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# RELATIONS BETWEEN KERNELS AND IMAGES OF REDUCED POWERS FOR SOME RIGHT $\mathcal{A}_p$-MODULES

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Abstract. We investigate the right action of the mod $p$ Steenrod algebra $\mathcal{A}_p$ on the homology $H_*(L^{\wedge s}, \mathbb{Z}_p)$ where $L = B\mathbb{Z}_p$ is the lens space. Following ideas of Ault and Singer we investigate the relation between intersection of kernels of the reduced powers $P^{p^i}$ and Bockstein element $\beta$ and the intersection of images of $P^{p^{i+1}-1}$ and of $\beta$. Namely one can check that $\bigcap_{i=0}^{k} \operatorname{im} P^{p^{i+1}-1} \subset \bigcap_{i=0}^{k} \ker P^{p^i}$ and $\bigcap_{i=0}^{k} \operatorname{im} P^{p^{i+1}-1} \cap \operatorname{im}\beta \subset \bigcap_{i=0}^{k} \ker P^{p^i} \cap \ker\beta$. We generalize Ault's homotopy systems to $p > 2$ and examine when the reverse inclusions are true.

## Introduction

Two natural questions arise when the action of the Steenrod algebra $\mathcal{A}_p$ on the cohomology $H^*(X, \mathbb{Z}_p)$ of a space $X$ is considered. One is to describe the *annihilating ideal* $I_X \subset \mathcal{A}_p$ which is defined as $I_X = \{\phi \in \mathcal{A}_p \mid \phi(H^*(X, \mathbb{Z}_p)) = 0\}$. The problem appeared to be difficult even for $p = 2$ and $X = \mathbb{R}P^\infty$, see [**1**].

The second question is to describe a minimal generating set of the $\mathcal{A}_p$-module $H^*(X, \mathbb{Z}_p)$ and the set of elements of the form $\phi x$, where $\phi \in \mathcal{A}_p$, $\deg\phi > 0$, $x \in H^*(X, \mathbb{Z}_p)$ (or the space spanned by such elements, this is the so called hit problem).

This question is far from being solved even for $X = \prod_{i=1}^{k} \mathbb{R}P^\infty$, though cases $k \leqslant 4$ are completely described in the papers [**4, 5, 6, 7**]

In this note we address the problem which is dual to the second question. Consider the conjugate action of the Steenrod algebra $\mathcal{A}_p$ on the $\mathbb{Z}_p$-homology of a space $X$. We say that $x \in H_*(X, \mathbb{Z}_p)$ is $P$-*annihilated* iff $xP^m = 0$ for all $m > 0$ and $\beta P$-*annihilated* iff $x$ is $P$-annihilated and $x\beta = 0$. The problem is to describe the subspace of all $P$-anihilated or $\beta P$-annihilated elements. Following ideas of Ault and Singer [**2, 3**], we study subspaces of *partially* annihilated elements ($\Delta_M(k)$ and $\Delta_M^\beta(k)$, see below). It appears that there is a structure (homotopy system, see Definitions 2.1 and 2.2) which relates these spaces with the spaces of so called

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*spike images* ($I_M(k)$ and $I_M^\beta(k)$, see below). The main results are Theorem 2.1 and Corollary 2.1. Also we show that such homotopy systems exist for $\tilde\Lambda$ (see Section 3 and Theorem 4.1). Here $\tilde\Lambda$ is a bigraded space which is given by (1.2), see below.

Throughout the paper all the (co)homologies are considered with $\mathbb{Z}_p$ coefficients.

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## 1. Lens space and the $\mathcal{A}_p$-module $\tilde\Lambda$.

Denote by $L$ the classifying space $B\mathbb{Z}_p$. The cohomology ring structure is described by the isomorphism $H^*(L) \equiv \mathbb{Z}_p[x] \otimes \Lambda_{\mathbb{Z}_p}(y)$, where $\deg x = 2$ and $\deg y = 1$. The action of the Steenrod algebra $\mathcal{A}_p$ on $H^*(L)$ is given by the following relations:

$$P^k x^m = \binom{m}{k} x^{k(p-1)+m}, \qquad \beta x^m = 0,$$
$$P^k x^m y = \binom{m}{k} x^{k(p-1)+m} y, \quad \beta x^m y = x^{m+1}.$$

There is the conjugate right action of the algebra $\mathcal{A}_p$ on the reduced homology of $L$. Denote by $[a]$, $a \geqslant 1$, the generator of $\tilde H_{2a}(L)$ and by $[\hat a]$, $a \geqslant 0$, the generator of $\tilde H_{2a+1}(L)$. The action of $\mathcal{A}_p$ on these elements is described by the formulas:

$$(1.1)\qquad \begin{aligned} P^k &= \binom{a-(p-1)k}{k}[a-(p-1)k], \qquad [a]\beta = [\widehat{a-1}],\\ [\hat a]P^k &= \binom{a-(p-1)k}{k}[a-\widehat{(p-1)}k], \qquad [\hat a]\beta = 0. \end{aligned}$$

Here we write a homomorphism to the right from its argument.

Consider the bigraded algebra $\tilde\Lambda = \{\tilde\Lambda_{*,*}\}$ which is defined by

$$(1.2)\qquad \tilde\Lambda_{n,k} = \begin{cases} \mathbb{Z}_p, & \text{if } n = k = 0;\\ \tilde H_k(L^{\wedge n}, \mathbb{Z}_p), & \text{if } n \geqslant 1;\\ 0, & \text{othervise.} \end{cases}$$

This algebra is a very natural module on which the right action of $\mathcal{A}_p$ should be studied.

## 2. Homotopy system

For a right $\mathcal{A}_p$-module $M$ define the following subspaces:

$$\Delta_M(k) = \bigcap_{i=0}^{k} \ker P^{p^i}, \qquad I_M(k) = \bigcap_{i=0}^{k} \operatorname{im} P^{p^{i+1}-1},$$
$$\Delta_M^\beta(k) = \Delta_M(k) \cap \ker\beta, \quad I_M^\beta(k) = I_M(k) \cap \operatorname{im}\beta.$$

The subspaces $\Delta_M(k)$ and $\Delta_M^\beta(k)$ are subspaces of partially annihilated elements. The subspaces $I_M(k)$ and $I_M^\beta(k)$ are subspaces of simultaneous spike images. For $M = \tilde\Lambda$, denote these modules by $\Delta(k)$, $\Delta^\beta(k)$, $I(k)$, $I^\beta(k)$.

LEMMA 2.1. *For an $\mathcal{A}_p$-module $M$ and $k \geqslant 0$ one has $I_M(k) \subset \Delta_M(k)$ and $I^\beta_M(k) \subset \Delta^\beta_M(k)$.*

REMARK 2.1. *Generally $I_M(k) \not\subset \Delta^\beta_M(k)$.*

PROOF. The statements easily follow from the relations $P^{p^{m+1}-1}P^{p^m} = 0$ and $\beta\beta = 0$. The last one is a particular case of the general relation $P^{pm-1}P^m = 0$. To check it, note that the Adem relation applies to $P^{pm-1}P^m$:

$$P^{pm-1}P^m = \sum_{j=0}^{m-1} (-1)^{pm-1+j} \binom{(p-1)(m-j)-1}{pm-1-pj} P^{(p+1)m-1-j}P^j.$$

The inequality $(p-1)(m-j)-1 < pm-1-pj$ follows easily from the restriction $j \leqslant m-1$. Hence all the binomial coefficients in the last sum vanish. □

In some cases one can prove the reverse inclusions $\Delta_M(k) \subset I_M(k)$ and $\Delta^\beta_M(k) \subset I^\beta_M(k)$. Now we describe an algebraic structure which is responsible for these inclusions.

DEFINITION 2.1. For a right $\mathcal{A}_p$-module $M$ define a *$k$-th order homotopy system* to be a *null space* $N \subset M$ and a collection of $\mathbb{Z}_p$-homomorphisms $Q_m : M \to M$ for $0 \leqslant m \leqslant k$ such that

(i) $Q_m(N) \subset N$ for $0 \leqslant m \leqslant k$;
(ii) for $1 \leqslant m \leqslant k$ and $l < p^m$ the following diagram comutes

$$\begin{array}{ccc} N & \xrightarrow{P^l} & M \\ {\scriptstyle Q_m}\downarrow & & \downarrow{\scriptstyle Q_m} \\ N & \xrightarrow{P^l} & M \end{array}$$

(iii) $(x)(P^{p^m}Q_m - Q_mP^{p^m}) = x$ for any $x \in N$.

DEFINITION 2.2. For a right $\mathcal{A}_p$-module $M$ define a *$k$-th order $\beta$-homotopy system* to be a $k$-th order homotopy system $\{Q_m\}$ with null space $N \subset M$, which is equipped with a $\mathbb{Z}_p$-homomorphism $\alpha : M \to M$ such that

(iv) $\alpha(N) \subset N$ and for any $x \in N$ one has $x\alpha\beta + x\beta\alpha = x$.

For a nonnegative integer $s$, denote by $0 \leqslant d_j(s) < p$ the coefficients of the expansion $s = \sum d_j(s)p^j$ in the base $p$. Recall Lucas's theorem which states that for any nonnegative integers $x$ and $y$ one has

$$\binom{x}{y} \equiv \prod_j \binom{d_j(x)}{d_j(x)} \mod p.$$

LEMMA 2.2. *For $1 \leqslant a \leqslant p-1$ one has*

$$(P^{p^0})^a(P^{p^1})^a \ldots (P^{p^{n-1}})^a(P^{p^n})^a = cP^{a(p^0+p^1+\cdots+p^{n-1}+p^n)},$$

*for some nonzero $c \in \mathbb{Z}_p$.*

Proof. First of all we show that for any $a$ such that $1 \leqslant a \leqslant p-1$ one has $(P^1)^a = cP^a$ for some nonzero $c \in \mathbb{Z}_p$. For $a = 1$ the statement is trivial. Assume the statement is true for $a = b$, where $b < p-1$. Then $(P^1)^{b+1} = (P^1)^b P^1 = cP^b P^1$, where $c \neq 0$. Apply the Adem relation to $P^b P^1$:

$$P^b P^1 = (-1)^b \binom{p-2}{b} P^{b+1},$$

hence $(P^1)^{b+1} = c(-1)^b \binom{p-2}{b} P^{b+1}$. One easily checks that the binomial coefficient $\binom{p-2}{b}$ is nonzero in $\mathbb{Z}_p$.

Now we proceed by induction to prove that for any integers $a$ and $b$ such that $1 \leqslant a \leqslant p-1$ and $0 \leqslant b < a$ one has

$$P^{a(p^0+p^1+\cdots+p^m)+bp^{m+1}} P^{p^{m+1}} = dP^{a(p^0+p^1+\cdots+p^m)+(b+1)p^{m+1}}$$

for some nonzero $d \in \mathbb{Z}_p$.

Denote $a(p^0 + p^1 + \cdots + p^m) + bp^{m+1}$ by $N$. Apply the Adem relation to the product $P^N P^{p^{m+1}}$:

$$P^N P^{p^{m+1}} = \sum_{t=0}^{[N/p]} (-1)^{N+t} \binom{(p-1)(p^{m+1}-t)-1}{N-pt} P^{N+p^{m+1}-t} P^t.$$

The coefficient for $t = 0$, by Lucas's theorem, is equal to $(-1)^N \binom{p-2}{b} \binom{p-1}{a}^{m+1}$ and hence is not zero.

It is left to prove that for $t > 0$, the binomial coefficient vanishes. To apply Lucas's theorem, it is enough to find $j$ such that $d_j((p-1)(p^{m+1}-t)-1) < d_j(N-pt)$. Fix $0 < t < [N/p]$. For simplicity let $a_j = d_j(t)$. Choose the smallest $j$ such that $a_j \neq 0$.

*Case 1.* If $1 \leqslant a_j \leqslant a$ then $d_j(N-pt) = a$ while $d_j((p-1)(p^{m+1}-t)-1) = a_j - 1$. But $a_j - 1 < a$.

*Case 2.* If $a_{j+1} \geqslant a_j > a$ then $d_{j+1}(N-pt) = p + a - a_j$, while $d_{j+1}((p-1)(p^{m+1}-t)-1) = a_{j+1} - a_j$. Note that $a_{j+1} - a_j < p + a - a_j$.

*Case 3.* Assume for some $k$ one has $a_{k+1} > a_k \geqslant a$ and $a_k \leqslant a_{k-1} \leqslant \cdots \leqslant a_{j+2} \leqslant a_{j+1} < a_j$. Then $d_{k+1}(N-pt) = p-1+a-a_k$ and $d_{k+1}((p-1)(p^{m+1}-t)-1) = -1 + a_{k+1} - a_k$. Note that $p-1+a-a_k > -1 + a_{k+1} - a_k$.

*Case 4.* Assume for some $k$ one has $a_{k+1} < a \leqslant a_k \leqslant a_{k-1} \leqslant \cdots \leqslant a_{j+2} \leqslant a_{j+1} < a_j$. Then $d_{k+1}(N-pt) = p-1+a-a_k$ and $d_{k+1}((p-1)(p^{m+1}-t)-1) = p-1+a_{k+1}-a_k$. Note that $p-1+a-a_k > p-1+a_{k+1}-a_k$. □

Theorem 2.1. *Suppose that for a right $\mathcal{A}_p$-module $M$ there exists a $k$-th order homotopy system $\{Q_m\}$ with a null space $N$. Then $\Delta_M(k) \cap N = I_M(k) \cap N$. Moreover, if $x \in \Delta_M(k) \cap N$ then $y = xQ_m^{p-1} Q_{m-1}^{p-1} \ldots Q_0^{p-1}$ satisfies $yP^{p^{m+1}-1} = cx$ for some nonzero $c \in \mathbb{Z}_p$.*

Proof. Denote for simplicity $P^{p^m}$ by $P_m$. Using standard technique for calculations with commutators involving creation-annihilation operators we prove for $j = 0, \ldots, m-1$ that

$$xQ_m^{p-1} Q_{m-1}^{p-1} \ldots Q_j^{p-1} P_j^{p-1} = -xQ_m^{p-1} Q_{m-1}^{p-1} \ldots Q_{j+1}^{p-1}.$$

The relation

$$xQ_m^{p-1}P_m^{p-1} = -x$$

is proved in the same way. Denote $xQ_m^{p-1}Q_{m-1}^{p-1}\dots Q_{j+1}^{p-1}$ by $W$. Since all $Q_n$'s left the subspace $N$ invariant and $P_j$ commutes with $Q_m,\dots,Q_{j+1}$ on the subspace $N$ one has $WP_j = 0$. Note that

$$[Q_j^n, P_j^n] = \sum_{s+t=n-1} P_j^s[Q_j^n, P_j]P_j^t.$$

Apply this relation, for $n = p-1$, to the product $W(Q_j)^{p-1}(P_j)^{p-1}$:

$$WQ_j^{p-1}P_j^{p-1} = WP_j^{p-1}Q_j^{p-1} + \sum_{s+t=p-2} WP_j^s[Q_j^{p-1}, P_j]P_j^t.$$

On the right-hand side, all the summands vanish except $W[Q_j^{p-1}, P_j]P_j^{p-2}$. Now use the equality

$$[Q_j^{p-1}, P_j] = \sum_{s+t=p-2} Q_j^s[Q_j, P_j]Q_j^t$$

and rewrite the product $W[Q_j^{p-1}, P_j]P_j^{p-2}$ as

$$W[Q_j^{p-1}, P_j]P_j^{p-2} = \sum_{s+t=p-2} WQ_j^s[Q_j, P_j]Q_j^tP_j^{p-2}.$$

Since $WQ_j^s \in N$, one has $WQ_j^s[Q_j, P_j] = -WQ_j^s$. Finally,

$$W[Q_j^{p-1}, P_j]P_j^{p-2} = \sum_{s+t=p-2} -WQ_j^sQ_j^tP_j^{p-2} = -(p-1)WQ_j^{p-2}P_j^{p-2}.$$

Proceeding in the same way, we obtain the equality

$$WQ_j^{p-1}P_j^{p-1} = (-1)^{p-1}(p-1)!W = -W.$$

Here we use simple part of the Wilson theorem which states that for a prime $p$ one has $(p-1)! + 1 \equiv 0 \mod p$.

By Lemma 2.2, $P^{p^{m+1}-1} = d\,P_0^{p-1}\dots P_m^{p-1}$ for some nonzero $d \in \mathbb{Z}_p$. Then

$$\begin{aligned} yP^{p^{m+1}-1} &= d\,xQ_m^{p-1}\dots Q_1^{p-1}Q_0^{p-1}P_0^{p-1}P_1^{p-1}\dots P_m^{p-1} \\ &= -d\,xQ_m^{p-1}\dots Q_1^{p-1}P_1^{p-1}\dots P_m^{p-1} \\ &= \dots = (-1)^m d\,xQ_m^{p-1}P_m^{p-1} = (-1)^{m+1}d\,x. \end{aligned}$$ □

Corollary 2.1. *Suppose that for a right $\mathcal{A}_p$-module $M$ there exists a $k$-th order $\beta$-homotopy system $\{Q_m\}$ with a null space $N$ and a $\mathbb{Z}_p$-homomorphism $\alpha$. Then $\Delta_M^\beta(k) \cap N = I_M^\beta(k) \cap N$. Moreover, if $x \in \Delta_M^\beta(k) \cap N$, then $y = x\alpha$ belongs to $N$ and satisfies $y\beta = x$.*

Proof. From $x\beta\alpha + x\alpha\beta = x$ and $x \in \ker\beta$ it follows that $y\beta = x\alpha\beta = x$. From Definition 2.2 one has $y \in N$. □

## 3. Shift maps in the homology of the smash-product power of the lens space

In this section we consider examples of homotopy systems for certain $\mathcal{A}_p$-modules. The reduced homology of $L^{\wedge s}$ is a vector space with the basis which consists of the elements $[u_1,\dots,u_s]$ of degree $\sum \deg u_j$, where every $u_j$ is $a$, $a\geqslant 1$, or $\hat a$, $a\geqslant 0$, and $\deg a=2a$, $\deg\hat a=2a+1$. Now we describe homotopy systems which appear naturally for $\tilde H_*(L^{\wedge s})$.

Define the shift map $\Psi_i^l:\tilde\Lambda_{s,*}\to\tilde\Lambda_{s,*+l}$ by formulas $[u_1,\dots,u_i,\dots,u_s]\Psi_i^l=[u_1,\dots,u_i+l,\dots,u_s]$. Here $u_j+l$ denotes $a_j+l$ if $u_j=a_j$ and $\widehat{a_j+l}$ if $u_j=\hat a_j$. Also define $[u_1,\dots,a_i,\dots,u_s]\alpha_i=0$ and $[u_1,\dots,\hat a_i,\dots,u_s]\alpha_i=[u_1,\dots,a_i+1,\dots,u_s]$. Clearly $(\Psi_i^{p^m})^k=\Psi_i^{kp^m}$.

Hereinafter for $s=1$ we denote $\Psi^l=\Psi_1^l$ and $\alpha=\alpha_1$.

Lemma 3.1. *We have*
(a) $[a]\Psi^{kp^m}P^n=[a]P^n\Psi^{kp^m}$ *for* $a\geqslant(p-1)n$, $p^m>n$, $1\leqslant k\leqslant p-1$;
(b) $[\hat a]\Psi^{kp^m}P^n=[\hat a]P^n\Psi^{kp^m}$ *for* $a\geqslant(p-1)n$, $p^m>n$, $1\leqslant k\leqslant p-1$;
(c) $[a]\Psi^{kp^m}\beta=[a]\beta\Psi^{kp^m}\neq 0$;
(d) $[\hat a]\Psi^{kp^m}\beta=0=[\hat a]\beta\Psi^{kp^m}$.

Proof. By formulas (1.1), one has

$$[a]\Psi^{kp^m}P^n=\binom{a+kp^m-(p-1)n}{n}[a+kp^m-(p-1)n],$$
$$[a]P^n\Psi^{kp^m}=\binom{a-(p-1)n}{n}[a+kp^m-(p-1)n].$$

Two binomial coefficients coincide under the assumptions of Lemma. Other statements are proved in the same way. □

Lemma 3.2. *We have*
(a) $[a]P^{p^m}\Psi^{(p-1)p^m}-[a]\Psi^{(p-1)p^m}P^{p^m}=[a]$ *for* $m\geqslant 0$ *and* $a\geqslant(p-1)p^m$;
(b) $[\hat a]P^{p^m}\Psi^{(p-1)p^m}-[\hat a]\Psi^{(p-1)p^m}P^{p^m}=[\hat a]$ *for* $m\geqslant 0$ *and* $a\geqslant(p-1)p^m$;
(c) $[a]\alpha\beta+[a]\beta\alpha=[a]$ *for* $a\geqslant 1$;
(d) $[\hat a]\alpha\beta+[\hat a]\beta\alpha=[\hat a]$ *for* $a\geqslant 0$.

Proof. By straightforward computation, one can get

$$[a]P^{p^m}\Psi^{(p-1)p^m}=\binom{a-(p-1)p^m}{p^m}[a],\quad [a]\Psi^{(p-1)p^m}P^{p^m}=\binom{a}{p^m}[a].$$

Assume $a=\sum b_kp^k$, where $0\leqslant b_k<p$. If $b_m=p-1$ then $\binom{a}{p^m}=p-1$ and $\binom{a-(p-1)p^m}{p^m}=0$. If $b_m<p-1$ then $\binom{a}{p^m}=b_m$ and $\binom{a-(p-1)p^m}{p^m}=b_m+1$. The proof of (b) is similar, and the statements (b) and (c) are clear. □

Lemma 3.3. *Assume* $x=[u_1,\dots,u_s]$, *where each* $u_j$ *is* $a_j$ *or* $\hat a_j$. *If* $p^m>n$, $a_i\geqslant(p-1)n$, $1\leqslant k\leqslant p-1$, *then one has* $x\Psi_i^{kp^m}P^n=xP^n\Psi_i^{kp^m}$.

PROOF. Without loss of generality one can assume $i = 1$. The case $s = 1$ was considered in Lemma 3.1, so assume $s > 1$. Let $y = [u_2, \dots, u_s]$. Then $x = [u_1] \times y$, and by Lemma 3.1 one has

$$
\begin{aligned}
([u_1] \times y)P^n \Psi_1^{kp^m} &= \sum_l ([u_1]P^l \times yP^{n-l})\Psi_1^{kp^m} \\
= \sum_l (-1)^{k \deg y}(xP^l \Psi_1^{kp^m} \times yP^{n-l}) &= \sum_l (-1)^{k \deg y}([u_1]\Psi_1^{kp^m} P^l \times yP^{n-l}) \\
&= (-1)^{k \deg y}([u_1]\Psi_1^{kp^m} \times y)P^n = x\Psi_1^{kp^m} P^n.
\end{aligned}
$$

Here we have used the standard sign convention. □

LEMMA 3.4. *Assume* $x = [u_1, \dots, u_s]$, *where each* $u_j$ *is* $a_j$ *or* $\hat{a}_j$. *If* $a_i \geqslant (p-1)p^m$, *then* $xP^{p^m} \Psi_i^{(p-1)p^m} - x\Psi_i^{(p-1)p^m} P^{p^m} = x$.

PROOF. Again one can consider only $i = 1$. For $s = 1$ the statement coincides with Lemma 3.2, so assume $s > 1$. Let $y = [u_2, \dots, u_s]$; then $x = [u_1] \times y$. By Lemma 3.1, one has

$$
\begin{aligned}
(3.1) \qquad & ([u_1] \times y)P^{p^m} \Psi_1^{(p-1)p^m} \\
&= \sum_{l=0}^{p^m-1} ([u_1]P^l \times yP^{p^m-l})\Psi_1^{(p-1)p^m} + ([u_1]P^{p^m} \times y)\Psi_1^{(p-1)p^m} \\
&= \sum_{l=0}^{p^m-1} ([u_1]P^l \Psi_1^{(p-1)p^m} \times yP^{p^m-l}) + ([u_1]P^{p^m} \Psi_1^{(p-1)p^m} \times y);
\end{aligned}
$$

$$
\begin{aligned}
(3.2) \qquad & ([u_1] \times y)\Psi_1^{(p-1)p^m} P^{p^m} \\
&= \sum_{l=0}^{p^m-1} ([u_1]\Psi_1^{(p-1)p^m} P^l \times yP^{p^m-l}) + ([u_1]\Psi_1^{(p-1)p^m} P^{p^m} \times y).
\end{aligned}
$$

Take the difference of (3.1) and (3.2) and note that the sums on the right-hand sides cancel by Lemma 3.1. The difference of the last summands in (3.1) and (3.2) is equal to $x$ by Lemma 3.2. □

LEMMA 3.5. *Let* $x = [u_1, \dots, u_s]$. *Then* $x\beta\alpha_i + x\alpha_i\beta = x$.

PROOF. Again one can consider only $i = 1$. For $s = 1$ the statement coincides with Lemma 3.2(b,c), so assume $s > 1$. Let $y = [u_2, \dots, u_s]$. Then $x = [u_1] \times y$, and by Lemma 3.1 one has

$$
\begin{aligned}
(3.3) \qquad ([u_1] \times y)\beta\alpha_1 &= ((-1)^{\deg y}[u_1]\beta \times y + [u_1] \times y\beta)\alpha_1 \\
&= [u_1]\beta\alpha_1 \times y + (-1)^{\deg y\beta}[u_1]\alpha_1 \times y\beta.
\end{aligned}
$$

$$
\begin{aligned}
(3.4) \qquad ([u_1] \times y)\alpha_1\beta &= (-1)^{\deg y}([u_1]\alpha_1 \times y)\beta \\
&= [u_1]\alpha_1\beta \times y + (-1)^{\deg y}[u_1]\alpha_1 \times y\beta.
\end{aligned}
$$

Summing (3.3) and (3.4), we obtain

$$(x)\beta\alpha_1 + (x)\alpha_1\beta = [u_1]\beta\alpha_1 \times y + [u_1]\alpha_1\beta \times y = ([u_1]\beta\alpha_1 + [u_1]\alpha_1\beta) \times y$$

which by Lemma 3.2(b,c) is equal to $[u_1] \times y = x$. □

## 4. Examples

For some $i = 1, \dots, s$ denote by $N_s(i,k)$ the subspace in $\tilde{\Lambda}_{s,*} = \tilde{H}_*(L^{\wedge s})$ spanned by $[u_1, \dots, u_s]$, where $u_i = a_i$ or $u_i = \hat{a}_i$ and $a_i \geqslant (p-1)p^k$.

Lemma 4.1. *For $s \geqslant 1$ and $k \geqslant 0$ the collection of maps $\{Q_m = \Psi_i^{(p-1)p^m} \mid m \leqslant k\}$ forms a $k$-th homotopy system for $\tilde{\Lambda}_{s,*}$ with the null space $N_s(i,k)$.*

Proof. The subspace $N_s(i,k)$ is stable under the action of $Q_m$. The properties (ii) and (iii) from Definition 2.1 are checked in Lemmas 3.3 and 3.3. The property (iv) from Definition 2.2 is checked in Lemma 3.5. □

Theorem 4.1. *Fix $s \geqslant 1$ and $k \geqslant 0$. Assume $x \in N_s(i,k)$ for all $1 \leqslant i \leqslant s$. Then $x \in \Delta(k)$ iff $x \in I(k)$ and $x \in \Delta^\beta(k)$ iff $x \in I^\beta(k)$.*

Proof. The statement follows immediately from Lemma 4.1, Theorem 2.1, and Corollary 2.1. □

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