)]'$, we have

$$
\begin{aligned}
I_1&=\frac{1}{\rho}\sum_{k=1}^{\infty}p_{n,k}(x,c)\int_0^{\infty}(1+ct)[t\Theta^{\rho}_{n,k}(t,c)]'(t-x)^m dt\\
&=\frac{(1+cx)}{\rho}\sum_{k=1}^{\infty}p_{n,k}(x,c)\int_0^{\infty}[t\Theta^{\rho}_{n,k}(t,c)]'(t-x)^m dt\\
&\quad+\frac{c}{\rho}\sum_{k=1}^{\infty}p_{n,k}(x,c)\int_0^{\infty}[t\Theta^{\rho}_{n,k}(t,c)]'(t-x)^{m+1}dt
\end{aligned}
$$

Now integrating by parts for $n\rho > (m+1)c$, we get

$$\begin{aligned}
I_1 &= \frac{(1+cx)}{\rho}\sum_{k=1}^{\infty} p_{n,k}(x)\Big[t\Theta_{n,k}^{\rho}(t,c)(t-x)^m\Big]_0^{\infty} \\
&\quad - \frac{m(1+cx)}{\rho}\sum_{k=1}^{\infty} p_{n,k}(x,c)\int_0^{\infty}[t\Theta_{n,k}^{\rho}(t,c)](t-x)^{m-1}dt \\
&\quad + \frac{c}{\rho}\sum_{k=1}^{\infty} p_{n,k}(x,c)\Big[t\Theta_{n,k}^{\rho}(t,c)(t-x)^{m+1}\Big]_0^{\infty} \\
&\quad - \frac{c(m+1)}{\rho}\sum_{k=1}^{\infty} p_{n,k}(x,c)\int_0^{\infty} t\Theta_{n,k}^{\rho}(t,c)(t-x)^{m}dt \\
&= -\frac{m(1+cx)}{\rho}\sum_{k=1}^{\infty} p_{n,k}(x,c)\int_0^{\infty}[t\Theta_{n,k}^{\rho}(t,c)](t-x)^{m-1}dt \\
&\quad - \frac{c(m+1)}{\rho}\sum_{k=1}^{\infty} p_{n,k}(x,c)\int_0^{\infty} t\Theta_{n,k}^{\rho}(t,c)(t-x)^{m}dt \\
&=: J_1 + J_2.
\end{aligned}$$

Proceeding in a similar manner and taking $t = [(t-x)+x]$, we get J_1 and J_2 as

$$J_1 = -\frac{m(1+cx)}{\rho}\mu_{n,m}(x) - \frac{mx(1+cx)}{\rho}\mu_{n,m-1}(x), \tag{2.4}$$

$$J_2 = -\frac{c(m+1)}{\rho}\mu_{n,m+1}(x) - \frac{c(m+1)x}{\rho}\mu_{n,m}(x). \tag{2.5}$$

Combining (2.1)–(2.5), we obtain

$$\begin{aligned}
x(1+cx)[\mu'_{n,m}(x) + m\mu_{n,m-1}(x)] &= -\frac{c(m+1)}{\rho}\mu_{n,m+1}(x) - \frac{c(m+1)x}{\rho}\mu_{n,m}(x) \\
&\quad - \frac{m(1+cx)}{\rho}\mu_{n,m}(x) - \frac{mx(1+cx)}{\rho}\mu_{n,m-1}(x) \\
&\quad + n\mu_{n,m+1}(x) + nx\mu_{n,m}(x) - nx\mu_{n,m}(x).
\end{aligned}$$

Hence,

$$\begin{aligned}
(n\rho - (m+1)c)\mu_{n,m+1}(x) &= \rho x(1+cx)\mu'_{n,m}(x) + [cx + m(1+2cx)]\mu_{n,m}(x) \\
&\quad + mx(1+cx)(1+\rho)\mu_{n,m-1}(x). \quad \square
\end{aligned}$$

Remark 2.2. From Lemma 2.2, we get that $L_{n,\rho}(t^m,x)$ is a polynomial in x of degree exactly m, for all $m \in \mathbb{N}^0$ and we can write as

$$\begin{aligned}
L_{n,\rho}(t^m,x) &= \frac{n(n+c)\dots[n+(m-1)c]}{(n\rho-c)\dots(n\rho-mc)}(\rho x)^m \\
&\quad + \frac{m(m-1)(1+\rho)}{2}\frac{n(n+c)\dots[n+(m-2)c]}{(n\rho-c)\dots(n\rho-mc)}(\rho x)^{m-1} + O(n^{-2}).
\end{aligned}$$

Lemma 2.3. *There exist the polynomials $q_{i,j,r}(x,c)$ independent of n and k such that*

$$[x(1+cx)]^r \frac{d^r}{dx^r}[p_{n,k}(x,c)] = \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i (k-nx)^j q_{i,j,r}(x,c) p_{n,k}(x,c).$$

The proof of the Lemma follows along the line of proof in [**11**].

3. Main Results

In this section, we state and prove the main results in simultaneous approximation.

Theorem 3.1. *If $r \in \mathbb{N}$, $f \in C_\gamma[0,\infty)$ for some $\gamma > 0$ and $f^{(r)}$ exists at a point $x \in (0,\infty)$, then*

$$\lim_{n\to\infty} L_{n,\rho}^{(r)}(f,x) = f^{(r)}(x). \tag{3.1}$$

Further, if $f^{(r)}$ exists and continuous on $(a-\eta, b+\eta) \subset (0,\infty)$, $\eta > 0$, then (3.1) *holds uniformly in $x \in [a,b]$.*

Proof. By Taylor's expansion of f, we have

$$f(t) = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^r,$$

where $\varepsilon(t,x) \to 0$ as $t \to x$.

$$L_{n,\rho}^{(r)}(f,x) = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} L_{n,\rho}^{(r)}((t-x)^i, x) + L_{n,\rho}^{(r)}(\varepsilon(t,x)(t-x)^r, x) =: I_1 + I_2.$$

In view of Remark 2.2, we have

$$\begin{aligned} I_1 &= \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j} L_{n,\rho}^{(r)}(t^j, x) \\ &= \frac{f^{(r)}(x)}{r!} \left(\frac{n(n+c)\dots(n+(r-1)c)}{(n\rho - c)\dots(n\rho - rc)} r!\rho^r \right) \\ &= f^{(r)}(x) \left(\frac{n(n+c)\dots(n+(r-1)c)}{(n\rho - c)\dots(n\rho - rc)} \rho^r \right) \to f^{(r)}(x) \text{ as } n \to \infty. \end{aligned}$$

Next, we estimate I_2 by using Lemma 2.3, we have

$$\begin{aligned} I_2 &= L_{n,\rho}^{(r)}(\varepsilon(t,x)(t-x)^r, x) \\ &= \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i \frac{q_{i,j,r}(x,c)}{x^r(1+cx)^r} \sum_{k=1}^{\infty} p_{n,k}(x,c)(k-nx)^j \int_0^\infty \Theta_{n,k}^{\rho}(t,c)\varepsilon(t,x)(t-x)^r dt \\ &\quad + (-1)^r \frac{\Gamma(\frac{n}{c}+r)}{\Gamma(\frac{n}{c})}(1+cx)^{-\frac{n}{c}-r}\varepsilon(0,x)(-x)^r \end{aligned}$$

$$|I_2| \leqslant \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i \frac{|q_{i,j,r}(x,c)|}{x^r(1+cx)^r} \sum_{k=1}^{\infty} p_{n,k}(x,c)|k-nx|^j \int_0^{\infty} \Theta_{n,k}^{\rho}(t,c)|\varepsilon(t,x)||t-x|^r dt$$
$$+ \frac{\Gamma(\frac{n}{c}+r)}{\Gamma(\frac{n}{c})}(1+cx)^{-\frac{n}{c}-r}|\varepsilon(0,x)||x|^r =: I_3 + I_4.$$

Since $\varepsilon(t,x) \to 0$ as $t \to x$, for a given $\varepsilon > 0$ there exists $\delta > 0$ such that $|\varepsilon(t,x)| < \epsilon$ whenever $|t-x| < \delta$, further if λ is any integer $> \max\{\gamma, r\}$ then we find a constant $K > 0$ such that $|\varepsilon(t,x)||t-x|^r \leqslant K|t-x|^{\lambda}$. Thus

$$I_3 = C_1 \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i \sum_{k=1}^{\infty} p_{n,k}(x,c)|k-nx|^j \bigg\{ \int_{|t-x|<\delta} \varepsilon \Theta_{n,k}^{\rho}(t,c)|t-x|^r dt$$
$$+ \int_{|t-x|\geqslant\delta} K\Theta_{n,k}^{\rho}(t,c)|t-x|^{\gamma} dt \bigg\}$$
$$=: I_5 + I_6.$$

Applying the Schwarz inequality for the integration and summation we have

$$I_5 \leqslant \varepsilon C_1 \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i \sum_{k=1}^{\infty} p_{n,k}(x,c)|k-nx|^j \left(\int_0^{\infty} \Theta_{n,k}^{\rho}(t,c)\, dt \right)^{1/2}$$
$$\times \left(\int_0^{\infty} \Theta_{n,k}^{\rho}(t,c)(t-x)^{2r} dt \right)^{1/2}$$
$$\leqslant \varepsilon C_1 \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i \left(\sum_{k=1}^{\infty} p_{n,k}(x,c)(k-nx)^{2j} \right)^{1/2}$$
$$\times \left(\sum_{k=1}^{\infty} p_{n,k}(x,c) \int_0^{\infty} \Theta_{n,k}^{\rho}(t,c)(t-x)^{2r} dt \right)^{1/2},$$

as $\int_0^{\infty} \Theta_{n,k}^{\rho}(t,c)\, dt = 1$. Making use of Lemma 2.2, we get

$$\sum_{k=1}^{\infty} p_{n,k}(x,c)(k-nx)^{2j} \tag{3.2}$$
$$= n^{2j} \left[\sum_{k=0}^{\infty} p_{n,k}(x,c)\Big(\frac{k}{n}-x\Big)^{2j} - (1+cx)^{-\frac{n}{c}}(-x)^{2j} \right]$$
$$= n^{2j}[O(n^{-j}) + O(n^{-s})] = O(n^{-j}) \quad \text{for any } s > 0.$$

Also, by using Lemma 2.2 and arguing as above, we have

$$\sum_{k=1}^{\infty} p_{n,k}(x,c) \int_0^{\infty} \Theta_{n,k}^{\rho}(t,c)(t-x)^{2r} dt = O(n^{-r}). \tag{3.3}$$

Thus

$$I_5 \leqslant \varepsilon C_1 \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i \cdot O(n^{j/2}) \cdot O(n^{-r/2}) = \varepsilon O(1).$$

Next, using the Schwarz inequality for the integration and summation, in view of (3.2) and (3.3), we have

$$\begin{aligned}
I_6 &\leqslant C_2 \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i \sum_{k=1}^{\infty} p_{n,k}(x,c)|k-nx|^j \int_{|t-x|\geqslant\delta} \Theta^{\rho}_{n,k}(t,c)|t-x|^{\gamma}\,dt\\
&\leqslant C_2 \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i \sum_{k=1}^{\infty} p_{n,k}(x,c)|k-nx|^j \Big(\int_{|t-x|\geqslant\delta} \Theta^{\rho}_{n,k}(t,c)\,dt\Big)^{1/2}\\
&\qquad\qquad \times \Big(\int_{|t-x|\geqslant\delta} \Theta^{\rho}_{n,k}(t,c)(t-x)^{2\gamma}dt\Big)^{1/2}\\
&\leqslant C_2 \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i \Big(\sum_{k=1}^{\infty} p_{n,k}(x,c)(k-nx)^{2j}\Big)^{1/2}\\
&\qquad\qquad \times \Big(\sum_{k=1}^{\infty} p_{n,k}(x,c)\int_0^{\infty} \Theta^{\rho}_{n,k}(t,c)(t-x)^{2\gamma}dt\Big)^{1/2}\\
&= \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i\cdot O(n^{j/2})\cdot O(n^{-m/2}) = O(n^{(r-m)/2}) = o(1),
\end{aligned}$$

where m is an integer $\geqq \lambda$. Thus due to the arbitrariness of ε, it follows that $I_3 = o(1)$. Also, $I_4 \to 0$ as $n\to\infty$ and hence $I_2 = o(1)$. Combining the estimates I_1 and I_2 we obtain the desired result (3.1). ☐

Theorem 3.2. *Let $f \in C_\gamma[0,\infty)$ for some $\gamma > 0$, admitting the derivative of order $(r+2)$ at a fixed $x \in (0,\infty)$. Let $f(t) = O(t^\gamma)$ as $t\to\infty$ for some $\gamma > 0$, then we have*

$$\begin{aligned}
(3.4)\qquad \lim_{n\to\infty} n\big[L^{(r)}_{n,\rho}(f,x) - f^{(r)}(x)\big] &= \Big(\frac{cr(r-1)}{2} + \frac{cr(r+1)}{2\rho}\Big) f^{(r)}(x)\\
&+ \Big(\frac{r(1+\rho)(1+2cx)}{2\rho} + \frac{cx}{\rho}\Big) f^{(r+1)}(x) + \frac{x(1+cx)(1+\rho)}{2\rho} f^{(r+2)}(x).
\end{aligned}$$

Proof. By Taylor's expansion of f, we have

$$f(t) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^{r+2},$$

where $\varepsilon(t,x)\to 0$ as $t\to x$ and $\varepsilon(t,x) = o((t-x)^\delta)$ as $t\to\infty$ for some $\delta > 0$,

Applying the r-th derivative of $L_{n,\rho}$, we can write

$$\begin{aligned}
n\big[L^{(r)}_{n,\rho}(f,x) - f^{(r)}(x)\big] &= n\Big[\sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} L^{(r)}_{n,\rho}((t-x)^i,x) - f^{(r)}(x)\Big]\\
&\quad + n\big[L^{(r)}_{n,\rho}(\varepsilon(t,x)(t-x)^{r+2},x)\big] =: I_1 + I_2.
\end{aligned}$$

By Lemma 2.2 and Remark 2.2, we have

$$
\begin{aligned}
I_1 &= n\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}\sum_{j=r}^{i}\binom{i}{j}(-x)^{i-j}L_{n,\rho}^{(r)}(t^j,x)-nf^{(r)}(x)\\
&= \frac{f^{(r)}(x)}{r!}n\big[L_{n,\rho}^{(r)}(t^r,x)-(r!)\big]\\
&\quad+\frac{f^{(r+1)}(x)}{(r+1)!}n\big\{(r+1)(-x)L_{n,\rho}^{(r)}(t^r,x)+L_{n,\rho}^{(r)}(t^{r+1},x)\big\}\\
&\quad+\frac{f^{(r+2)}(x)}{(r+2)!}n\Big\{\frac{(r+2)(r+1)}{2}x^2L_{n,\rho}^{(r)}(t^r,x)\\
&\qquad\qquad+(r+2)(-x)L_{n,\rho}^{(r)}(t^{r+1},x)+L_{n,\rho}^{(r)}(t^{r+2},x)\Big\}\\
&= n\Big[\frac{n(n+c)\dots[n+(r-1)c]}{(n\rho-c)\dots(n\rho-rc)}\rho^r-1\Big]f^{(r)}(x)\\
&\quad+n\frac{f^{(r+1)}(x)}{(r+1)!}\Big\{(r+1)(-x)\frac{n(n+c)\dots[n+(r-1)c]}{(n\rho-c)\dots(n\rho-rc)}\rho^r r!\\
&\qquad\qquad+\frac{n(n+c)\dots[n+rc]}{(n\rho-c)\dots(n\rho-(r+1)c)}\rho^{r+1}(r+1)!x\\
&\qquad\qquad+\frac{r(r+1)(1+\rho)}{2}\frac{n(n+c)\dots[n+(r-1)c]}{(n\rho-c)\dots(n\rho-(r+1)c)}\rho^r r!\Big\}\\
&\quad+n\frac{f^{(r+2)}(x)}{(r+2)!}\Big\{\frac{(r+2)(r+1)}{2}x^2\frac{n(n+c)\dots[n+(r-1)c]}{(n\rho-c)\dots(n\rho-rc)}\rho^r r!\\
&\qquad\qquad-(r+2)x\Big(\frac{n(n+c)\dots[n+rc]}{(n\rho-c)\dots(n\rho-(r+1)c)}\rho^{r+1}(r+1)!x\\
&\qquad\qquad\qquad+\frac{r(r+1)(1+\rho)}{2}\frac{n(n+c)\dots[n+(r-1)c]}{(n\rho-c)\dots(n\rho-(r+1)c)}\rho^r r!\Big)\\
&\qquad\qquad+\frac{n(n+c)\dots[n+(r+1)c]}{(n\rho-c)\dots(n\rho-(r+2)c)}\rho^{r+2}\frac{(r+2)!}{2}x^2\\
&\qquad+\frac{(r+1)(r+2)(1+\rho)}{2}\frac{n(n+c)\dots(n+rc)}{(n\rho-c)\dots(n\rho-(r+2)c)}\rho^{r+1}(r+1)!x\Big\}\\
&\qquad\qquad\qquad\qquad+O(n^{-2}).
\end{aligned}
$$

Now the coefficients of $f^{(r)}(x)$, $f^{(r+1)}(x)$ and $f^{(r+2)}(x)$ in the above expression are respectively $\frac{cr(r-1)}{2}+\frac{cr(r+1)}{2\rho}$, $\frac{r(1+\rho)(1+2cx)}{2\rho}$ and $\frac{x(1+cx)(1+\rho)}{2\rho}$ respectively, which follows by using induction hypothesis on r and taking the limits as $n\to\infty$. Hence in order to prove (3.4), it suffices to show that $I_2\to 0$ as $n\to\infty$, which follows on proceeding along the lines in the estimation of I_2 as done in Theorem 3.1. □

REMARK 3.1. In the case $c=0$ the above theorem is given in [**16**]. Another special case of the above asymptotic formula in simultaneous approximation for $c=0$ and $\rho=1$ was discussed by Agrawal and Gupta in [**1**].

Theorem 3.3. *Let $f \in C_\gamma[0,\infty)$ for some $\gamma > 0$ and $r \leqslant m \leqslant r+2$. If $f^{(m)}$ exists and is continuous on $(a-\eta, b+\eta) \subset (0,\infty)$, $\eta > 0$, then for n sufficiently large*

$$\big\|L_{n,\rho}^{(r)}(f,.)-f^{(r)}\big\|_{C[a,b]} \leqslant C_1 n^{-1}\sum_{i=r}^{m}\|f^{(i)}\|_{C[a,b]}+C_2 n^{-1/2}\omega(f^{(m)},n^{-1/2})+O(n^{-2}),$$

where C_1, C_2 are constants independent of f and n, $\omega(f^{(m)},\delta)$ is the modulus of continuity of $f^{(m)}$ on $(a-\eta,b+\eta)$ and $\|.\|_{C[a,b]}$ denotes the sup-*norm on $[a,b]$.*

Proof. By Taylor's expansion of f, we have

$$f(t)=\sum_{i=0}^{m}\frac{f^{(i)}(x)}{i!}(t-x)^i+\frac{f^{(m)}(\xi)-f^{(m)}(x)}{m!}(t-x)^m\chi(t)+h(t,x)(1-\chi(t)),$$

where ξ lies between t and x and $\chi(t)$ is the characteristic function on the interval $(a-\eta,b+\eta)$.

Now,

$$\begin{aligned}L_{n,\rho}^{(r)}(f,x)-f^{(r)}(x) &= \bigg\{\sum_{i=0}^{m}\frac{f^{(i)}(x)}{i!}L_{n,\rho}^{(r)}((t-x)^i,x)-f^{(r)}(x)\bigg\}\\ &\quad+L_{n,\rho}^{(r)}\Big(\frac{f^{(m)}(\xi)-f^{(m)}(x)}{m!}(t-x)^m\chi(t),x\Big)\\ &\quad+L_{n,\rho}^{(r)}(h(t,x)(1-\chi(t)),x)=:E_1+E_2+E_3.\end{aligned}$$

By using Lemma 2.2 and Remark 2.2, we have

$$\begin{aligned}E_1 &= \sum_{i=0}^{m}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}\frac{d^r}{dx^r}\Big(\frac{n(n+c)\dots[n+(j-1)c]}{(n\rho-c)\dots(n\rho-jc)}(\rho x)^j\\ &\quad+\frac{j(j-1)(1+\rho)}{2}\frac{n(n+c)\dots[n+(j-2)c]}{(n\rho-c)\dots(n\rho-jc)}(\rho x)^{j-1}+O(n^{-2})\Big)-f^{(r)}(x).\end{aligned}$$

Consequently,

$$\|E_1\|_{C[a,b]}\leqslant C_1 n^{-1}\sum_{i=r}^{m}\|f^{(i)}\|_{C[a,b]}+O(n^{-2}),\ \text{uniformly on } [a,b].$$

Next, we estimate E_2 as follows

$$\begin{aligned}|E_2| &\leqslant \sum_{k=1}^{\infty}p_{n,k}^{(r)}(x,c)\int_0^\infty\Theta_{n,k}^{\rho}(t,c)\bigg\{\Big|\frac{f^{(m)}(\xi)-f^{(m)}(x)}{m!}\Big||t-x|^m\chi(t)\bigg\}dt\\ &\quad+p_{n,0}^{(r)}(x,c)\Big|\frac{f^{(m)}(\xi)-f^{(m)}(x)}{m!}\Big||x|^m\\ &\leqslant \frac{\omega(f^{(m)},\delta)}{m!}\bigg[\sum_{k=1}^{\infty}|p_{n,k}^{(r)}(x,c)|\int_0^\infty\Theta_{n,k}^{\rho}(t,c)\Big(1+\frac{|t-x|}{\delta}\Big)|t-x|^m dt\\ &\qquad+p_{n,0}^{(r)}(x,c)\Big(1+\frac{|x|}{\delta}\Big)|x|^m\bigg]\end{aligned}$$

$$\leqslant \frac{\omega(f^{(m)},\delta)}{m!}\Big[\sum_{k=1}^{\infty}|p_{n,k}^{(r)}(x)|\int_0^\infty \Theta_{n,k}^{\rho}(t,c)(|t-x|^m+\delta^{-1}|t-x|^{m+1})\,dt + p_{n,0}^{(r)}(x,c)(|x|^m+\delta^{-1}|x|^{m+1})\Big].$$

Using the Schwarz inequality for both integration and summation, we get

$$\begin{aligned}(3.5)\qquad &\sum_{k=1}^{\infty}p_{n,k}(x,c)|k-nx|^j\int_0^\infty \Theta_{n,k}^{\rho}(t,c)|t-x|^m dt\\ &\leqslant \sum_{k=1}^{\infty}p_{n,k}(x,c)|k-nx|^j\Big(\int_0^\infty \Theta_{n,k}^{\rho}(t,c)\,dt\Big)^{1/2}\times\Big(\int_0^\infty \Theta_{n,k}^{\rho}(t,c)(t-x)^{2m}dt\Big)^{1/2}\\ &\leqslant \Big(\sum_{k=1}^{\infty}p_{n,k}(x,c)(k-nx)^{2j}\Big)^{1/2}\times\Big(\sum_{k=1}^{\infty}p_{n,k}(x,c)\int_0^\infty \Theta_{n,k}^{\rho}(t,c)(t-x)^{2m}dt\Big)^{1/2}\\ &= O(n^{j/2})\cdot O(n^{-m/2})=O(n^{(j-m)/2}),\quad \text{uniformly on } [a,b].\end{aligned}$$

Therefore, by Lemma 2.3 and (3.5), we get

$$\begin{aligned}(3.6)\quad &\sum_{k=1}^{\infty}|p_{n,k}^{(r)}(x,c)|\int_0^\infty \Theta_{n,k}^{\rho}(t,c)|t-x|^m dt\\ &\leqslant \sum_{k=1}^{\infty}\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i|k-nx|^j\frac{q_{i,j,r}(x,c)}{x^r(1+cx)^r}p_{n,k}(x,c)\times\int_0^\infty \Theta_{n,k}^{\rho}(t,c)|t-x|^m dt\\ &\leqslant \Big(\sup_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}\frac{q_{i,j,r}(x,c)}{x^r(1+cx)^r}\Big)\times\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i\Big(\sum_{k=1}^{\infty}p_{n,k}(x,c)|k-nx|^j\int_0^\infty \Theta_{n,k}^{\rho}(t,c)|t-x|^m dt\Big)\\ &= C\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i O(n^{(j-m)/2})=O(n^{(r-m)/2}),\quad \text{uniformly on } [a,b].\end{aligned}$$

where $C=\sup_{2i+j\leqslant r;\, i,j\geqslant 0}\frac{|q_{i,j,r}(x)|}{x^r(1+x)^r}$, $\forall x\in[0,\infty)$.

Choosing $\delta=n^{-1/2}$ and applying (3.6), we obtain

$$\begin{aligned}|E_2| &\leqslant \frac{\omega(f^{(m)},n^{-1/2})}{m!}\big[O(n^{(r-m)/2})+n^{1/2}O(n^{(r-m-1)/2})+O(n^{-m})\\ &\leqslant C_2 n^{-(r-m)/2}\omega(f^{(m)},n^{-1/2}).\end{aligned}$$

Since $t \in [0,\infty) \setminus (a-\eta, b+\eta)$, we can choose δ such that $|t-x| \geqslant \delta$ for all $x \in [a,b]$. Thus by Lemma 2.3, we have

$$\begin{aligned}|E_3| \leqslant C \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i \sum_{k=1}^{\infty} p_{n,k}(x,c)|k-nx|^j \int_0^\infty \Theta_{n,k}^{\rho}(t,c)|h(t,x)| \\ + \frac{(n+r-1)!}{(n-1)!}(1+cx)^{-n-r}|h(0,x)|.\end{aligned}$$

For $|t-x| \geqslant \delta$, we can find the a constant M such that $|h(t,x)| \leqslant M|t-x|^{\beta}$, where β is an integer $\geqq \{\gamma, m\}$. Hence using the Schwarz inequality for both integration and summation, (3.2) and (3.3), it easily follows that $I_3 = O(n^{-s})$ for any $s > 0$, uniformly on $[a,b]$.

Combing the estimates of E_1, E_2, E_3, the required result is immediate. □

REMARK 3.2. Very recently, Verma et al. [**20**] proposed the Stancu-type generalization of Baskakov–Durrmeyer type operators. One can consider the similar generalization of the operators (1.1) further based on two more parameters, but that will just provide extension of work without much gain as far as the convergence is concerned.

References

1. P. N. Agrawal, V. Gupta, *On convergence of derivatives of Phillips operators*, Demonstr. Math. **27**(2) (1994), 501–510.
2. P. N. Agrawal, K. J. Thamer, *Approximation of unbounded functions by a new sequence of linear positive operators*, J. Math. Anal. Appl. **225** (1998), 660–672.
3. M. M. Derriennic, *Sur l'approximation de fonctions integrable sur* $[0,1]$ *par des polynômes de Bernstein modifies*, J. Approx. Theory **31** (1981), 325–343.
4. Z. Ditzian, K. Ivanov, *Bernstein-type operators and their derivatives*, J. Approx. Theory **56** (1989), 72–90.
5. Z. Finta, V. Gupta, *Direct and inverse estimates for Phillips type operators*, J. Math. Anal. Appl. **303**(2) (2005), 627–642.
6. H. H. Gonska, X. L. Zhou, *A global inverse theorem on simultaneous approximation by Bernstein-Durrmeyer operators*, J. Approx. Theory **67** (1991), 284–302.
7. V. Gupta, *A note on modified Baskakov type operators*, Approximation Theory Appl. **10**(3) (1994), 74–78.
8. V. Gupta, R. P. Agarwal, *Convergence Estimates in Approximation Theory*, Springer 2014.
9. V. Gupta, D. K. Verma, P. N. Agrawal, *Simultaneous approximation by certain Baskakov–Durrmeyer–Stancu operators*, J. Egypt. Math. Soc. **20**(3) (2012), 183–187.
10. N. Ispir, I. Yuksel, *On the Bezier variant of Srivastava–Gupta operators*, Appl. Math. E-Notes **5** (2005), 129–137.
11. H. S. Kasana, P. N. Agrawal, V. Gupta, *Inverse and saturation theorems for linear combination of modified Baskakov operators*, Approximation Theory Appl. **7**(2) (1991), 65–82.
12. G. G. Lorentz, *Bernstein Polynomials, Univ. Toronto Press*, Toronto, 1953.
13. G. V. Milovanović, M. Th. Rassias (eds.), *Analytic Number Theory, Approximation Theory and Special Functions: In Honor of Hari M. Srivastava*, Springer, New York, 2014, XI, 880p.
14. G. V. Milovanović, M. Th. Rassias, *Some properties of a hypergeometric function which appear in an approximation problem*, J. Glob. Optim. **57** (2013), 1173–1192.
15. R. Păltănea, *Modified Szász–Mirakjan operators of integral form*, Carpathian J. Math. **24**(3) (2008), 378–385.

16. ______, *Simultaneous approximation by a class of Szász–Mirakjan operators*, J. Appl. Funct. Anal. **9**(3–4) (2014), 356–368.
17. R. S. Phillips, *An inversion formula for Laplace transforms and semi-groups of linear operators*, Ann. Math. (2) **59**, (1954), 325–356.
18. H. M. Srivastava, V. Gupta, *A certain family of summation integral type operators*, Math. Comput. Modelling **37**(12–13) (2003), 1307–1315.
19. D. K. Verma, P. N. Agrawal, *Convergence in simultaneous approximation for Srivastava–Gupta operators*, Math. Sci., Springer **6** (2012), Article ID 22, 8p.
20. D. K. Verma, V. Gupta, P. N. Agrawal, *Some approximation properties of Baskakov–Durrmeyer–Stancu operators*, Appl. Math. Comput. **218**(11) (2012), 6549–6556.
21. S. Wigert, *Sur l'approximation par polynomes des fonctions continues*, Ark. Mat. Astron. Fys. B **22**(9) (1932), 1–4.
22. R. Yadav, *Approximation by modified Srivastava–Gupta operators*, Appl. Math. Comput. **226**, (2014), 61–66.

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