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A NOTE ON THE ASYMPTOTICS OF PERTURBED EXPANDING MAPS

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Abstract: Given any analytic expanding map $f\colon M \to M$ on a compact manifold M, it is well-known that $f : M \to M$ is exponentially mixing with respect to the smooth invariant measure μ. Our first result is that although for linear expanding maps on tori the rate of mixing is arbitrarily fast, generically this is *not* the case.

For random compositions of ϵ-close analytic expanding maps $g : M \to M$ which also preserve μ and we show that the rate of mixing for the composition has an upper bound which can be made arbitrarily close to that for the single transformation f by choosing $\epsilon > 0$ sufficiently small.

1 – Expanding maps and rates of mixing

Let M be a compact manifold (without boundary) and let $f\colon M \to M$ be a locally distance expanding map i.e. $\exists\, 0 < \theta < 1$ such that $d(fx, fy) \geq \frac{1}{\theta} d(x, y)$, for $x, y \in M$ sufficiently close. Furthermore, we shall assume that $f\colon M \to M$ is real analytic (i.e. we can choose some neighbourhood $M \subset U \subset M^{\mathbf{C}}$ in the complexification $M^{\mathbf{C}}$ to which f has an analytic extension). It is well known that the map $f : M \to M$ preserves a unique smooth probability measure μ (cf. [Ma], for example). We can assume, without loss of generality, that μ is the volume on M (otherwise this can be achieved by a simple conformal change in the Riemannian metric on M).

We let $C^\omega(M, \mathbf{C})$ denote the space of functions on M which have a uniformly bounded analytic extension to the neighbourhood U, and for $F, G \in C^\omega(M, \mathbf{C})$ we denote the *correlation function* by

$$\rho_f(N) = \int F \circ f^N \cdot G \, d\mu - \int F \, d\mu \cdot \int G \, d\mu \quad \text{for} \quad N \geq 1 \, .$$

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Definition. We define the *rate of mixing* to be

$$\rho = \sup\Big\{\limsup_{N\to\infty} |\rho_f(N)|^{\frac{1}{N}} : \; F, G \in C^\omega(M, \mathbf{C}) \;\; \text{with} \;\; \int F\, d\mu = 0 = \int G\, d\mu\Big\} \; .$$

To understand this quantity we introduce the *transfer operator* $\mathcal{L}_f : C^\omega(M, \mathbf{C}) \to C^\omega(M, \mathbf{C})$ defined by $(\mathcal{L}_f G)(x) = \sum_{fy=x} d_f(y)\, G(y)$ where $d_f(x) = \frac{1}{|\mathrm{Det}(D_x f)|} \in C^\omega(M)$. (We may reduce the size of the neighbourhood U, if necessary, to ensure this operator is well-defined). Using the standard identity $\int F \circ f \cdot G\, d\mu = \int F \cdot \mathcal{L}_f\, G\, d\mu$ (cf. [Ru1]) we can write

$$\rho_f(N) = \int F \cdot (\mathcal{L}_f^N G)\, d\mu - \int F\, d\mu \cdot \int G\, d\mu \quad \text{for} \;\; N \geq 1 \; . \tag{1.1}$$

This simple identity makes it clear that the spectrum of $\mathcal{L}_g$ influences rate of mixing.

Proposition 1. *The spectrum of the operator* $\mathcal{L}_f \colon C^\omega(M, \mathbf{C}) \to C^\omega(M, \mathbf{C})$ *has the following properties*

- **(a)** *There is a maximal positive eigenvalue* $\beta = \beta(f) > 0$*;*
- **(b)** *The rest of the spectrum consists of isolated eigenvalues of finite multiplicity (accumulating at zero), all of modulus strictly less than* β*.*

[Ru1], [Ru2].

In the particular case of interest, where μ is the unique absolutely continuous invariant measure, the maximal eigenvalue β is *always* equal to unity. We immediately have the following question: *Are there any other non-zero eigenvalues for* $\mathcal{L}_g \colon C^\omega(M) \to C^\omega(M)$ *than* β *?* By identity (1.1) the existance of such an eigenvalue is equivalent to the rate of mixing *not* being arbitrarily fast.

To illustrate the solution we consider the case where M is the usual flat torus and μ is the Haar measure.

Theorem 1.

- **(i)** *If* $f : \mathbf{T}^d \to \mathbf{T}^d$ *is an orientation preserving linear expanding map on the flat torus* $\mathbf{T}^d = \mathbf{C}^d/\mathbf{C}^d$*, then rate of mixing is arbitrarily fast i.e.* $\rho = 0$ *(equivalently,* 1 *is the only non-zero eigenvalue for* $\mathcal{L}_f \colon C^\omega(\mathbf{T}^d) \to C^\omega(\mathbf{T}^d)$*);*
- **(ii)** *There exists a neighbourhood* $f \in \mathcal{U} \subset C^\omega(\mathbf{T}^d, \mathbf{T}^d)$ *such that for an open dense set of* $g \in \mathcal{U}$ *the rate of mixing is non-zero i.e.* $\rho \neq 0$ *(equivalently, the operator* $\mathcal{L}_f$ *has other non-zero eigenvalues than unity).*

Proof: We begin by recalling that the linear operators $\mathcal{L}_f : C^\omega(\mathbf{T}^d) \to C^\omega(\mathbf{T}^d)$ are *trace class* (cf. [Ru1], [G] and [My]) (i.e. the eigenvalues $\{\lambda_i\}_{i=1}^\infty$ for $\mathcal{L}_f$ are summable). Furthermore, each of the traces

$$\operatorname{trace}(\mathcal{L}_f^n) := \sum_{i=1}^\infty \lambda_i^n, \qquad \text{for } \; n \geq 1$$

is finite, and we have the identities

$$\operatorname{trace}(\mathcal{L}_f^n) = \sum_{f^n x = x} \frac{1}{\operatorname{Det}(D_x f^n - 1)} \tag{1.2}$$

for each $n \geq 1$, where the sum on the right hand side of this identity is over all periodic points of period n (cf. [Ru1], [My]).

Since we are considering a *linear* expanding map $f\colon \mathbf{T}^d \to \mathbf{T}^d$, we have:

(a) If $\alpha > 1$ is the number of pre-images, under $f : M \to M$ of any point $x \in M$ then $\operatorname{Det}(D_x f^n) = \alpha^n$ for $n \geq 1$;

(b) The number of periodic points (of order n) is given by $\operatorname{Det}(D_x f^n) - 1$.

For part (a), we observe that $D_x f$ (and thus each $D_x f^n$) is constant, and then the value $D_x f^n = \alpha^n$, for $n \geq 1$, comes from the change of volume.

For part (b), we need only apply the Lefschetz fixed point theorem, where for the torus we can identify $Df = f^*$ with the action on homology. In particular, for $0 \leq j \leq d$ the j-th homology group takes the form $H_j(\mathbf{T}^d, \mathbf{C}) = \bigoplus_0^{\binom{d}{j}} \mathbf{C}^d$, and the induced action $\bigoplus_0^{\binom{d}{j}} f^*\colon \bigoplus_0^{\binom{d}{j}} \mathbf{C}^d \to \bigoplus_0^{\binom{d}{j}} \mathbf{C}^d$.

Assume that the matrix $D_x f^n$ has eigenvalues $\beta_1, \dots, \beta_d$, then by the Lefschetz formula the number of fixed points is given by the alternating sum

$$\begin{aligned}\sum_{j=0}^d \operatorname{trace}\Big(\bigoplus_0^{\binom{d}{j}} (f^n)^*\Big) &= \sum_{j=0}^d (-1)^{j+1} \sum_{\text{distinct } i_1,\dots,i_j} \beta_{i_1} \cdots \beta_{i_j} \\ &= \operatorname{Det}(D_x f^n - 1) \ .\end{aligned}$$

Substituting (a) and (b) into the identity (1.2) gives that

$$\sum_{i=1}^\infty \lambda_i^n = \operatorname{tr}(\mathcal{L}_i^n) = 1, \qquad \text{for } \; n \geq 1 \ .$$

We conclude from this family of identities, that there is exactly one non-zero eigenvalue, and this must take the value unity. This completes the proof of part (i).

For part (ii), we consider the identity (1.2) for fixed n. To be definite, we shall choose $n = 1$. It is clear, from the right hand side of this identity, that for generic (small) C^ω perturbations g of the linear map f, we can arrange that

$$\mathrm{trace}(\mathcal{L}_f) = \sum_{gx=x} \frac{1}{\mathrm{Det}(D_x g) - 1} \neq 1$$

(where we are implicitly using the fact that by structural stability there is a correspondence between the fixed points). Thus, by identity (1.2), we have that $\sum_{i=1}^\infty \lambda_i = \mathrm{trace}(\mathcal{L}_g) \neq 1$ and we conclude that there are other non-zero eigenvalues than just $\beta = 1$. This proves part (ii). ∎

Clearly, the proof of part (ii) of the theorem works equally well for any compact manifold.

Corollary 1.1. *Let $f: \mathbf{T}^1 \to \mathbf{T}^1$ a map on the unit circle $\mathbf{T}^1$ defined by $f(z) = z^n$, for some $n \geq 2$, where $\mathbf{T}^1 = \{z \in \mathbf{C}\colon |z| = 1\}$ is the unit circle, then the corresponding weight function is $d_f = \frac{1}{n}$.*

- **(i)** *$\beta = 1$ is the only non-zero eigenvalue for $\mathcal{L}_f\colon C^\omega(\mathbf{T}^1) \to C^\omega(\mathbf{T}^1)$;*
- **(ii)** *There exists a neighbourhood $f \in \mathcal{U} \subset C^\omega(\mathbf{T}^1, \mathbf{T}^1)$ such that for an open dense set of $g \in \mathcal{U}$ has other non-zero eigenvalues than just unity.*

Remark 1. Theorem 1 also has implications for the spectrum of the operator $L_g: C^k(M) \to C^k(M)$ acting on C^k functions, for $k \geq 1$. For $1 \leq r < \infty$, the spectrum in the region $|z| > \beta\theta^r + \epsilon$ consists only of isolated eigenvalues (of finite multiplicity and nullity), for any $\epsilon > 0$ [Ru2], [Ta]. For sufficiently large $r > 0$, we can find expanding maps $g\colon \mathbf{T}^d \to \mathbf{T}^d$ (arbitrarily close to f) such that $\mathcal{L}_g: C^r(\mathbf{T}^d) \to C^r(\mathbf{T}^d)$ has other eigenvalues in $|z| > \beta\theta^r$. To see this, we first choose an analytic function g, as in part (ii) of Theorem 1. If we assume that λ is an eigenvalue for $\mathcal{L}_g\colon C^\omega(\mathbf{T}^d) \to C^\omega(\mathbf{T}^d)$ which is different to 0 or 1. Since we can choose an eigenfunction $h \in C^\omega(\mathbf{T}^n)$ associated to the eigenvalue λ, the same value is also an eigenvalue for $\mathcal{L}_g: C^r(\mathbf{T}^d) \to C^r(\mathbf{T}^d)$, for any $r \geq 1$. Provided $r \geq 1$ is sufficiently large that $\beta\theta^r < |\lambda|$, the value λ is an *isolated* eigenvalue.

Remark 2. If we considered manifolds with boundary, then the analogues of this corollary would be slightly different. For example, given any $k \times k$ matrix A with entries 0 or 1, we can associate a piecewise linear map g on the union I of the intervals $\{I_{ij} = [\frac{i-1}{k} + \frac{j-1}{kn_i}, \frac{i}{k} + \frac{j}{kn_i}]|\colon i = 1, \ldots, k \text{ and } j = 1, \ldots, n_i\}$ where $n_j = \mathrm{Card}\{1 \leq i \leq k \mid A(i,j) = 1\}$, which linearly maps I_{ij} onto $\{I_i = [\frac{i-1}{k}, \frac{i}{k}]\colon i = 1, \ldots, k\}$. With the choice of weight function $d = 1$, the spectrum of the operator $\mathcal{L}\colon C^\omega(I) \to C^\omega(I)$ contains the eigenvalues of the matrix A.

2 – A general result on operator norms

We want to formulate a general result for a bounded linear operator $T\colon B \to B$ on Banach space $(B, \|\cdot\|_B)$. Given a bounded linear operator $T: B \to B$ we define the operator norm by $\|T\| = sup\{\|Tv\|_B : v \in B$ with $\|v\|_B \le 1\}$. We begin with the following definitions.

Definitions.

(i) The *spectral radius* of the operator T is defined to be

$$\sigma(T) = \limsup_{n\to\infty} \|T^n\|^{1/n} \ ;$$

(ii) Given $\delta > 0$ we define the *δ-neighbourhood spectral radius*

$$\sigma_\delta(T) = \sup\Big\{ \limsup_{n\to\infty} \|T_n \ldots T_1\|^{1/n} \colon \ \|T - T_i\| \le \delta \Big\} \ .$$

As is well-known, the spectral radius $\sigma(T)$ is finite (being bounded by the norm of the operator i.e. $\sigma(T) \le \|T\|$) and the operator $(\lambda - T): B \to B$ is invertible whenever $|\lambda| > \sigma(T)$.

Proposition 2. *For any $\eta > 0$, we can choose $\delta_0 > 0$ such that $\sigma_\delta \le \sigma + \eta$ whenever $0 < \delta \le \delta_0$.*

Proof: Assume we are given $\eta > 0$ and that we choose δ_0 sufficiently small, as described below. Consider a product $T_{j_1} \ldots T_{j_n}$ formed from a sequence of bounded linear operators $T_{j_1}, \ldots, T_{j_n} : B \to B$, for $n \ge 1$, each satisfying $\|T - T_{j_i}\| \le \delta_0$, for $0 \le i \le n$.

By applying the triangle inequality for the norm $\|\cdot\|$ (on bounded linear operators on the Banach space), we get the upper bound

(2.1)

$$\begin{aligned}
\|T_{j_1} \ldots T_{j_n}\| &\le \|T^n\| + \|(T_{j_1} \ldots T_{j_n}) - T^n\| \\
&\le \|T^n\| + \Big\| \sum_{k=2}^{n} T_{j_1} \ldots T_{j_{k-1}}.(T - T_{j_k}).T^{n-k} + (T_1 - T).T^{n-1} \Big\| \\
&\le \|T^n\| + \sum_{k=2}^{n} \|T_{j_1} \ldots T_{j_{k-1}}\|.\|T - T_{j_k}\|.\|T^{n-k}\| + \|(T_1 - T).T^{n-1}\| \ .
\end{aligned}$$

We want to fix a few values

(1) Fix any values ρ, β such that $\sigma < \rho < \beta < \sigma + \eta$.

(2) By definition of the spectral radius σ of the operator T, there exists a constant $C > 0$ with $\|T^n\| \le C\rho^n$, $\forall n \ge 1$.

(3) Fix any value $K > C > 0$.

(4) Choose δ_0 sufficiently small that:

$$C\Big(1+\frac{\delta_0}{\rho}\Big) + K\,\delta_0\Big(\frac{1}{\beta}+\frac{C\rho}{\beta^2(1-\frac{\rho}{\beta})}\Big) \le K \quad \text{and} \quad C\rho+\delta_0 \le K\beta\ .$$

We claim that $\|T_{j_1}\dots T_{j_n}\| \le K\beta^n$ for $n \ge 1$, and our proof will be by induction.

To start the induction we observe that $\|T_{j_1}\| \le \|T\| + \delta_0 \le C\rho + \delta_0 \le K\beta$ (by (2) and (4) above). To prove the inductive step, we assume that for some $n \ge 1$ we have that

$$\|T_{j_1}\dots T_{j_r}\| \le K\beta^n \quad \text{for all} \quad 1 \le r \le n-1\ .$$

Substituting these bounds into the identity (2.1) we get the estimate

(2.2)

$$\begin{aligned}
\|T_{j_1}\dots T_{j_n}\| &\le \|T^n\| + \sum_{k=2}^{n}\|T_{j_1}\dots T_{j_{k-1}}\|.\|T-T_{j_k}\|.\|T^{n-k}\| + \|T_1 - T\|.\|T^{n-1}\| \\
&\le C\rho^n + \sum_{k=2}^{n}(K\beta^{k-1})\,(\delta_0)\,(C\rho^{n-k}) + C\rho^{n-1}\,\delta_0 \\
&= C\rho^n + C\rho^{n-1}\delta_0 + K\,C\,\delta_0\sum_{k=2}^{n-1}\beta^{k-1}\,\rho^{n-k} + K\,\beta^{n-1}\,\delta_0 \\
&= C\rho^n\Big(1+\frac{\delta_0}{\rho}\Big) + K\,C\,\beta^{n-1}\,\delta_0\sum_{k=2}^{n-1}\Big(\frac{\rho}{\beta}\Big)^{n-k} + K\,\beta^{n-1}\,\delta_0 \\
&= C\rho^n\Big(1+\frac{\delta_0}{\rho}\Big) + K\,C\,\beta^{n-1}\,\delta_0\,\frac{\rho}{\beta}\Big(\frac{(1-(\frac{\rho}{\beta})^{n-3})}{(1-\frac{\rho}{\beta})}\Big) + K\,\beta^{n-1}\delta_0 \\
&\le C\Big(1+\frac{\delta_0}{\rho}\Big) + K\,\delta_0\Big(\frac{1}{\beta}+\frac{C}{\beta^2(1-\frac{\rho}{\beta})}\Big)\beta^n \\
&\le K\,\beta^n\ ,
\end{aligned}$$

where for the last inequality we have used (4). This completes the inductive step, and the proof of the claim.

We therefore conclude that whenever $\delta \le \delta_0$, we have that

$$\begin{aligned}
\sigma_\delta(T) &= \sup\Big\{\limsup_{n\to\infty}\|T_{j_1}\dots T_{j_n}\|^{1/n} :\ \|T-T_i\| \le \delta\Big\} \\
&\le \limsup_{n\to\infty}(K\,\beta^n)^{1/n} = \beta < \sigma + \eta\ .
\end{aligned}$$

This completes the proof of the proposition. ∎

3 – Random compositions of expanding maps

For any $\epsilon > 0$, we denote by $B_\mu(f,\epsilon)$ the space of all locally distance expanding maps $g\colon M \to M$ which are ϵ-close to f in the C^ω topology and *which preserve the same measure* μ. Let $\bigoplus_1^N B_\mu(f,\epsilon)$ be the direct sum of N copies of this neighbourhood space, for $N = 1,2,\ldots,\infty$, and let $\pi_{N,N'}\colon \bigoplus_1^N B_\mu(f,\epsilon) \to \bigoplus_1^{N'} B^r_\mu(f,\epsilon)$, for $N \geq N'$, be the natural map (by truncating sequences).

Definition. Given $F, G \in C^\omega(M)$ and $\underline{f} = (f_n)_{n=0}^\infty \in \bigoplus_0^\infty B^r_\mu(f,\epsilon)$, and $N \geq 1$, we define a *stochastic correlation function* by

$$\rho_{\underline{f}}(N) = \int \Big(\pi_{\infty,N}(\underline{f})^* F\Big).G\,d\mu - \int F\,d\mu.\int G\,d\mu \quad \text{for} \quad N \geq 1 , \tag{3.1}$$

where

$$(\pi_{\infty,N}(\underline{f})^* F)(x) = F(f_N \circ \ldots \circ f_0 x) .$$

In the special case where $f_n = f$, for all $n \geq 0$, this reduces to the usual correlation function (for the single expanding map $f\colon M \to M$).

Providing $\epsilon > 0$ is sufficiently small the transfer operators $\mathcal{L}_g\colon C^\omega(M) \to C^\omega(M)$ are well-defined. Moreover, since each $g\colon M \to M$ preserves the measure μ we have $\int F \circ g.G\,d\mu = \int F.\mathcal{L}_g G\,d\mu$, $\forall g \in B_\mu(f,\epsilon)$, and we can write

$$\rho_{\underline{f}}(N) = \int F.\Big(\mathcal{L}_{f_N}\mathcal{L}_{f_{N-1}}\ldots\mathcal{L}_{f_2}\mathcal{L}_{f_1}G\Big)\,d\mu - \int F\,d\mu.\int G\,d\mu . \tag{3.2}$$

For each $g \in B^r_\mu(f)$, we know that

- **(i)** The volume $d(Vol)$ on M is the eigenfunction for the simple eigenvalue 1 of the dual operator $\mathcal{L}_g^*$;
- **(ii)** The constant functions $\mathbf{C}$ are common eigenfunctions for $\mathcal{L}_g$ with associated eigenvalue 1. The associated eigenprojection $P\colon C^\omega(M) \to C^\omega(M)$ takes the form $P(F) = \int F d(Vol)$.

In particular, the operators $\mathcal{L}_g\colon C^\omega(M) \to C^\omega(M)$ have the same eigenprojection $P\colon C^r(M) \to C^r(M)$ for the eigenvalue 1. In order to eliminate the common maximal eigenvalue 1, so that we can study the remainder of the spectrum which determines the *rate* of convergence of the various correlation functions, we consider the restriction $\mathcal{L}_g\colon B \to B$ where $B \subset C^\omega(M)$ denote the co-dimension one subspace $B = \{F \in C^\omega(M)\colon \int F d(Vol) = 0\}$.

For the correlation function $\rho_f(N)$ we have

$$\rho = \sup\left\{\limsup_{N\to\infty}\left|\int F.(\mathcal{L}_{f_N}\mathcal{L}_{f_{N-1}}\ldots\mathcal{L}_{f_2}\mathcal{L}_{f_1}G)\,d\mu\right|^{\frac{1}{N}}\colon F,G \in B\right\} ,$$

where ρ is the spectral radius σ of the operator $\mathcal{L}_f : B \to B$.

We would like to apply Proposition 2 from the preceeding section. This requires knowing that as $\epsilon \to 0$, the difference $\|\mathcal{L}_f - \mathcal{L}_g\|$ (in the operator norm on some appropriate Banach space of functions) tends to zero for $g \in B_\mu(f,\epsilon)$. For analytic functions it is easy to check (using Cauchy's theorem) that we have the strong estimate: *for any* $\epsilon > 0$ *there exists* $C > 0$ *such that for* $g \in B_\mu(f,\epsilon)$ *we have that* $\|\mathcal{L}_f - \mathcal{L}_g\| \le C.\|f - g\|$. This brings us to the following result.

Theorem 2. *Assume that* $f: M \to M$ *is a* C^ω *expanding map with rate of mixing* ρ, *then the rate of mixing of any composition of maps in* $B_\mu(f,\epsilon)$ *has an upper bound which can be made arbitrarily close to* ρ *for sufficiently small* $\epsilon > 0$ *(i.e.* $\forall\, \rho' > \rho$, $\exists\, \epsilon > 0$ *such that* $\forall\, F, G \in C^\omega(M)$, $\exists C' > 0$ *with* $\rho_{\underline{f}}(N) \le C'(\rho')^N$, $\forall N \ge 1$, $\forall \underline{f} \in \bigoplus_0^\infty B_\mu(f,\epsilon)$*).*

After writing this note, I received the pre-print [BY] in which the authors there prove a similar result to Theorem 2 above (and many other results besides). The referee informs me that our Theorem 2 can be deduced from their work, and I thank him for this information.

Proof of Theorem: We first choose $\rho' - \rho > \eta > 0$ and then choose δ_0 sufficiently small that for $\|\mathcal{L}_f - \mathcal{L}_g\|_B \le \delta \le \delta_0$ we have $\sigma_\delta < \sigma + \eta$. We then set $\epsilon_0 = \frac{\delta_0}{C}$, where $C > 0$ is the Lipshitz constant.

By the identity (3.2) we observe that

$$\begin{aligned} |\rho_{\underline{f}}(N)|^{\frac{1}{N}} &= \limsup_{N\to+\infty} \left| \int F.\left(\mathcal{L}_{f_N}\mathcal{L}_{f_{N-1}} \dots \mathcal{L}_{f_2}\mathcal{L}_{f_1} G\right)\, d\mu \right|^{\frac{1}{n}} \\ &\le \|F\|_\infty \|G\|_\infty \|\mathcal{L}_{f_N}\mathcal{L}_{f_{N-1}} \dots \mathcal{L}_{f_2}\mathcal{L}_{f_1}\|^{\frac{1}{N}}, \qquad \forall\, F, G \in B\ . \end{aligned} \tag{3.3}$$

The identity (3.3) allows us to apply Proposition 2, and to deduce that

$$\limsup_{n\to\infty} |\rho_{\underline{f}}(n)|^{\frac{1}{n}} \le \sigma_\delta \le \sigma + \eta\ .$$

This completes the proof. ∎

Remark. In [BY] there is a section which treats certain types of random C^k maps. Unfortunately, the corresponding "Lispschitz" estimate for $C^r(M)$ does not hold. The nearest approximations are estimates $\|(\mathcal{L}_f - \mathcal{L}_g)h\|_{C^k} \le C.\|f - g\|_{C^k} \|h\|_{C^{k+1}}$. I am grateful to Viviane Baladi for pointing out this difficulty to me.

4 – An application to interval maps

In [Ke], Keller considered perturbations of the Transfer operator associated to interval maps $f\colon I \to I$ and an invariant probability measure μ. In this context, it is possible to take the Banach space B of functions $g : I \to \mathbf{C}$ of bounded variation (i.e. $\mathrm{var}(g) = \sup\{\sum_{i=1}^n |f(a_i) - f(a_{i-1})| \colon a_0 < a_1 < ... < a_n\} < +\infty$) with the norm

$$\|g\| = \mathrm{var}(g) + \|g\|_1 \ .$$

Assuming that m is an atomless invariant measure m we can associate a Transfer operator $\mathcal{L}_f\colon B \to B$ defined by $\mathcal{L}_f g = \frac{d}{dm}\,(T(f.m))$. For details of the spectrum of this operator we refer to [Ke]. Keller introduced an interesting notion of distance in the space of such transformations (motivated by the Skorohod metric)

$$\begin{aligned} d(f_1,f_2) = \inf\Big\{ & \epsilon > 0 \colon \exists\, A \subset I,\ m(A) > m(I) - \epsilon, \\ & \exists \text{ a diffeomorphism } \sigma\colon I \to I \text{ with } f_1|A = f_2 \circ \sigma|A \\ & \text{and } \forall\, x \in A\colon |\sigma(x) - x| < \epsilon,\ \Big|\frac{1}{\sigma'(x)} - 1\Big| < \epsilon \Big\} \ . \end{aligned}$$

With this metric, Keller established the following relation between the Transfer operators $\mathcal{L}_{f_i}$ associated to two maps f_i $(i = 1, 2)$:

$$\|L_{f_1} - L_{f_2}\| \le 12.\, d(f_1,f_2) \ .$$

We can repeat our argument as above, except that now we want to let $B_m(f)$ be a neighbourhood of the interval map $f : I \to I$ with respect to the above metric. In this context, the following analogue of Theorem 2 is true.

Proposition 3. *Assume that $f\colon I \to I$ and m are as defined above, then the rate of mixing of any composition of maps in $B_\mu(f,\epsilon)$ has an upper bound which can be made arbitrarily close to ρ for sufficiently small $\epsilon > 0$ (i.e. $\forall \rho' > \rho$, $\exists \epsilon > 0$ sufficiently small that $\forall F, G \in C^0(M)$, $\exists C' > 0$ such that $\rho_{\underline{f}}(N) \le C'(\rho')^N$, for all $N \ge 1$ and $\forall \underline{f} \in \bigoplus_0^\infty B_m(f,\epsilon)$.)*

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