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SEMI-GENERALIZED CONTINUOUS MAPS IN TOPOLOGICAL SPACES

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1 – Introduction

Throughout this paper we adopt the notations and terminology of [6], [1] and [3] and the following conventions: (X,τ), (Y,σ) and (Z,γ) (or simply X, Y and Z) will always denote topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of (X,τ). A subset A of X is said to be semi-open [6] if for some open set 0, $0 \subset A \subset \mathrm{Cl}(0)$, where $\mathrm{Cl}(0)$ denotes the closure of 0 in X. The complement of a semi-open set is called semi-closed [2]. The family of all semi-open (resp. semi-closed) sets in (X,τ) is denoted by $S0(X,\tau)$ (resp. $SC(X,\tau)$). The intersection of all semi-closed sets containing A is called the semi-closure of A [5] and is denoted by $\mathrm{sCl}(A)$. A map $f\colon (X,\tau)\to(Y,\sigma)$ is said to be semi-continuous [6] (resp. irresolute [5]) if the inverse image of every open subset (resp. semi-open subset) of (Y,σ) is semi-open in (X,τ).

Levine [7] has defined a subset A to be g-closed if $\mathrm{Cl}(A)\subset 0$ when $A\subset 0$ and 0 is open. The complement of a g-closed set is called g-open. The purpose of this paper is to introduce and study the concepts of two new class of maps, namely sg-continuous maps, which includes the class of continuous maps; and the class of sg-irresolute maps defined analogous irresolute maps. Moreover we introduce the concepts of sg-compactness and sg-connectedness of topological spaces. Among the theorems proved are the following:

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(A) *The following are equivalent:*

i) *X is* sg*-connected;*

ii) *X and ϕ are the only subsets of X which are both* sg*-open and* sg*-closed;*

iii) *Each* sg*-continuous map of X into a discrete space Y with at least two points is a constant map.*

(B) sg*-connectedness is preserved under* sg*-irresolute surjections.*

2 – Semi-generalized continuous maps

Here we introduce the concept of semi-generalized continuous maps. To state this, we recall some definitions and properties.

Definition 2.1. A subset A of a space X is said to be semi-generalized closed (written in short as sg-closed set) [1] if $\mathrm{sCl}(A) \subset 0$ whenever $A \subset 0$ and $0 \in S0(X, \tau)$. A subset A of X is said to be a semi-generalized open set (written in short as sg-open) if, its complement A^c is sg-closed in X.

Example 2.1:

i) [1]. Let τ be the usual topology on the real line $\mathbb{R}$ and let A be the open interval (a, b). Then A is sg-closed but not g-closed.

ii) Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, X\}$. If $A = \{a, b\}$, then A is g-closed but not sg-closed.

Remark 2.1.

i) Example 2.1 shows that g-closed and sg-closed sets are, in general, independent.

ii) Every semi-closed set is sg-closed but the converse is not true as may be seen from the following example.

Example 2.2: Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{c, d\}, X\}$. If $A = \{a, b, d\}$, then $\mathrm{sCl}(A) = X$ and so A is not semi-closed. Since X is the only semi-open set containing A, A is sg-closed.

In [1] has proved that the intersection and the union of two sg-closed sets is generally not a sg-closed. Hence, we have the following definition.

Definition 2.2. The intersection of all sg-closed sets containing a set A is called the semi-generalized-closure of A and is denoted by $\mathrm{sgCl}(A)$.

If A is a sg-closed set, then $\mathrm{sgCl}(A) = A$. The converse is not true, since the intersection of sg-closed sets need not be sg-closed.

Lemma 2.1. *If $A \subset X$, then $A \subset \mathrm{sgCl}(A) \subset \mathrm{sCl}(A) \subset \mathrm{Cl}(A)$.*

Proof: A closed (semi-closed) set is sg-closed. ∎

Definition 2.3. A map $f\colon X \to Y$ is said to be semi-generalized continuous (abbreviated by sg-continuous) if, for every closed set F of Y the inverse image $f^{-1}(F)$ is sg-closed in X.

Clearly it is proved that a map $f\colon X \to Y$ is sg-continuous if and only if the inverse image of every open set in Y is sg-open in X.

Remark 2.2. Every semi-continuous map $f\colon X \to Y$ (in particular, continuous) is sg-continuous, but the converse is not true as may be seen from the following examples.

Example 2.3: [6]. Let $X = Y = [0,1]$. Let $f\colon X \to Y$ as follows: $f(x) = 1$ if $0 \le x \le \frac{1}{2}$ and $f(x) = 0$ if $\frac{1}{2} \le x \le 1$. Then f is semi-continuous, therefore by Remark 2.2, sg-continuous but it is not continuous.

Example 2.4: Let $X = \{a,b,c,d\}$, $\tau = \{\phi, \{c,d\}, X\}$, $Y = \{p,q\}$, $\sigma = \{\phi, \{q\}, Y\}$. Let $f : (X,\tau) \to (Y,\sigma)$ be defined by $f(a) = f(b) = f(d) = p$, $f(c) = q$. Since if $A = \{a,b,d\}$ then, by Example 2.2, A is sg-closed. But A not semi-closed. Therefore f is sg-continuous but it is not semi-continuous.

Remark 2.3. When X is semi $T_{1/2}$, the concepts of semi-continuity and sg-continuity coincide (see [1] and [4] for the concept and property of semi $T_{1/2}$-spaces).

Theorem 2.1. *Let $f\colon X \to Y$ be a map from a topological space X into a topological space Y.*

- **i)** *The following statements are equivalent.*
  - **a)** *f is* sg-*continuous.*
  - **b)** *The inverse image of each open set in Y is* sg-*open in X.*
- **ii)** *If $f : X \to Y$ is* sg-*continuous, then $f(\mathrm{sgCl}(A) \subset \mathrm{Cl}(f(A)))$ for every subset A of X.*

iii) *The following statements are equivalent.*

a) *For each point x in X and each open set V in Y with $f(x) \in V$, there is a* sg*-open set U in X such that $x \in U$, $f(U) \subset V$.*

b) *For every subset A of X, $f(\mathrm{sgCl}(A)) \subset \mathrm{Cl}(f(A))$ holds.*

c) *For each subset B of Y, $\mathrm{sgCl}(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}(B))$.*

Proof: i) a)⇔b): See Definition 2.3.

ii) Since $A \subset f^{-1}f(A)$, we have $A \subset f^{-1}(\mathrm{Cl}(f(A)))$. Now $\mathrm{Cl}(f(A))$ is a closed set in Y and hence $f^{-1}(\mathrm{Cl}(f(A)))$ is a sg-closed set containing A. Consequently $\mathrm{sgCl}(A) \subset f^{-1}(\mathrm{Cl}(f(A)))$. Therefore $f(\mathrm{sgCl}(A)) \subset ff^{-1}(\mathrm{Cl}(f(A))) \subset \mathrm{Cl}(a(A))$.

iii) a)⇔b): Suppose that a) holds and let $y \in f(\mathrm{sgCl}(A))$ and let V be any open neighbourhood of y. Then there exists a point $x \in X$ and a sg-open U such that $f(x) = y$, $x \in U$, $x \in \mathrm{sgCl}(A)$ and $f(U) \subset V$. Since $x \in \mathrm{sgCl}(A)$, $U \cap A \neq \emptyset$ holds and hence $f(A) \cap V \neq \emptyset$. Therefore we have $y = f(x) \in \mathrm{Cl}(f(A))$.

Conversely, if b) holds and let $x \in X$ and let V be any open set containing $f(x)$. Let $A = f^{-1}(V^c)$, then $x \notin A$. Since $f(\mathrm{sgCl}(A)) \subset \mathrm{Cl}(f(A)) \subset V^c$, it is shown that $\mathrm{sgCl}(A) = A$. Then, since $x \notin \mathrm{sgCl}(A)$, there exists a sg-open set U containing x such that $U \cap A = \emptyset$ and hence $f(U) \subset f(A^c) \subset V$.

b)⇔c): Suppose that b) holds and let B be any subset of Y. Replacing A by $f^{-1}(B)$ we get from b) $f(\mathrm{sgCl}(f^{-1}(B))) \subset \mathrm{Cl}(ff^{-1}(B)) \subset (B)$. Hence $\mathrm{sgCl}(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}(B))$.

Conversely, suppose that c) holds, let $B = f(A)$ where A is a subset of X. Then $\mathrm{sgCl}(A) \subset \mathrm{sgCl}(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}(f(A)))$. Therefore $f(\mathrm{sgCl}(A)) \subset \mathrm{Cl}(f(A))$. This completes the proof. ∎

3 – Relation between sg-continuous maps and sg-irresolute maps

Analogous to irresolute maps in topological spaces, we introduce the class of semi-generalized-irresolute (or sg-irresolute) maps which is included in the class of sg-continuous maps. In this section we investigate basic properties of sg-irresolute maps.

Definition 3.1. A map $f : X \to Y$ from a topological space X into a topological space Y is called sg-irresolute if the inverse image of every sg-closed set in Y is sg-closed in X.

Theorem 3.1. *A map $f\colon X \to Y$ is* sg*-irresolute if and only if, for every* sg*-open A of Y, $f^{-1}(A)$ is* sg*-open in X.*

Proof: *Necessity*. If $f : X \to Y$ is sg-irresolute, then for every sg-closed B of Y, $f^{-1}(B)$ is sg-closed in X. If A is any sg-open subset of Y, then A^c is sg-closed. Thus $f^{-1}(A^c)$ is sg-closed, but $f^{-1}(A^c) = (f^{-1}(A))^c$ so that $f^{-1}(A)$ is sg-open.

Sufficiency. If for all sg-open subsets A of Y, $f^{-1}(A)$ is sg-open in X, and if B is any sg-closed subset of Y, then B^c is sg-open. Also $f^{-1}(B^c) = (f^{-1}(B))^c$ is sg-open. Thus $f^{-1}(B)$ is sg-closed. ∎

Theorem 3.2. *If a map $f\colon X \to Y$ is* sg*-irresolute, then it is* sg*-continuous but not conversely.*

Proof: Since every closed set is sg-closed, it is proved that f is sg-continuous. The converse need not be true as seen from the following example. ∎

Example 3.1: Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$ and $\sigma = \{\phi, \{a\}, Y\}$. Let $f : (X, \tau) \to (Y, \tau)$ be defined by $f(a) = f(c) = b$ and $f(b) = c$. Then f is sg-continuous. However, $\{b\}$ is sg-closed in Y but $f^{-1}(\{b\})$ is not sg-closed in X. Therefore, f is not sg-irresolute.

Theorem 3.3. *If $f\colon X \to Y$ and $g\colon Y \to Z$ are both* sg*-irresolute, then $g_0 f\colon X \to Z$ is* sg*-irresolute.*

Proof: If $A \subset Z$ is sg-open, then $g^{-1}(A)$ is sg-open and $f^{-1}(g^{-1}(A))$ is sg-open since g and f are sg-irresolute. Thus $(g_0 f)^{-1}(A) = f^{-1}(g^{-1}(A))$ is sg-open, and $g_0 f$ is sg-irresolute. ∎

Theorem 3.4. *Let X, Y and Z be any topological spaces. For any* sg*-irresolute map $f\colon X \to Y$ and any* sg*-continuous map $g\colon Y \to Z$, the composition $g_0 f\colon X \to Y$ is* sg*-continuous.*

Proof: It follows from definitions. ∎

Theorem 3.5. *Let (Y, σ) be a topological space where "every semi-closed subset is closed". If $f : (X, \tau) \to (Y, \sigma)$ is bijective, pre-semi-open (i.e., for all $O \in SO(X, \tau)$, $f(O) \in SO(Y, \sigma)$) and* sg*-continuous, then f is* sg*-irresolute.*

Proof: Let A be a sg-closed set in (Y, σ). Let $f^{-1}(A) \subset O$ where $O \in SO(X, \tau)$. Therefore, $A \subset f(O)$ holds. Since $f(O)$ is semi-open in Y and A is sg-closed in Y, $\mathrm{sCl}(A) \subset f(O)$ holds and hence $f^{-1}(\mathrm{sCl}(A)) \subset f^{-1}f(O) = O$. Since f is sg-continuous, and $\mathrm{sCl}(A)$ is closed in Y, $f^{-1}(\mathrm{sCl}(A))$ is sg-closed. Therefore $\mathrm{sCl}(f^{-1}(\mathrm{sCl}(A))) \subset 0$ and so $\mathrm{sCl}(f^{-1}(A)) \subset 0$. Hence $f^{-1}(A)$ is sg-closed in X. Then f is sg-irresolute. ∎

Example 3.2: Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b, c\}, Y\}$. Let $f\colon (X, \tau) \to (Y, \sigma)$ the identity map. We have that in (Y, σ) every semi-closed subset is closed. f is sg-continuous bijective but it is not pre-semi-open and so f is not sg-irresolute.

Theorem 3.6. *If a map $f\colon X \to Y$ is* sg*-irresolute, then, for every subset A of X, $f(\mathrm{sgCl}(A)) \subset \mathrm{sCl}(f(A))$.*

Proof: If $A \subset X$, then consider $\mathrm{sCl}(f(A))$ which is sg-closed in Y. Thus by Definition 3.1, $f^{-1}(\mathrm{sCl}(f(A)))$ is sg-closed in X. Furthermore, $A \subset f^{-1}(f(A)) \subset f^{-1}(\mathrm{sCl}(f(A)))$. Therefore by the destination of sg-closure $\mathrm{sgCl}(A) \subset f^{-1}(\mathrm{sCl}(f(A)))$, and consequently, $f(\mathrm{sgCl}(A)) \subset f(f^{-1}(\mathrm{sCl}(f(A)))) \subset \mathrm{sCl}(f(A))$. ∎

The following two examples show that the concepts of irresolute maps and sg-irresolute maps are independent of each other.

Example 3.3: Let (X, τ) and (Y, σ) be the space in Example 3.2. Then, the identity map $f\colon (X, \tau) \to (Y, \sigma)$ is irresolute. However $\{a, b\}$ is sg-closed in Y but is not sg-closed in X. Therefore f is not sg-irresolute.

Example 3.4: Let X, Y and f be as in Example 2.4. Now $\{p\}$ is semi-closed in Y and $A = \{a, b, d\}$ sg-closed, but it is not semi-closed in X. Therefore f is sg-irresolute but it is not irresolute.

4 – sg-compactness and sg-connectedness

Definition 4.1. A collection $\{A_\alpha\colon \alpha \in \nabla\}$ of sg-open sets in a topological space X is called a sg-open cover of a subset B of X if $B \subset \bigsqcup\{A_\alpha\colon \alpha \in \nabla\}$ holds.

Definition 4.2. A topological space X is semi-generalized-compact (or sg-compact) if every sg-open cover of X has a finite subcover.

Definition 4.3. A subset B of a topological space X is said to be sg-compact relative to X if, for every collection $\{A_\alpha\colon \alpha \in \nabla\}$ of sg-open subsets of

X such that $B \subset \bigsqcup\{A_\alpha \colon \alpha \in \nabla\}$, there exists a finite subset ∇_0 of ∇ such that $B \subset \bigsqcup\{A_\alpha \colon \alpha \in \nabla_0\}$.

Definition 4.4. A subset B of a topological space X is said to be sg-compact if B is sg-compact as a subspace of X.

Theorem 4.1. *Every* sg-*closed subset of a* sg-*compact space X is* sg-*compact relative to X.*

Proof: Let A be a sg-closed subset of X. Then A^c is sg-open in X. Let $M = \{G_\alpha \colon \alpha \in \nabla\}$ be a cover of A by sg-open subsets in X. Then $M^* = M \sqcup A^c$ is a sg-open cover of X, i.e., $X = (\bigsqcup\{G_\alpha \colon \alpha \in \nabla\}) \sqcup A^c$. By hypothesis, X is sg-compact, hence M^* is reducible to a finite cover of X, say $X = G_{\alpha_1} \sqcup G_{\alpha_2} \sqcup ... \sqcup G_{\alpha_m} \sqcup A^c$, $G_{\alpha_k} \in M$. But A and A^c are disjoint; hence $A \subset G_{\alpha_1} \sqcup ... \sqcup G_{\alpha_m}$, $G_{\alpha_k} \in M$. We have just shown that any sg-open cover M of A contains a finite subcover, i.e., A is sg-compact relative to X. ∎

Theorem 4.2.

i) *A* sg-*continuous image of a* sg-*compact space is compact.*

ii) *If a map $f \colon X \to Y$ is* sg-*irresolute and a subset B of X is* sg-*compact relative to X, then the image $f(B)$ is* sg-*compact relative to Y.*

Proof: **i)** Let $f \colon X \to Y$ be a sg-continuous map from a sg-compact space X onto a topological space Y. Let $\{A_\alpha \colon \alpha \in \nabla\}$ be an open cover of Y. Then $\{f^{-1}(A_\alpha); \alpha \in \nabla\}$ is a sg-open cover of X. Since X is sg-compact, it has a finite subcover, say $\{f^{-1}(A_1), ..., f^{-1}(A_n)\}$. Since f is onto $\{A_1, ..., A_n\}$ is a cover of Y and so Y is compact.

ii) Let $\{A_\alpha : \alpha \in \nabla\}$ be any collection of sg-open subsets of Y such that $f(B) \subset \bigsqcup\{A_\alpha \colon \alpha \in \nabla\}$. Then $B \subset \bigsqcup\{f^{-1}(A_\alpha) \colon \alpha \in \nabla\}$ holds. By hypothesis there exists a finite subset ∇_0 of ∇ such that $B \subset \bigsqcup\{f^{-1}(A_\alpha) : \alpha \in \nabla_0\}$. Therefore we have $f(B) \subset \bigsqcup\{A_\alpha \colon \alpha \in \nabla_0\}$ which shows that $f(B)$ is sg-compact relative to Y. ∎

Definition 4.5. A topological space X is said to be sg-connected if X can not be written as a disjoint union of two non-empty sg-open sets. A subset of X is sg-connected if it is sg-connected as a subspace.

In view of the Definition 4.5, we can give a characterization of sg-connected spaces.

Theorem 4.3. *For a topological space X, the following are equivalent.*

i) *X is* sg-*connected.*

ii) *X and ϕ are the only subsets of X which are both* sg-*open and* sg-*closed.*

iii) *Each* sg-*continuous map of X into a discrete space Y with at least two points is a constant map.*

Proof: i)⇒ii): Let O be a sg-open and sg-closed subset of X. Then O^c is both sg-open and sg-closed. Since X is the disjoint union of the sg-open sets O and O^c, one of these must be empty, that is $O = \phi$ or $O = X$.

ii)⇒i): Suppose that $X = A \cup B$ where A and B are disjoint non-empty sg-open subsets of X. Then A is both sg-open and sg-closed. By assumption, $A = \phi$ or X. Therefore X is sg-connected.

ii)⇒iii): Let $f\colon X \to Y$ be a sg-continuous map then X is covered by sg-open and sg-closed covering $\{f^{-1}(y):\ y \in Y\}$. By assumption $f^{-1}(y) = \phi$ or X for each $y \in Y$. If $f^{-1}(y) = \emptyset$ for all $y \in Y$, then f fails to be map. Then, there exists only one point $y \in Y$ such that $f^{-1}(y) \neq \emptyset$ and hence $f^{-1}(y) = X$. This shows that f is a constant map.

iii)⇒ii): Let O be both sg-open and sg-closed in X. Suppose $O \neq \emptyset$. Let $f\colon X \to Y$ be a sg-continuous map defined by $f(O) = \{y\}$ and $f(O^c) = \{w\}$ for some distinct points y and w in Y. By assumption f is constant. Therefore we have $O = X$. ∎

It is obvious that every sg-connected space is connected. The following example shows that the converse is not true.

Example 4.1: Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Then the topological space (X, τ) is connected. However, since $\{a\}$ is both sg-open and sg-closed, X is not sg-connected by Theorem 4.3.

As a direct consequence of Theorem 4.3, we have:

Corollary 4.1. *In a topological space (X, τ) with at least two points, if $SO(X, \tau) = SC(X, \tau)$, X is not* sg-*connected.*

Proof: Using the hypothesis and Theorem 5 due to in [1] there is a proper non-empty subset of X which is both sg-open and sg-closed in X. By Theorem 4.3, X is not sg-connected. ∎

Finally, we proved sg-connectedness is preserved under sg-irresolute surjections.

Theorem 4.4.

i) *$f\colon X \to Y$ is a* sg*-continuous surjection and X is* sg*-connected, then Y is connected.*

ii) *If $f\colon X \to Y$ is* sg*-irresolute surjection and X is* sg*-connected, then Y is* sg*-connected.*

Proof: **i)** Suppose that Y is not connected. Let $Y = A \sqcup B$ where A and B are disjoint non-empty open set in Y. Since f is sg-continuous and onto, $X = f^{-1}(A) \sqcup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty and sg-open in X. This contradicts the fact that X is sg-connected. Hence Y is connected.

ii) The argument is a minor modification of the proof i). ∎

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