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HAHN–BANACH THEOREM IMPLIES RIESZ THEOREM

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The following result is proved in almost all textbooks on Functional Analysis. See [2], [3] or [4]:

Riesz Theorem. *Let E be a normed space. If E is locally compact then it is finite dimensional.*

Proof: We present here what seems to be a new and more natural (than the usual one) proof. Use will be made of the Hahn–Banach Theorem.

Let $S^1 = \{v \in E \mid \|v\| = 1\}$ be the unit sphere in E and let $\mathbf{H}$ be the family of all closed hyperplanes in E. A well known geometric corollary of Hahn–Banach Theorem states that $\bigcap \mathbf{H} = \{0\}$. Therefore $\mathbf{H}^1 = \{H \cap S^1 \mid H \in \mathbf{H}\}$ is a family of closed sets in S^1 with $\bigcap \mathbf{H}^1 = \emptyset$; the local compactness of E implies that S^1 is compact and therefore there are finitely many closed hyperplanes $H_1, ..., H_n$ such that $H_1 \cap ... \cap H_n \cap S^1 = \emptyset$; this is equivalent to $H_1 \cap ... \cap H_n = \{0\}$. But then, since hyperplanes have codimension one, E has dimension at most n. ∎

More generally, it is well known that any locally compact Hausdorff topological vector space is finite dimensional ([1], Chap. I, §2, Thm. 3). Our proof generalizes but we need to assume local convexity.

Riesz Theorem (Locally convex case)**.** *Let E be a Hausdorff locally convex space. If E is locally compact then E is finite dimensional.*

Proof: Note first that if U is a compact nbhd of 0 and F is a closed vector subspace of E such that $F \subseteq U$ then F is compact; but then any nbhd of 0 in F absorbs F and the nbhd has to be equal to F. Therefore F is indiscrete and because it is Hausdorff necessarily $F = \{0\}$. Hence $F \neq \{0\}$ implies $F \cap \mathrm{int}(U) \neq \emptyset$ and $F \cap (E - U) \neq \emptyset$; but, since F is connected, $F \cap \partial U \neq \emptyset$. Let $\mathbf{H}$ be as before.

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By the locally convex version of the Hahn–Banach Theorem ([1], Chap. II, §3, Cor. 3), $\bigcap \mathbf{H} = \{0\}$. Therefore $\mathbf{H}^U = \{H \cap \partial U \mid H \in \mathbf{H}\}$ is a family of closed sets in ∂U with $\bigcap \mathbf{H}^U = \emptyset$. By the compactness of ∂U there exist $H_1, ..., H_n$ in $\mathbf{H}$ such that $H_1 \cap ... \cap H_n \cap \partial U = \emptyset$; therefore $H_1 \cap ... \cap H_n = \{0\}$ and again E has dimension at most n. ∎

The same proof applies replacing local convexity by the slightly more general hypothesis that the dual of E separates points, this being equivalent to $\bigcap \mathbf{H} = \{0\}$.

It would be nice to have a simple proof that a locally compact Hausdorff topological vector space is locally convex since that, taken with the present results, would provide a simple proof of the general case of Riesz Theorem.

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