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INFINITE A-TENSOR PRODUCT ALGEBRAS

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In this paper we study similarities between the spectra of a finite tensor product of topological algebras, as presented in [6], and those of the "infinite topological **A**-tensor products".

Namely given an arbitrary family of locally convex **A**-algebras, i.e. locally convex algebras with "coefficients" from a locally convex algebra **A** (not necessarily equal to $\mathbb{C}$), the infinite topological **A**-tensor product algebra is defined by means of the direct system of the finite projective tensor product locally convex **A**-algebras (Definition 1.1). This type of tensor product is different from that already defined in [7], where we consider finite inductive tensor product **A**-algebras. Moreover this is an extension of the usual (complex) infinite tensor product of [4, 11, 14] (cf. also [5]), where the given algebras are unital and their "coefficients" are taken from $\mathbb{C}$. The relationship between these two tensor products is given by Proposition 1.1.

Furthermore we consider the relationship between the numerical spectrum (Gel'fand space) and the generalized **A**-spectrum of the infinite topological **A**-tensor product of a family of unital locally convex **A**-algebras and the corresponding spectra of the factor algebras. Thus we get analogous results to the finite case, extending those of [6, 10, 11, 15].

On the other hand, the investigation of the complete infinite topological **A**-tensor product algebras enables us to take results of [12, 13, 15] within the present more general context (cf. §5).

1. Let $(E_i)_{i\in K}$ be a family of locally convex **A**-algebras (where **A** is a given locally convex algebra; cf. [6]) and $\mathcal{F}(K)$ the set of finite subsets of K. For any $\alpha \in \mathcal{F}(K)$ consider $E_\alpha \equiv \bigotimes^{\pi}_{\mathbf{A}\, i\in\alpha} E_i$ as the finite projective tensor prod-

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uct (locally convex) **A**-algebra being (within an isomorphism of locally convex **A**-algebras) uniquely defined, independently of any "enumeration" of $\alpha \in \mathcal{F}(K)$ (cf. [3: Chap. I, p. 50 ff.], [6: Proposition 2.1] and also [11: p. 216]). Moreover, let $0 \neq \widetilde{x} \equiv (\widetilde{x}_i) \in \prod_{i\in K} E_i$ be an idempotent element of the algebra $\widetilde{E} \equiv \prod_{i\in K} E_i$, i.e. $(\widetilde{x})^2 = \widetilde{x}$. For each $\alpha, \beta \in \mathcal{F}(K)$, with $\alpha \subseteq \beta$, one defines the map

$$f_{\beta\alpha}\colon E_\alpha \to E_\beta\colon x_\alpha \equiv \bigotimes_{i\in\alpha}{}_{\mathbf{A}} x_i \mapsto y_\beta \equiv \bigotimes_{j\in\beta}{}_{\mathbf{A}} y_j \ , \tag{1.1}$$

with $y_j = x_i$ if $j = i \in \alpha$ and $y_j = \widetilde{x}_j$ if $j \in \beta \cap \mathcal{C}\alpha$, that is

$$y_\beta = \bigotimes_{i\in\beta}{}_{\mathbf{A}} y_i = \Big(\bigotimes_{j\in\alpha}{}_{\mathbf{A}} x_i\Big) \otimes_{\mathbf{A}} \Big(\bigotimes_{j\in\beta\cap\mathcal{C}\alpha}{}_{\mathbf{A}} \widetilde{x}_j\Big) \equiv x_\alpha \otimes_{\mathbf{A}} \widetilde{x}_{\beta\cap\mathcal{C}\alpha} \ . \tag{1.2}$$

The map (1.1) is a continuous **A**-algebra morphism such that the family $(E_\alpha, f_{\beta\alpha})$ defines a *direct system of locally convex* **A**-*algebras*, in such a way that we set the following.

Definition 1.1. Let $(E_i)_{i\in K}$ be a family of locally convex **A**-algebras with idempotent elements. We call (stabilized) infinite (projective) topological **A**-tensor product algebra the corresponding locally convex direct limit **A**-algebra of the system $(E_\alpha, f_{\beta\alpha})$ (cf. [8]); that is we have

$$\bigotimes_{i\in K}{}_{\mathbf{A}} E_i \equiv \varinjlim_{\alpha\in\mathcal{F}(K)} \Big(\bigotimes_{i\in\alpha}{}^{\pi}_{\mathbf{A}} E_i\Big) \ . \tag{1.3}$$

In case E_i, $i \in K$, are locally convex **A**-algebras with continuous multiplications, then $\bigotimes\limits_{i\in\alpha}{}^{\pi}_{\mathbf{A}} E_i$, $\alpha \in \mathcal{F}(K)$ is an algebra of the same type, such that (1.3) yields the following isomorphism of locally convex **A**-algebras

$$\widehat{\bigotimes_{i\in K}}{}_{\mathbf{A}} E_i = \widehat{\varinjlim_{\alpha}}\Big(\widehat{\bigotimes_{i\in\alpha}}{}^{\pi}_{\mathbf{A}} E_i\Big) \tag{1.4}$$

("$\widehat{\ }$" means the completion; cf. also [8: §1, (1.8)]). Moreover one has an analogous argument for locally m-convex **A**-algebras by considering on (1.3) the respective final locally m-convex topology (cf. [10: Chapter II , Definition 9.1] and also [8]).

In a similar manner we have already defined another type of infinite (inductive) topological **A**-tensor product algebra (cf. [7: (4.3)]). These two infinite tensor products are isomorphic for suitable algebras, for instance in case of Fréchet locally convex **A**-algebras $(E_i)_{i\in K}$.

One gets a realization of the preceding analogous to the finite case (cf. [6: §2]) within the usual (complex) infinite topological tensor product algebra (cf. [10: Chapter X, Definition 5.2]).

For $\mathbf{A} = \mathbb{C}$ we obtain via (1.3) the infinite topological ($\mathbb{C}$-) tensor product algebra which coincides with that of [9: Chapter X, Definition 5.1] for $\widetilde{x}_i = 1_i$, $i \in K$ (identities of E_i, $i \in K$); i.e. one gets

$$\bigotimes_{i\in K} E_i = \lim_{\substack{\rightarrow \\ \alpha\in\mathcal{F}(K)}} \Big(\bigotimes_{i\in\alpha}^{\pi} E_i\Big), \quad \alpha \in \mathcal{F}(K) , \tag{1.5}$$

where, for each $\alpha \in \mathcal{F}(K)$, $\bigotimes\limits_{i\in\alpha}^{\pi} E_i$ is the (usual) finite projective ($\mathbb{C}$-) tensor product algebra.

Now, let E_i, $i \in K$, be locally convex $\mathbf{A}$-algebras with idempotent elements and $\bigotimes\limits_{i\in\alpha}^{\pi} E_i$, $\alpha \in \mathcal{F}(K)$, the finite projective tensor product algebra. For each $a \in \mathbf{A}$, $\bigotimes\limits_{i\in\alpha} x_i \in \bigotimes\limits_{i\in\alpha}^{\pi} E_i$, $\alpha \in \mathcal{F}(K)$, the relation

$$a \cdot \Big(\bigotimes_{i\in\alpha} x_i\Big) = \bigotimes_{i\in\alpha} y_i , \tag{1.6}$$

with $y_j = a \cdot x_j$, $j \in \alpha$, and $y_j = x_i$, $i \neq j$, constitutes $\bigotimes\limits_{i\in\alpha}^{\pi} E_i$ a locally convex $\mathbf{A}$-algebra such that $(\bigotimes\limits_{i\in\alpha}^{\pi} E_i, f_{\beta\alpha})$ defines a direct system of locally convex $\mathbf{A}$-algebras. On the other hand, for every $\alpha \in \mathcal{F}(K)$, let I_α be the closed vector subspace of $\bigotimes\limits_{i\in\alpha}^{\pi} E_i$ generated by the set

$$\left\{ \bigotimes_{i\in\alpha} y_i - \bigotimes_{i\in\alpha} z_i : \begin{array}{l} y_i = x_i,\ i\neq k \\ z_i = x_i,\ i \neq k \end{array} \text{ and } \begin{array}{l} y_j = a_j \\ z_k = a x_k \end{array} (j\neq k):\ a\in\mathbf{A},\ \bigotimes_{i\in\alpha} x_i \in \bigotimes_{i\in\alpha}^{\pi} E_i \right\}.$$

In particular, this is a closed 2-sided $\mathbf{A}$-ideal of $\bigotimes\limits_{i\in\alpha}^{\pi} E_i$, such that $(I_\alpha)_{\alpha\in\mathcal{F}(K)}$ constitutes a direct system of closed 2-sided $\mathbf{A}$-ideals of $(\bigotimes\limits_{i\in\alpha}^{\pi} E_i)_\alpha$.

Proposition 1.1. *Let $(E_i)_{i\in K}$ be a family of locally convex $\mathbf{A}$-algebras with idempotent elements and $I := \varinjlim I_\alpha$ the direct limit of the above system $(I_\alpha)_\alpha$. Then, one gets*

$$\bigotimes_{i\in K}{}_{\mathbf{A}} E_i = \bigotimes_{i\in K} E_i / I \tag{1.7}$$

(cf. (1.3), (1.5)) within an isomorphism of locally convex $\mathbf{A}$-algebras. For Fréchet locally convex $\mathbf{A}$-algebras E_i, $i \in K$, we have the following isomorphism of (complete) locally convex $\mathbf{A}$-algebras

$$\widehat{\bigotimes_{i\in K}}{}_{\mathbf{A}} E_i = \widehat{\bigotimes_{i\in K}} E_i / I . \tag{1.8}$$

Before the proof we set the next.

Lemma 1.1. *Let $(E_\alpha, f_{\beta\alpha})$ be a direct system of locally convex* **A**-*algebras and $(I_\alpha, t_{\beta\alpha} \equiv f_{\beta\alpha}|_{I_\alpha})$ a direct system of closed 2-sided* **A**-*ideals of $(E_\alpha, f_{\beta\alpha})$, such that $t_{\beta\alpha}(I_\alpha) \subseteq I_\beta$, $\alpha \leq \beta$ in K. Then $(E_\alpha/I_\alpha)_{\alpha\in K}$ is a direct system of locally convex* **A**-*algebras, such that*

$$\varinjlim(E_\alpha/I_\alpha) = \varinjlim E_\alpha / \varinjlim I_\alpha$$

within an isomorphism of locally convex **A**-*algebras.*

Proof: For $\alpha \leq \beta$, I_α is the inverse image of $t^{-1}_{\beta\alpha}(I_\beta)$, such that the induced mapping $\widetilde{f}_{\beta\alpha} : E_\alpha/I_\alpha \to E_\beta/I_\beta$ is a monomorphism of **A**-algebras. Thus the projections $p_\alpha : E_\alpha \to E_\alpha/I_\alpha$ induce a continuous projection $p : \varinjlim E_\alpha \to \varinjlim(E_\alpha/I_\alpha)$ whose kernel is equal to $\varinjlim I_\alpha$. So we have a continuous isomorphism of locally convex **A**-algebras $u\colon \varinjlim E_\alpha/\varinjlim I_\alpha \to \varinjlim(E_\alpha/I_\alpha)$ which is also open for the relative final topologies. ∎

Proof of Proposition 1.1: For each $\alpha \in \mathcal{F}(K)$, $\bigotimes^{\pi}_{i\in\alpha} E_i/I_\alpha$ is a locally convex **A**-algebra, such that $\bigotimes^{\pi}_{i\in\alpha} E_i/I_\alpha = \bigotimes^{\pi}_{\mathbf{A}\, i\in\alpha} E_i$, within an isomorphism of locally convex **A**-algebras (cf. [6: Proposition 2.1]).

Hence, Lemma 1.1 and (1.3), (1.5) imply (1.7). Moreover, concerning (1.8) *I is a closed 2-sided ideal* since $I := \varinjlim I_\alpha = \varinjlim \ker(p_\alpha) \cong \ker p$ (see also proof of Lemma 1.1 and [1: Chap. 3, §7, no. 6, (27)]). ∎

2. In this Section we examine the (numerical) spectrum (Gel'fand space [10]) of the infinite **A**-tensor product algebra as above, taking an analogous decomposition of it as in the finite case.

Throughout the sequel we suppose that E_i, $i \in K$, are unital such the identities are considered as the idempotent elements $\widetilde{x}_i$, $i \in K$.

The next theorem extends results of this author [6: Theorem 3.1] as well as of A. Mallios [10: Chapter XII, Theorem 2.1 and Corollary 2.1].

Theorem 2.1. *Let $(E_i)_{i\in K}$ be a family of (unital) locally convex* **A**-*algebras and $\bigotimes_{\mathbf{A}\, i\in K} E_i$ the corresponding infinite topological* **A**-*tensor product algebra. Then one has*

$$\mathcal{M}\Big(\bigotimes_{\mathbf{A}\, i\in K} E_i\Big) = \underset{i\in K}{\times}{}_{\mathcal{M}(\mathbf{A})^+} \mathcal{M}(E_i) \tag{2.1}$$

within a homeomorphism. Moreover, let the algebras E_i, $i \in K$, have continuous multiplications and locally equicontinuous spectra, such that for all except of finite many of them the corresponding spectra are equicontinuous. Then, one has the next homeomorphism

$$\mathcal{M}\Big(\widehat{\bigotimes_{i\in K}}{}_{\mathbf{A}} \widehat{E}_i\Big) = \mathop{\times}_{i\in K}{}_{\mathcal{M}(\mathbf{A})^+} \mathcal{M}(E_i) \; . \tag{2.2}$$

Proof: By (1.1) and [10: Chapter V, Theorem 3.1] we have

$$\mathcal{M}\Big(\bigotimes_{i\in K}{}_{\mathbf{A}} E_i\Big) = \varprojlim_{\alpha\in\mathcal{F}(K)} \mathcal{M}\Big(\bigotimes_{i\in\alpha}{}^{\pi}_{\mathbf{A}} E_i\Big) \tag{2.3}$$

within a homeomorphism. Moreover, the next relation constitutes an extension of [6: Theorem 3.1]

$$\mathcal{M}\Big(\bigotimes_{i\in\alpha}{}^{\pi}_{\mathbf{A}} E_i\Big) = \mathop{\times}_{i\in\alpha}{}_{\mathcal{M}(\mathbf{A})^+} \mathcal{M}(E_i) \; , \tag{2.4}$$

where the equality presents a homeomorphism while the second member means the pullback of $\mathcal{M}(E_i)$, $i \in \alpha$, over $\mathcal{M}(\mathbf{A})^+$ $(:= \mathcal{M}(\mathbf{A}) \cup \{0\})$. Thus, $(\mathop{\times}_{i\in\alpha}{}_{\mathcal{M}(\mathbf{A})^+} \mathcal{M}(E_i))_\alpha$ defines an inverse system of topological spaces whose limit is the pullback of $\mathcal{M}(E_i)$ over $\mathcal{M}(\mathbf{A})^+$ for all $i \in K$, that is

$$\mathop{\times}_{i\in K}{}_{\mathcal{M}(\mathbf{A})^+} \mathcal{M}(E_i) = \varprojlim_{\alpha} \Big(\mathop{\times}_{i\in\alpha}{}_{\mathcal{M}(\mathbf{A})^+} \mathcal{M}(E_i)\Big) \tag{2.5}$$

within a homeomorphism. So, (2.1) is an immediate consequence of (2.3), (2.4) and (2.5). On the other hand, the algebra $\bigotimes_{i\in K}{}_{\mathbf{A}} E_i$ has a locally equicontinuous spectrum (cf. [6: Theorem 2.1] and also [10: Chapter XII, Theorem 2.2]) such that

$$\mathcal{M}\Big(\widehat{\bigotimes_{i\in K}}{}_{\mathbf{A}} \widehat{E}_i\Big) = \mathcal{M}\Big(\bigotimes_{i\in K}{}_{\mathbf{A}} E_i\Big) \tag{2.6}$$

within a homeomorphism (cf. (1.4), [10: Chapter V, Theorem 2.1 and Lemma 2.2]). So (2.2) follows from (2.1), (1.4). ∎

Remark 2.1. Let $(E_i)_{i\in K}$ be a family of locally convex $\mathbf{A}$-algebras and $\mu_i\colon \mathcal{M}(E_i) \to \mathcal{M}(\mathbf{A})^+$, $i \in K$, the continuous maps defined via

$$\mu_i(X_i)(a) := \frac{X_i(a x_i)}{X_i(x_i)} \; , \tag{2.7}$$

where $X_i \in \mathcal{M}(E_i)$, $a \in \mathbf{A}$ and $x_i \in E_i$ with $X_i(x_i) \neq 0$ (cf. [6: (3.5)]). Then for any $\alpha \in \mathcal{F}(K)$ the (finite cartesian product) continuous maps

$$\mu_\alpha \equiv \prod_{i\in\alpha} \mu_i : \ \prod_{i\in\alpha} \mathcal{M}(E_i) \underset{\text{homeo.}}{\cong} \mathcal{M}\Big(\bigotimes_{i\in\alpha}^{\pi} E_i\Big) \to (\mathcal{M}(\mathbf{A})^+)^\alpha \tag{2.8}$$

(cf. [10: Chapter XII, p. 216, (2.13)]) define an inverse system of continuous maps, such that the relative inverse limit (continuous map) is given by

$$\mu := \varprojlim_\alpha \mu_\alpha : \ \varprojlim_\alpha \Big(\prod_{i\in\alpha} \mathcal{M}(E_i)\Big) \underset{\text{homeo.}}{\cong} \prod_{i\in K} \mathcal{M}(E_i) \ \to \ \to \ \varprojlim_\alpha (\mathcal{M}(\mathbf{A})^+)^\alpha \cong (\mathcal{M}(\mathbf{A})^+)^K \tag{2.9}$$

(cf. [10: Chapter XII, Lemma 2.1]). If (Δ_α^+) is the inverse system of diagonals of $(\mathcal{M}(\mathbf{A})^+)^\alpha$, $\alpha \in \mathcal{F}(K)$ and Δ^+ the diagonal of $(\mathcal{M}(\mathbf{A})^+)^K$, then $\Delta^+ \underset{\text{homeo.}}{\cong} \varprojlim \Delta_\alpha^+$ (cf. [6]), such that one has

$$\mu^{-1}(\Delta^+) = \varprojlim_\alpha (\mu_\alpha^{-1}(\Delta_\alpha^+)) \ , \tag{2.10}$$

within a homeomorphism (cf. also [1: p. 80, Proposition 2]).

On the other hand, by considering the direct system of closed 2-sided ideals $(I_\alpha)_{\alpha\in\mathcal{F}(K)}$ as in Proposition 1.1, the family $(h(I_\alpha))_\alpha$ consisting of the hull of I_α (the set of those elements of $\mathcal{M}(E_\alpha)$ vanishing on I_α, cf. [10: Chapter IX, Definition 1.1]), constitutes an inverse system of topological spaces such that

$$h(\varinjlim I_\alpha) = \varprojlim h(I_\alpha) = \varprojlim (\mu_\alpha^{-1}(\Delta_\alpha^+)) = \mu^{-1}(\Delta^+) \tag{2.11}$$

within homeomorphisms (cf. (2.9) and also [6: Theorem 2.1], [7: Scholium 2.1]). Thus (2.1), (2.10), (2.12) yield

$$\mathcal{M}\Big(\bigotimes_{i\in K}{}_{\mathbf{A}} E_i\Big) = h(\varinjlim I_\alpha) = \mu^{-1}(\Delta^+) = \underset{i\in K}{\times}{}_{\mathcal{M}(\mathbf{A})^+} \mathcal{M}(E_i) \tag{2.12}$$

within homeomorphisms. Moreover, by considering the completions of the above algebras the next homeomorphisms follow from (2.2), (2.7), (2.12)

$$\mathcal{M}\Big(\widehat{\bigotimes_{i\in K}}{}_{\mathbf{A}} \widehat{E}_i\Big) = h(\varinjlim I_\alpha) = \mu^{-1}(\Delta^+) = \underset{i\in K}{\times}{}_{\mathcal{M}(\mathbf{A})^+} \mathcal{M}(\widehat{E}_i) \ . \tag{2.13}$$

The last relations (2.12), (2.13) generalize relative applications analogous for the finite case (cf. [6: Theorem 2.1]).

3. In the present section we are interested in the generalized **A**-spectrum of $\bigotimes_{i\in K}{}_{\mathbf{A}} E_i$ and its completion with respect to (w.r.t.) a locally convex **A**-algebra G, getting thus analogous decompositions to [6: Proposition 3.1] in the present infinite case.

So, consider a family $(E_i)_{i\in K}$, of (unital) locally convex **A**-algebras and $\mathcal{M}_{\mathbf{A}}(E_i, G)$ the generalized **A**-spectrum of E_i w.r.t. a unital locally convex **A**-algebra G with continuous multiplication; i.e. the set of (non-zero) continuous **A**-morphisms of E_i to G endowed with the simple convergence topology on E_i from $\mathcal{L}_s(E_i, G)$ (cf. [6: §3]). Then for any $\alpha \in \mathcal{F}(K)$ consider Q_α the (closed) subset of $\prod_{i\in\alpha} \mathcal{M}_{\mathbf{A}}(E_i, G)$ consisting of all elements $(f_i)_{i\in\alpha}$ such that

$$\bigodot_{i\in\alpha} f_{\sigma(i)}(x_{\sigma(i)}) = \bigodot_{i\in\alpha} f_{\tau(i)}(x_{\tau(i)}) \equiv \bigodot_{i\in\alpha} f_i(x_i)\ . \tag{3.1}$$

Here $(x_i) \in \prod_{i\in\alpha} E_i$, $\sigma, \tau \in \mathrm{Aut}(\alpha)$ (: automorphisms of α) and "$\bigodot$" denotes (the ring) multiplication in G. The last relation remains "the same" w.r.t. each "enumeration" of the finite set $\alpha \in \mathcal{F}(K)$ (cf. [2: Chapter I, p. 45, Corollary of the Proposition 2]) such that one gets (3.1) by an easy extension (to finite many factors) of [6: p. 53, (3.6)]. Moreover, for any $\alpha \subseteq \beta$ in $\mathcal{F}(K)$, if ${}^{\mathrm{t}}f_{\beta\alpha}$ is the transpose map of (1.1) in $\mathcal{M}_{\mathbf{A}}(\bigotimes^{\pi}_{\mathbf{A}, i\in\alpha} E_i, G)$ the family

$$(Q_\alpha, {}^{\mathrm{t}}f_{\beta\alpha|Q_\alpha}) \tag{3.2}$$

defines an inverse system of topological spaces (cf. also [8: (3.1), (3.2)]) such that

$$\varprojlim_{\alpha} Q_\alpha \subseteq \varprojlim_{\alpha}\Big(\prod_{i\in\alpha} \mathcal{M}_{\mathbf{A}}(E_i, G)\Big) \underset{\text{homeo.}}{\cong} \prod_{i\in K} \mathcal{M}_{\mathbf{A}}(E_i, G) \tag{3.3}$$

(cf. [6: Chapter XII, Lemma 2.1]).

Theorem 3.1. *Let $(E_i)_{i\in K}$ be a family of (unital) locally convex* **A***-algebras and G a unital locally convex* **A***-algebra with continuous multiplication. Then,*

$$\mathcal{M}_{\mathbf{A}}\Big(\bigotimes_{i\in K}{}_{\mathbf{A}} E_i, G\Big) = \varprojlim_{\alpha} Q_\alpha \underset{\rightarrow}{\subseteq} \prod_{i\in K} \mathcal{M}_{\mathbf{A}}(E_i, G) \tag{3.4}$$

within homeomorphisms (cf. (3.3)). In particular, for G commutative, one has the next homeomorphism

$$\mathcal{M}_{\mathbf{A}}\Big(\bigotimes_{i\in K}{}_{\mathbf{A}} E_i, G\Big) = \prod_{i\in K} \mathcal{M}_{\mathbf{A}}(E_i, G)\ . \tag{3.5}$$

Proof: By (1.1) and [8: (3.1)] one obtains

$$\mathcal{M}_{\mathbf{A}}\Big(\bigotimes_{i\in K}{}_{\mathbf{A}} E_i, G\Big) = \varprojlim_{\alpha} \mathcal{M}_{\mathbf{A}}\Big(\bigotimes^{\pi}_{\mathbf{A}, i\in\alpha} E_i, G\Big) \tag{3.6}$$

within a homeomorphism. Moreover, for any $\alpha \in \mathcal{F}(K)$, an extension of [6: Proposition 3.1] to finite many factors gives the next homeomorphism

$$\mathcal{M}_{\mathbf{A}}\Big(\bigotimes_{i\in\alpha}{}^{\pi}_{\mathbf{A}} E_i, G\Big) = Q_\alpha \tag{3.7}$$

(cf. also (3.1)), such that (3.3), (3.6), (3.7) imply (3.4). Furthermore for G commutative, (3.5) is an immediate consequence of (3.1), (3.3), (3.4). ∎

4. In this Section we extend several results of [10: Chapter VI, §§1, 2], [12,13] concerning the local equicontinuity of the generalized **A**-spectra of topological **A**-algebras through the corresponding Gel'fand map. These results are useful for the study of $\widehat{\bigotimes_{i\in K}{}_{\mathbf{A}} \widehat{E}_i}$ in connection with the functor $\mathcal{M}_{\mathbf{A}}(\cdot, G)$ (cf. §5 below).

Thus, if E, G are topological **A**-algebras the *generalized Gel'fand* **A**-*map* is defined as the **A**-morphism

$$\mathcal{G}\colon E \to \mathcal{C}_c(\mathcal{M}_{\mathbf{A}}(E,G),G)\colon \quad x \mapsto \mathcal{G}(x) := \widehat{x} \; (\colon f \mapsto \widehat{x}(f) := f(x)) \;, \tag{4.1}$$

where in the range of $\mathcal{G}$ we mean the **A**-algebra of continuous maps of $\mathcal{M}_{\mathbf{A}}(E,G)$ into G, endowed with the topology of compact convergence. Moreover, one gets

$$x \,\widehat{\otimes_{\mathbf{A}}}\, y = \widehat{x} \otimes_{\mathbf{A}} \widehat{y}\,|_{Q} \tag{4.2}$$

(cf. [6: Theorem 3.1]), where $\widehat{x}$, $\widehat{y}$, $x \,\widehat{\otimes_{\mathbf{A}}}\, y$ are the generalized Gel'fand **A**-transforms of $x \in E$, $y \in F$, $x \otimes_{\mathbf{A}} y \in E \widehat{\otimes}^{\tau}_{\mathbf{A}} F$ respectively. (We use here the hypotheses of [6: Theorem 3.1]). Furthermore, the map (4.1) is continuous if, and only if, every compact subset of $\mathcal{M}_{\mathbf{A}}(E,G)$ is equicontinuous (cf. [10: Chapter VI, Theorem 1.1 and Remark 5.1]). On the other hand, if E, G are locally convex **A**-algebras with G semi-Montel, then

(4.3) *$\mathcal{M}_{\mathbf{A}}(E,G)$ is locally equicontinuous if, and only if, $\mathcal{M}_{\mathbf{A}}(E,G)$ is locally compact and the map (4.1) is continuous.*

(cf. [10: Chapter VI, Corollary 1.3] for an analogous instance).

Now, given E, G two topological **A**-algebras, E is called **A**-*spectrally barrelled* if the equicontinuous and weakly bounded subsets of $\mathcal{M}_{\mathbf{A}}(E,G)$ coincide (cf. [13: Definition 2.1] for $\mathbf{A} = \mathbb{C}$). One has this situation by considering, for instance, Fréchet locally convex **A**-algebras. In this respect one obtains the continuity of (4.1) and moreover we have an extension of [13: Lemma 4.3] as follows.

Lemma 4.1. *Let E, F, G be unital topological* **A***-algebras with G having continuous multiplication. Moreover, let τ be a compatible* **A***-tensor product topology on $E \otimes_{\mathbf{A}} F$ satisfying the conditions*

i) *The canonical map of* $E \times F$ *into* $E \otimes^{\tau}_{\mathbf{A}} F$ *is (jointly) continuous;*

ii) *For any pair* $(f, g) \in \mathcal{M}_{\mathbf{A}}(E, G) \times \mathcal{M}_{\mathbf{A}}(F, G)$, $f \otimes_{\mathbf{A}} g \in \mathcal{L}_{\mathbf{A}}(E \otimes^{\tau}_{\mathbf{A}} F, G)_s$*;*

(cf. [6: (3.1), (3.2)]). Furthermore consider the following assertions.

1) *E, F are* **A***-spectrally barrelled;*

2) $E \otimes^{\tau}_{\mathbf{A}} F$ *is* **A***-spectrally barrelled.*

Then 1)⇒2). In particular, 1)⇔2) whenever G is commutative.

Proof: By [6: Proposition 3.1] there exists a homeomorphism into

$$(4.4) \qquad u\colon \mathcal{M}_{\mathbf{A}}(E \otimes^{\tau}_{\mathbf{A}} F, G) \hookrightarrow \mathcal{M}_{\mathbf{A}}(E, G) \times \mathcal{M}_{\mathbf{A}}(F, G)\ ,$$

such that $\operatorname{Im} u = Q$ (cf. also (3.1) for $a = \{1, 2\}$). Thus if S is a bounded subset of $\mathcal{M}_{\mathbf{A}}(E \otimes^{\tau}_{\mathbf{A}} F, G)$, by (4.4) $u(S)$ is a bounded subset of $Q \subseteq \mathcal{L}_s(E, G) \times \mathcal{L}_s(F, G)$ (cf. [10]) such that $\mathrm{pr}_1(u(S))$, $\mathrm{pr}_2(u(S))$ (pr_i $(i = 1, 2)$ are the canonical projections of the last cartesian product) are bounded subsets of $\mathcal{M}_{\mathbf{A}}(E, G)$, $\mathcal{M}_{\mathbf{A}}(F, G)$ respectively (cf. (4.2)) and hence equicontinuous. Thus, the set

$$\begin{aligned} S \subseteq u^{-1}\Big(\Big(\mathrm{pr}_1(u(S)) \times \mathrm{pr}_2(u(S))\Big) \cap Q\Big) &= \\ &= \mathrm{pr}_1(u(S)) \otimes_{\mathbf{A}} \mathrm{pr}_2(u(S)) \subseteq \mathcal{M}_{\mathbf{A}}(E \otimes^{\tau}_{\mathbf{A}} F, G) \end{aligned}$$

is equicontinuous (cf. [13] for the present case). The direction 2)⇒1) is analogous to [13: Lemma 4.3]. ∎

Remark 4.1. The definition of a generalized Gel'fand **A**-map and the results following it are also valid by replacing the generalized **A**-spectra with the generalized $(\mathbf{A}, \mathbf{B})$-spectra; i.e. the sets of continuous non-zero $(\mathbf{A}, \mathbf{B})$-morphisms of a topological **A**-algebra into a topological **B**-algebra. The last sets are endowed with the simple convergence topology defined as in the generalized **A**-spectra.

Lemma 4.2. *Let* $(E_\alpha, f_{\beta\alpha})$ *be a direct system of unital topological* $(\mathbf{A}_\alpha, \sigma_{\beta\alpha})$*-algebras* ($\alpha < \beta$ *in* J) *(cf. [8]) and* G *a topological* $\mathbf{A}\,(= \varinjlim \mathbf{A}_\alpha)$*-algebra. Moreover, for each* $\alpha \in J$*, assume the continuity of the generalized Gel'fand* $(\mathbf{A}_\alpha, \mathbf{A})$*-maps* $\mathcal{G}_\alpha$ *of* E_α *(cf. (4.1), Remark 4.1). Then the generalized Gel'fand* **A***-map* $\mathcal{G}$ *of* $E = \varinjlim E_\alpha$ *is also continuous.*

Proof: For each $\alpha \in K$, if X_α is the canonical map of $\varprojlim \mathcal{M}_{(\mathbf{A}_\alpha, \mathbf{A})}(E_\alpha, G)$ into $\mathcal{M}_{(\mathbf{A}_\alpha, \mathbf{A})}(E_\alpha, G)$ (cf. [14: (3.1)]) the map

$$\theta_\alpha\colon \mathcal{C}_c\Big(\mathcal{M}_{(\mathbf{A}_\alpha, \mathbf{A})}(E_\alpha, G), G\Big) \to \mathcal{C}_c\Big(\varprojlim \mathcal{M}_{(\mathbf{A}_\alpha, \mathbf{A})}(E, G), G\Big)\colon\ s \mapsto \theta_\alpha(s) := s{\circ}X_\alpha\ ,$$

is continuous such that

$$\theta_\alpha \circ \mathcal{G}_\alpha = \mathcal{G} \circ f_\alpha\,, \qquad \alpha \in K\ , \tag{4.5}$$

with f_α being the canonical map of E_α into E. Hence, the assertion follows from (4.5) in connection with the definition of the direct limit topology on $E = \varinjlim E_\alpha$ (cf. also [10: Chapter IV, §2]). ∎

5. The main objective of this Section is to give the generalized spectrum of $\widehat{\bigotimes_{i\in K}}_{\mathbf{A}} \widehat{E}_i$ (w.r.t. a topological algebra G) as a cartesian product of the corresponding spectra of the factor algebras, getting thus an analogous result to that of [6: Theorem 3.1].

First we set some applications concerning to the local equicontinuity of $\mathcal{M}_{\mathbf{A}}(\bigotimes_{i\in K}{}_{\mathbf{A}} E_i, G)$.

Lemma 5.1. *Let $(E_i)_{i\in K}$ be a family of (unital) locally convex $\mathbf{A}$-algebras and G a unital commutative locally convex semi-Montel $\mathbf{A}$-algebra with continuous multiplication. Moreover, let E_i, $i \in K$, have locally equicontinuous generalized $\mathbf{A}$-spectra, such that for all except of finite many of them the corresponding spectra are equicontinuous. Then $\mathcal{M}_{\mathbf{A}}(\bigotimes_{i\in K}{}_{\mathbf{A}} E_i, G)$ is locally equicontinuous.*

Proof: By [6: Lemma 3.1], extending it in a finite case, $\mathcal{M}_{\mathbf{A}}(\bigotimes_{i\in\alpha}{}^{\pi}_{\mathbf{A}} E_i, G)$, $\alpha \in \mathcal{F}(K)$, is locally equicontinuous such that (4.3), Lemma 4.2 imply the continuity of the generalized Gel'fand $\mathbf{A}$-map of $\bigotimes_{i\in K}{}_{\mathbf{A}} E_i$. On the other hand, from the relations (3.5), (4.3) $\mathcal{M}_{\mathbf{A}}(E_i, G)$, $i \in K$, are locally compact or compact if they are locally equicontinuous or equicontinuous respectively. Hence $\mathcal{M}_{\mathbf{A}}(\bigotimes_{i\in K}{}_{\mathbf{A}} E_i, G)$ is locally compact (Tychonov's Theorem), such that the continuity of the generalized Gel'fand $\mathbf{A}$-map and the condition (4.3) yield the assertion. ∎

Corollary 5.1. *Let $(E_i)_{i\in K}$ be a family of locally convex $\mathbf{A}$-spectrally barrelled algebras and G an algebra as in Lemma 5.1. Then, $\mathcal{M}_{\mathbf{A}}(\bigotimes_{i\in K}{}_{\mathbf{A}} E_i, G)$ is locally equicontinuous if, and only if, the algebras E_i, $i \in K$, have locally equicontinuous generalized $\mathbf{A}$-spectra, such that for all except of finite many of them the corresponding spectra are equicontinuous.*

Proof: By [6: Appendix] $\bigotimes_{i\in\alpha}{}^{\pi}_{\mathbf{A}} E_i$, $\alpha \in \mathcal{F}(K)$ is an $\mathbf{A}$-spectrally barrelled algebra such that $\bigotimes_{i\in K}{}_{\mathbf{A}} E_i = \varinjlim_{\alpha}(\bigotimes_{i\in\alpha}{}^{\pi}_{\mathbf{A}} E_i)$ is also $\mathbf{A}$-spectrally barrelled (the proof is analogous to [13: Lemma 4.2]). Now, if $\mathcal{M}_{\mathbf{A}}(\bigotimes_{i\in K}{}_{\mathbf{A}} E_i, G)$ is locally equicontinuous then it is also locally compact (cf. (4.3)) such that the "if" part of the assertion follows from the Tychonov's Theorem (see also the comments before Lemma 1.1 as well as (4.3)). The rest is a consequence of the Lemma 5.1. (The **A**-spectrally barrelledness of E_i, $i \in K$, is not necessary for this part of proof). ∎

Now, we are in the position to set the next main result.

Theorem 5.1. *Let the hypotheses of Lemma 5.1 be satisfied where moreover the algebras E_i, $i \in K$, have continuous multiplications and G is complete. Then, one has*

$$\mathcal{M}_{\mathbf{A}}\Big(\widehat{\bigotimes_{i\in K}}{}_{\mathbf{A}} \widehat{E}_i, G\Big) = \prod_{i\in K} \mathcal{M}_{\mathbf{A}}(\widehat{E}_i, G) \tag{5.1}$$

within a homeomorphism.

Proof: by Lemma 5.1, $\mathcal{M}_{\mathbf{A}}(\bigotimes_{i\in K}{}_{\mathbf{A}} E_i, G)$ is locally equicontinuous such that

$$\mathcal{M}_{\mathbf{A}}\Big(\widehat{\bigotimes_{i\in K}}{}_{\mathbf{A}} E_i, G\Big) = \mathcal{M}_{\mathbf{A}}\Big(\bigotimes_{i\in K}{}_{\mathbf{A}} E_i, G\Big) \tag{5.2}$$

within a homeomorphism (cf. (1.4) and also [6: (3.11)]). Thus the relation (5.2), Theorem 3.1 and [6: (3.11)] prove the assertion. ∎

In another paper we extend the previous results examining the infinite tensor product sheaf (resp. bundle) of a family of topological **A**-algebra sheaves (resp. bundles) in relation with the holomorphic functions as well. Moreover in [9] we study the infinite topological **A**-tensor product algebra as above in relation with the theory of continuous central **A**-morphisms.

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