PORTUGALIAE MATHEMATICA
Vol. 53 Fasc. 1 – 1996

EVOLUTION PROBLEMS ASSOCIATED WITH NONCONVEX CLOSED MOVING SETS WITH BOUNDED VARIATION

C. Castaing and Manuel D.P. Monteiro Marques

Abstract: We consider the following new differential inclusion

$$-du \in N_{C(t)}(u(t)) + F(t, u(t)) ,$$

where $u: [0,T] \to \mathbb{R}^d$ is a right-continuous function with bounded variation and du is its Stieltjes measure; $C(t) = \mathbb{R}^d \setminus \operatorname{Int} K(t)$, where $K(t)$ is a compact convex subset of $\mathbb{R}^d$ with nonempty interior; $N_{C(t)}$ denotes Clarke's normal cone and $F(t,u)$ is a nonempty compact convex subset of $\mathbb{R}^d$. We give a precise formulation of the inclusion and prove the existence of a solution, under the following assumptions: $t \mapsto K(t)$ has right-continuous bounded variation in the sense of Hausdorff distance; $u \mapsto F(t,u)$ is upper semicontinuous and $t \mapsto F(t,u)$ admits a Lebesgue measurable selection (Theorem 3.4); F is bounded (Theorem 3.2) or has sublinear growth (Remark 3.3). In particular, these results extend the Theorem 4.1 in [6].

1 – Introduction

In this paper, we deal with perturbations of evolution equations governed by the sweeping process, i.e., with differential inclusions of the form

$$-\frac{du}{dt}(t) \in N_{C(t)}(u(t)) + F(t, u(t)) , \qquad u(t) \in C(t) , \tag{1.1}$$

where $N_{C(t)}$ denotes an outward normal cone to the set $C(t)$ and F is a multifunction (set-valued function). The unperturbed problem

$$-\frac{du}{dt}(t) \in N_{C(t)}(u(t)) , \qquad u(t) \in C(t) , \tag{1.2}$$

Received: January 10, 1995; *Revised*: April 21, 1995.
Mathematics Subject Classification (1991): 35K22, 34A60, 34G20.
Keywords: Evolution problems, Sweeping processes, Nonconvex sets, Clarke's normal cone, Bounded variation, Scorza-Dragoni's theorem, Dugundji's extension theorem.

with $C(t)$ a (moving) convex set, was thoroughly studied in the 70's mainly by Moreau (e.g. [14]) who named it the sweeping process. Its applications to Mechanics (for instance: quasistatical evolution, plasticity) are well known. In [16], Valadier introduced the finite-dimensional case of a complement of a convex set $K(t)$, i.e., $C(t) = \mathbb{R}^d \setminus \operatorname{Int} K(t)$, the normal cone being taken in the sense of Clarke. Such a situation may be visualized as a point $u(t)$ moving outside $\operatorname{Int} K(t)$ and being pushed by the boundary of that convex set when contact is established.

The addition of perturbations — roughly corresponding to the consideration of external forces, in a mechanical setting — is quite natural. Under convexity assumptions (on both $C(\cdot)$ and $F(\cdot,\cdot)$) the fixed point technique is quite efficient (see e.g. [12]). This is true under usual assumptions on F such as separate measurability with respect to t and upper semicontinuity (closed graph) with respect to u. One of the purposes of this paper is to weaken this type of requirement on F, even in situations that are not suited to the fixed point approach.

However, the main objective is to consider perturbations of discontinuous problems; to be precise, $t \mapsto C(t)$ is only assumed to have bounded variation, with respect to Hausdorff distance. For convex $C(t)$ this was studied in [12] and another such study is found in [1, chapter III]. Here we deal with the harder case of a complement of a possibly discontinuous moving convex set with bounded variation, so that the study of (1.1) needs a new mathematical formulation and nonclassical techniques, which provide deeper results. The lipschitzean case, which is treated in [6], then follows as a corollary; it should be noted, however, that [6] contains a finer estimate on the solution.

The paper is organized as follows. In section 2, some fundamental results have to be recalled. In section 3, we give existence results for the considered problem of a nonconvex moving set (either with bounded variation or lipschitzean).

2 – Auxiliary results

Let us recall the following new multivalued version of Scorza–Dragoni theorem.

Theorem 2.1. *([7]) Let $I = [0,T]$, $T > 0$ and λ be the Lebesgue measure on I, with σ-algebra $\mathcal{L}(I)$. Let X be a Polish space and Y be a compact metric space. Let $F\colon I \times X \to \mathrm{c}(Y)$ (nonempty closed subsets of Y) be a multifunction that satisfies the following hypotheses:*

i) $\forall\, t \in I$, $\operatorname{graph} F_t = \{(x,y) \in X \times Y \mid y \in F(t,x)\}$ *is closed in* $X \times Y$*;*

ii) $\forall\, x \in X$, *the multifunction* $t \mapsto F(t,x)$ *admits a* $(\mathcal{L}(I), \mathcal{B}(Y))$*-measurable selection.*

Then, there exists a multifunction F_0 *from* $I \times X$ *to* $\mathrm{c}(Y) \cup \{\emptyset\}$ *whose graph belongs to* $\mathcal{L}(I) \otimes \mathcal{B}(X) \otimes \mathcal{B}(Y)$ *and which has the following properties:*

(**1**) *there is a* λ*-null set* N*, independent of* (t,x)*, such that*

$$F_0(t,x) \subset F(t,x)\,, \qquad \forall\, t \notin N, \quad \forall\, x \in X\ ;$$

(**2**) *if* $u\colon\ I \to X$ *and* $v\colon\ I \to Y$ *are* $\mathcal{L}(I)$*-measurable functions with* $v(t) \in F(t,u(t))$ *a.e., then* $v(t) \in F_0(t,u(t))$ *a.e.;*

(**3**) *for every* $\varepsilon > 0$*, there is a compact subset* $J_\varepsilon \subset I$ *such that* $\lambda(I \backslash J_\varepsilon) < \varepsilon$*, the graph of the restriction* $F_0|_{J_\varepsilon \times X}$ *is closed and* $\emptyset \neq F_0(t,x) \subset F(t,x)$*,* $\forall\, (t,x) \in J_\varepsilon \times X$*.*

A convex version is also useful and it is immediately available. We denote by $\mathrm{ck}(Y)$ the set of nonempty compact convex subsets of a suitable set Y.

Corollary 2.2. *Let* I*,* λ *and* X *be as in Theorem 2.1. Let* Y *be a compact convex metrizable subset of a Hausdorff locally convex space. Let* $F\colon\ I \times X \to \mathrm{ck}(Y)$ *be a multifunction such that:* $\forall\, t \in I$, $\operatorname{graph} F_t$ *is closed in* $X \times Y$ *and* $\forall\, x \in X$, $t \mapsto F(t,x)$ *admits a* $(\mathcal{L}(I), \mathcal{B}(Y))$*-measurable selection. Then, there exists a measurable multifunction* $F_0 :\ I \times X \to \mathrm{ck}(Y) \cup \{\emptyset\}$*, which has the properties (1)–(3) in the preceding theorem.*

Proof: Applying Theorem 2.1 to F, we obtain a measurable multifunction G_0 with properties (1)–(3). Then we take $F_0(t,x) = \overline{\mathrm{co}}\, G_0(t,x)$ (or we verify that G_0 must take convex values, by (2)). ∎

This kind of version of Scorza–Dragoni theorem was first given in [11] and [15] (but the essential nonemptiness in part (3) is missing). Applications to viability theory are given e.g. in [5], [8], [10].

We close this section with a multivalued version of Dugundji's "single-valued" extension theorem ([9]), communicated by H. Benabdellah:

Theorem 2.3. *Let* E *and* X *be two Banach spaces and* $K \subset E$*,* $D \subset X$ *be nonempty and closed. Let* F *be an upper semicontinuous multifunction defined in* $K \times D$ *with values in* $\mathrm{cwk}(X)$ *(nonempty convex weakly compact subsets of* X*), such that*

$$\forall\, (t,x) \in K \times D\,, \qquad F(t,x) \subset c(t)\,(1 + \|x\|)\overline{B}_X\ ,$$

for some positive function $c\colon K \to [0,\infty[$ *(*$\overline{B}$ *denotes the closed unit ball). Let* $(U_\lambda)_{\lambda\in\Lambda}$ *be a locally finite open covering of* $E\backslash K$ *such that, for all* λ, $0 < \operatorname{diam} U_\lambda \le d(U_\lambda, K) := \inf\{\|t'-s\| : (t',s) \in U_\lambda \times K\}$. *Let* $(\psi_\lambda)_{\lambda\in\Lambda}$ *be a continuous partition of unity of* $E\backslash K$ *with* $\operatorname{supp}\psi_\lambda \subset U_\lambda$. *For every* $\lambda \in \Lambda$, *choose* $t_\lambda \in K$ *such that* $\operatorname{dist}(t_\lambda, U_\lambda) < 2d(U_\lambda, K)$. *Then the multifunction* $\widetilde{F}$ *defined in* $E \times D$ *by*

$$\widetilde{F}(t,x) = F(t,x), \quad \text{if } t \in K, \quad x \in D ,$$

$$\widetilde{F}(t,x) = \sum_\lambda \psi_\lambda(t)\, F(t_\lambda, x), \quad \text{if } t \in E\backslash K, \quad x \in D ,$$

is an upper semicontinuous extension of F *to* $E \times D$, *with values in* $\operatorname{cwk}(X)$. *Moreover, we have* $\widetilde{F}(E \times D) \subset \operatorname{co} F(K \times D)$ *and, if* c *is constant,* $\widetilde{F}(t,x) \subset c(1+\|x\|)\,\overline{B}_X$. *In particular, if for all* (t,x), $F(t,x) \subset C$ *where* C *is a convex set, then the extension still satisfies* $\widetilde{F}(t,x) \subset C$.

3 – The evolution of a nonconvex set with bounded variation

We consider perturbations of possibly discontinuous problems. To be precise, we want to find a function u such that $u(t) \in C(t)$, where

$$C(t) = \mathbb{R}^d \backslash \operatorname{Int} K(t) , \tag{3.1}$$

$K(t)$ being a compact convex subset of $\mathbb{R}^d$ with nonempty interior, which may not depend continuously on t. We assume that there is a positive (Radon) measure $d\mu$ such that:

$$h(K(s), K(t)) \le d\mu(]s,t]) , \quad \forall\, 0 \le s \le t \le T , \tag{3.2}$$

where h denotes Hausdorff distance between (closed) sets. We say that the multifunction K has right-continuous bounded variation or that it is rcbv. Then the same is true for C since, by Lemme 4 in [16], $h(C(s), C(t)) \le h(K(s), K(t))$ and so

$$h(C(s), C(t)) \le d\mu(]s,t]) , \quad \forall\, 0 \le s \le t \le T . \tag{3.3}$$

If such is the case, it is reasonable to expect that $u\colon I := [0,T] \to \mathbb{R}^d$ is only a right-continuous function with bounded variation (rcbv, for short), its differential or Stieltjes measure being denoted by du. Since the works of Moreau (e.g. [14]) it is well known that the differential inclusion

$$-du \in N_{C(t)}(u(t)) , \tag{3.4}$$

(with convex $C(t)$) has a precise meaning "in the sense of differential measures". In this so-called sweeping process by a convex set, it is required that the density of du with respect to some positive measure $d\nu$ satisfies

$$-\frac{du}{d\nu}(t) \in N_{C(t)}(u(t)), \quad d\nu\text{-a.e. ;} \tag{3.5}$$

we may simply take $d\nu = |du|$, the measure of total variation of du. The same definition applies to the case (3.1), which was recently considered in [3], [4] and which may be called the *pushing process* by a discontinuous convex set $K(t)$.

Here we consider a more general problem, in that we accept the presence of a perturbation, mathematically expressed by means of a multifunction $(t,u) \mapsto F(t,u)$ with compact convex values. It is assumed that this perturbation is effective Lebesgue almost everywhere. This is in analogy with a dynamical problem treated by the second author (see [13, chapter 3]) where F is a single-valued force.

In the interpretation of the differential inclusion, which is formally written as

$$-du \in N_{C(t)}(u(t)) + F(t,u(t)) \ , \tag{3.6}$$

we must then account for the presence of two possibly unrelated measures, namely the Stieltjes measure du and Lebesgue measure λ (or dt). The following formulation is adequate:

Problem 3.1: We say that a function $u\colon I = [0,T] \to \mathbb{R}^d$ is a solution to (3.6) if it is rcbv (right-continuous with bounded variation), $u(t) \in C(t)$, $\forall\, t \in I$, and there exist a positive Radon measure $d\nu$ and a function $z \in L^1(I, dt, \mathbb{R}^d)$ satisfying the following conditions:

$$h(C(s),C(t)) \le d\nu(]s,t]) \,, \qquad \forall\, 0 \le s \le t \le T \ , \tag{3.7}$$

(3.8) du and dt have densities with respect to $d\nu$,

$$z(t) \in F(t,u(t)), \quad dt\text{-a.e. ,} \tag{3.9}$$

$$-\frac{du}{d\nu}(t) - z(t)\,\frac{dt}{d\nu}(t) \in N_{C(t)}(u(t)), \quad d\nu\text{-almost everywhere in } I \ , \tag{3.10}$$

where the r.h.s. is Clarke's normal cone.

Some comments are in order. In the l.h.s. of (3.10), we consider the densities announced in (3.8). We shall see that $d\nu = d\mu + dt$ is a good choice. In the lipschitzean case, considered below, we have $d\mu = k_1\,dt$ so that we can take $d\nu = (k_1+1)\,dt$; then, the right-hand side of (3.10) being conical, we see that (3.10) is equivalent to

$$-\frac{du}{dt}(t) - z(t) \ \in \ N_{C(t)}(u(t))$$

dt-almost everywhere, which together with (3.9) gives the usual inclusion

$$-\frac{du}{dt}(t) \;\in\; N_{C(t)}(u(t)) + F(t,u(t)), \quad dt\text{-a.e. } . \tag{3.11}$$

The following existence theorem for Problem 3.1 is obtained through a discretization procedure. Notice that the use of fixed point theorems is precluded by the nonconvexity of the sets $C(t)$.

Theorem 3.2. *Let $I = [0,T]$ and let $K\colon I \to \mathrm{ck}(\mathbb{R}^d)$ take compact convex values with nonempty interior. Assume that there exists a positive measure $d\mu$ on I such that (3.2) holds and define $C(t)$ by (3.1). Let $F\colon I\times\mathbb{R}^d \to \mathrm{ck}(\mathbb{R}^d)$ be an upper semicontinuous multifunction with nonempty compact convex values, which is bounded:*

$$\exists\, M>0\colon\; F(t,u) \subset M\overline{B}, \quad \forall\,(t,u)\in I\times\mathbb{R}^d \;, \tag{3.12}$$

where $\overline{B}$ is the closed unit ball of $\mathbb{R}^d$. Let $u_0 \in C(0)$ be given.

Then, there is an rcbv solution of Problem 3.1 such that $u(0) = u_0$. Moreover, we may take $d\nu = d\mu + dt$ and the following estimate holds for a constant c (e.g. $c = 2M+1$):

$$\|u(t)-u(s)\| \le c\, d\nu(]s,t])\,, \qquad \forall\, 0 \le s \le t \le T \;. \tag{3.13}$$

Proof: 1) Algorithm – We discretize in "time" t, but we must proceed very carefully, since there is an interplay between Lebesgue measure and a general measure $d\mu$, which for instance may have atoms. Let us define:

$$d\nu = d\mu + dt \;, \tag{3.14}$$

$$v(t) = \int_{]0,t]} d\nu = d\nu(]0,t])\,, \qquad t\in I \;, \tag{3.15}$$

$$V = d\nu(]0,T]) = v(T) \;. \tag{3.16}$$

Then v is a nondecreasing right-continuous function with $v(0)=0$. Moreover, inequality (3.3) implies (3.7), since $d\mu \le d\nu$; i.e.:

$$h(C(s),C(t)) \le d\nu(]s,t]) = v(t)-v(s)\,, \qquad \forall\, s\le t \;. \tag{3.17}$$

For every integer $n\ge 1$, we consider nodes of discretization $t_{n,i}$ obtained in the following manner. The pre-images

$$J_{n,j} := v^{-1}\left(\left[\frac{j}{n}V, \frac{j+1}{n}V\right[\right) ,$$

with $j=0,...,n$, are intervals, closed on their left and relatively open on the right in I; they may be empty or reduce to a point (if $j=n$). The nonempty $J_{n,j}$ form a partition of I. We order their left endpoints and denote them by

$$t_{n,0}=0<t_{n,1}<...<t_{n,p_n}=T\ , \tag{3.18}$$

where $p_n \le n$. Since two consecutive nodes $t_{n,i}$ and $t_{n,i+1}$ are the endpoints of some $J_{n,j}$ and in this set v grows less than $\frac{V}{n}$, it follows that:

$$\forall\, t\in [t_{n,i},t_{n,i+1}[\colon\ \ d\nu(]t_{n,i},t])=v(t)-v(t_{n,i})<\frac{V}{n}\ . \tag{3.19}$$

It also follows that:

$$t_{n,i+1}-t_{n,i}\le d\nu(]t_{n,i},t_{n,i+1}[)\le \frac{V}{n}\ . \tag{3.20}$$

Notice also that every atom t of $d\mu$, i.e., every discontinuity point of the multifunction C is one of the nodes $t_{n,i}$ for every n large enough (depending on t). In fact for such a t, there exists m such that $d\mu(\{t\})>V/m$ and so $v(t)-v^-(t)>V/n$, $\forall\, n\ge m$ (v^- denotes the left-limit of v); hence, by (3.19) t is not an interior point of some $J_{n,i}$.

We proceed by induction, defining finite sequences $(u_{n,i})$ and $(z_{n,i})$ by:

$$u_{n,0}=u_0\ , \tag{3.21}$$

$$z_{n,i}\in F(t_{n,i},u_{n,i})\,,\qquad i\ge 0\ , \tag{3.22}$$

$$u_{n,i+1}\in \operatorname{proj}_{C(t_{n,i+1})}\Big(u_{n,i}-(t_{n,i+1}-t_{n,i})\,z_{n,i}\Big)\ , \tag{3.23}$$

where $\operatorname{proj}_C(u)$ denotes the set of proximal points y of u in the set C, i.e. of those $y\in C$ such that $\|y-u\|=\operatorname{dist}(u,C)$.

Then we define the following functions $u_n,z_n\colon\ I\to \mathbb{R}^d$:

$$\begin{aligned} u_n(t)=u_{n,i}+{}&\frac{d\nu(]t_{n,i},t])}{d\nu(]t_{n,i},t_{n,i+1}])}\Big(u_{n,i+1}-u_{n,i}+(t_{n,i+1}-t_{n,i})\,z_{n,i}\Big)\\ &-(t-t_{n,i})\,z_{n,i}\,,\qquad \forall\, t\in [t_{n,i},t_{n,i+1}]\ ; \end{aligned} \tag{3.24}$$

$$z_n(t)=z_{n,i}\,,\qquad \forall\, t\in [t_{n,i},t_{n,i+1}[\ . \tag{3.25}$$

2) Estimates and properties – The functions u_n are rcbv and their Stieltjes measures are given by

$$du_n=\Big(\sum_{i=0}^{n-1}\frac{w_{n,i}}{d\nu(]t_{n,i},t_{n,i+1}])}\,\chi_{]t_{n,i},t_{n,i+1}]}\Big)d\nu-z_n\,dt\ , \tag{3.26}$$

where χ_A denotes the characteristic function of A and we introduce a simplifying notation

$$w_{n,i} := u_{n,i+1} - \Big(u_{n,i} - (t_{n,i+1} - t_{n,i})\, z_{n,i}\Big) \;\in\; -N_{C(t_{n,i+1})}(u_{n,i+1})\ , \tag{3.27}$$

by definition (3.23) and a property of proximal points.

Notice that (3.25), (3.22) and (3.12) imply that

$$\|z_n(t)\| \le M\,, \qquad \forall\, t, n\ . \tag{3.28}$$

From (3.26) we have an expression for the density

$$\frac{du_n + z_n\, dt}{d\nu}(t) = \frac{1}{d\nu(]t_{n,i}, t_{n,i+1}])}\, w_{n,i}\,, \qquad \forall\, t \in\,]t_{n,i}, t_{n,i+1}]\ . \tag{3.29}$$

By (3.27) and (3.23)

$$\|w_{n,i}\| = \operatorname{dist}\Big(u_{n,i} - (t_{n,i+1} - t_{n,i})\, z_{n,i}, C(t_{n,i+1})\Big)$$

and by construction $u_{n,i} \in C(t_{n,i})$. Hence, by (3.3), (3.12), (3.14) and (3.22):

$$\begin{aligned}
\|w_{n,i}\| &\le h\Big(C(t_{n,i}), C(t_{n,i+1})\Big) + (t_{n,i+1} - t_{n,i})\, \|z_{n,i}\| \\
&\le d\mu(]t_{n,i}, t_{n,i+1}]) + M(t_{n,i+1} - t_{n,i}) \\
&\le (M+1)\, d\nu(]t_{n,i}, t_{n,i+1}])\ .
\end{aligned}$$

So (3.29) leads to the following estimate

$$\left\|\frac{du_n + z_n\, dt}{d\nu}(t)\right\| \le M + 1\,, \qquad \forall\, t \in I\ . \tag{3.30}$$

Since dt has a density with respect to $d\nu$ with $0 \le \frac{dt}{d\nu} \le 1$, the l.h.s. of (3.29) is a sum of densities and we obtain:

$$\left\|\frac{du_n}{d\nu}(t)\right\| \le M + 1 + \|z_n(t)\| \le 2M + 1, \quad d\nu\text{-a.e.}\ . \tag{3.31}$$

Let us define step-functions $\theta_n, \delta_n \colon I \to I$ by $\delta_n(0) = t_{n,1}$ and

$$\theta_n(t) = t_{n,i}, \qquad \text{if} \;\; t \in [t_{n,i}, t_{n,i+1}[\ , \tag{3.32}$$

$$\delta_n(t) = t_{n,i+1}, \qquad \text{if} \;\; t \in [t_{n,i}, t_{n,i+1}[\ . \tag{3.33}$$

Then it is clear by (3.20) that

$$\theta_n(t) \uparrow t, \quad \delta_n(t) \downarrow t \qquad \text{uniformly on} \;\; I\ . \tag{3.34}$$

By construction, the functions u_n and z_n have the following properties (see (3.21)–(3.25), (3.27) and (3.29))

$$u_n(0) = u_{n,0} = u_0 \ , \tag{3.35}$$

$$u_n(\theta_n(t)) = u_{n,i} \ \in \ C(\theta_n(t)) \ , \tag{3.36}$$

$$z_n(t) \in F(\theta_n(t), u_n(\theta_n(t))) \ , \tag{3.37}$$

$$\begin{aligned} -\frac{du_n + z_n\,dt}{d\nu}(t) \ \in \ & \frac{1}{d\nu(]t_{n,i}, t_{n,i+1}])} N_{C(t_{n,i+1})}(u_{n,i+1}) = \\ & = N_{C(\delta_n(t))}(u_n(\delta_n(t))) \ , \end{aligned} \tag{3.38}$$

since $d\nu(]t_{n,i}, t_{n,i+1}]) > 0$ and the r.h.s. is a cone.

3) Extraction of subsequences – By (3.28) we may extract a subsequence still denoted by (z_n) such that

$$z_n \to z \quad \text{ in } \ \sigma\Big(L^\infty(I, dt; \mathbb{R}^d), L^1(I, dt; \mathbb{R}^d)\Big) \ . \tag{3.39}$$

By (3.31) we have

$$|du_n| \le \left\| \frac{du_n}{d\nu} \right\| d\nu \le (2M+1)\,d\nu \ , \tag{3.40}$$

in the sense of the order of real measures. Since $u_n(0) = u_0$, then by a compactness result (see e.g. [13, Theorem 0.3.4]) we can extract a subsequence still denoted by (u_n) which converges pointwisely to an rcbv function $u\colon\ I \to \mathbb{R}^d$. Moreover the measure of total variation satisfies

$$|du| \le (2M+1)\,d\nu \ , \tag{3.41}$$

so that (3.8) and (3.13) hold.

4) Existence of a solution – Clearly $u(0) = \lim u_n(0) = u_0$ and all we have to check is (3.9), $u(t) \in C(t)$ and (3.10).

First we show that

$$u_n(\theta_n(t)) \to u(t)\,, \qquad \forall\, t \in]0, T] \ . \tag{3.42}$$

By (3.31) and (3.19) we have

$$\|u_n(\theta_n(t)) - u_n(t)\| \le \int_{]\theta_n(t),t]} \left\|\frac{du_n}{d\nu}\right\| d\nu \le (2M+1)\,d\nu(]\theta_n(t), t]) \le (2M+1)\frac{V}{n} \ ,$$

so that $\lim u_n(\theta_n(t)) = \lim u_n(t) = u(t)$.

Then (3.34), (3.37), (3.39), (3.42) and the assumption that the closed convex valued multifunction F is globally upper semicontinuous classically imply that (3.9) holds dt-almost everywhere.

By (3.36), (3.17) and (3.19):

$$\operatorname{dist}(u_n(\theta_n(t)), C(t)) \le h(C(\theta_n(t)), C(t)) \le d\nu(]\theta_n(t), t]) < \frac{V}{n} \ .$$

Hence by using (3.42) and the fact that $C(t)$ is a closed set, we obtain:

$$u(t) \in C(t)\,, \quad \forall\, t \in I \ . \tag{3.43}$$

Concerning (3.10), we already know that

$$\xi_n(t) := -\frac{du_n}{d\nu}(t) - z_n(t)\,\frac{dt}{d\nu}(t) \tag{3.44}$$

satisfies (3.38); to be precise, if we recall (3.30)

$$\xi_n(t) \in N_{C(\delta_n(t))}(u_n(\delta_n(t))) \cap (M+1)\,\overline{B} \tag{3.45}$$

$d\nu$-almost everywhere (densities being so defined). By construction, we have

$$\xi_n \to \xi := -\frac{du}{d\nu} - z\,\frac{dt}{d\nu} \quad \text{in } \ \sigma\Big(L^\infty(I, d\nu; \mathbb{R}^d), L^1(I, d\nu; \mathbb{R}^d)\Big) \ . \tag{3.46}$$

In fact, the pointwise convergence of (u_n) to u implies the weak-$*$ convergence of $\frac{du_n}{d\nu}$ to $\frac{du}{d\nu}$ in $L^\infty(I, d\nu; \mathbb{R}^d)$ (use the test functions $\chi_{]s,t]}$); while, if $g \in L^1(I, d\nu; \mathbb{R}^d)$ — hence g is also Lebesgue-integrable — then by (3.39)

$$\int z_n\,\frac{dt}{d\nu}\,g\,d\nu = \int z_n\,g\,dt \ \to \ \int z\,g\,dt = \int z\,\frac{dt}{d\nu}\,g\,d\nu \ ,$$

that is, $z_n\,\frac{dt}{d\nu} \to z\,\frac{dt}{d\nu}$ in $\sigma(L^\infty(I, d\nu; \mathbb{R}^d), L^1(I, d\nu; \mathbb{R}^d))$.

Moreover,

$$u_n(\delta_n(t)) \to u(t) \ , \tag{3.47}$$

since $u_n(t) \to u(t)$ and

$$\|u_n(\delta_n(t)) - u_n(t)\| \le (2M+1)\,d\nu(]t, \delta_n(t)]) \to 0 \ ,$$

because $]t, \delta_n(t)] \downarrow \emptyset$.

It is known ([3], [4]) that the compact convex valued multifunction

$$\phi(t,u) := N_{C(t)}(u) \cap (M+1)\,\overline{B} \quad\ (t \in I, \quad u \in \mathbb{R}^d) \tag{3.48}$$

has a closed graph in $I_r \times \mathbb{R}^d \times \mathbb{R}^d$ (I_r denoting I endowed with the right-topology). Since (3.45) is rewritten as $(\delta_n(t), u_n(\delta_n(t)), \xi_n(t)) \in \operatorname{graph}\phi$, then (3.34), (3.46) and (3.47) imply that $\xi(t) \in \phi(t, u(t))$, $d\nu$-a.e.; that is, (3.10) holds. ∎

Remark 3.3. From the existence of extensions of Gronwall inequality to discontinuous cases (see [12, Lemme 4] and [2, §3]) one might expect that the conclusion of Theorem 3.2 still holds true under sublinear growth assumptions on F. If for instance

$$F(t,u) \subset c(1 + \|u\|)\,\overline{B}\ ,$$

where $c > 0$ is fixed, then a simple argument applies. Notice that it suffices to show that z_n, u_n remain bounded (inequality (3.28) was essential in the above proof). By (3.23) and (3.17):

$$\begin{aligned}
\|u_{n,i+1} - u_{n,i}\| &\le \operatorname{dist}\Big(u_{n,i} - (t_{n,i+1} - t_{n,i})\, z_{n,i}, C(t_{n,i+1})\Big) + (t_{n,i+1} - t_{n,i})\, \|z_{n,i}\| \\
&\le h\Big(C(t_{n,i}), C(t_{n,i+1})\Big) + 2(t_{n,i+1} - t_{n,i})\, \|z_{n,i}\| \\
&\le \Big(1 + 2\, \|z_{n,i}\|\Big)\, d\nu(]t_{n,i}, t_{n,i+1}])\ .
\end{aligned}$$

But $z_{n,i} \in F(t_{n,i}, u_{n,i})$ implies $\|z_{n,i}\| \le c(1 + \|u_{n,i}\|)$, so

$$\|u_{n,i+1}\| \le (1 + 2\,c)\,\alpha_i + 2\,c\alpha_i\, \|u_{n,i}\|\ ,$$

where $\alpha_i := d\nu(]t_{n,i}, t_{n,i+1}])$ satisfies $\sum_{i=0}^{n-1} \alpha_i = d\nu([0,T]) = V$. By induction, this is easily seen to imply that

$$\|u_{n,i+1}\| \le \Big[\|u_{n,0}\| + (1 + 2\,c) \sum_{j=0}^{i} \alpha_j\Big] \exp\Big(2\,c \sum_{j=0}^{i} \alpha_j\Big)\ .$$

Thus,

$$\forall\, i, \quad \|u_{n,i}\| \le \Big[\|u_0\| + (1 + 2\,c)\,V\Big]\, e^{2cV} =: c_1\ ,$$

and $\|z_n(t)\| \le M_1 := c(1 + c_1)$, $\forall\, t$, $\forall\, n$.

By using the auxiliary results of §2 we are able to extend Theorem 3.2:

Theorem 3.4. *The conclusion of Theorem 3.2 holds true if we replace the assumption of global upper semicontinuity of $F : I \times \mathbb{R}^d \to \operatorname{ck}(\mathbb{R}^d)$ by the following hypotheses:*

(3.49) $\quad \forall\, u \in \mathbb{R}^d\colon\ t \mapsto F(t,u)$ *admits a $\mathcal{L}(I)$-measurable selection ;*

(3.50) $\quad \forall\, t \in I\colon\ u \mapsto F(t,u)$ *is upper semicontinuous .*

Proof: By Corollary 2.2, there is a multifunction $F_0\colon I{\times}X \to \mathrm{ck}(Y)\cup\{\emptyset\}$ where $Y = M\overline{B}$, which is measurable and has the properties (1)–(3) in Theorem 2.1. That is,

(**1**) there is a set $N \subset I$, independent of (t,u), such that N has zero (Lebesgue) measure and $F_0(t,u) \subset F(t,u)$, $\forall\, t \in I\backslash N$, $\forall\, u \in \mathbb{R}^d$;

(**2**) if $u: I \to \mathbb{R}^d$ and $v: I \to \mathbb{R}^d$ are $\mathcal{L}(I)$-measurable functions with $v(t) \in F(t,u(t))$ a.e., then $v(t) \in F_0(t,u(t))$ a.e.;

(**3**) for every $\varepsilon > 0$, there is a compact subset $J_\varepsilon \subset I$ such that $\lambda(I\backslash J_\varepsilon) < \varepsilon$, the graph of the restriction $F_0|_{J_\varepsilon\times\mathbb{R}^d}$ is closed and $\emptyset \neq F_0(t,u) \subset F(t,u)$, $\forall\,(t,u) \in J_\varepsilon \times \mathbb{R}^d$.

By property (3), there exists a sequence of compact sets $J_n \subset I$ with $\lambda(I\backslash J_n) = \varepsilon_n \to 0$ such that the restriction of F_0 to $J_n \times \mathbb{R}^d$ has closed graph (i.e., it is upper semicontinuous, since it takes compact values) and has nonempty values; we may also assume that (J_n) is increasing. By Theorem 2.3, there is an upper semicontinuous extension $\widetilde{F}_n$ of $F_0|_{J_n\times\mathbb{R}^d}$ to $I \times \mathbb{R}^d$, with $\widetilde{F}_n(t,u) \subset M\overline{B}$, for all $(t,u) \in I \times \mathbb{R}^d$.

We now apply Theorem 3.2 with $\widetilde{F}_n$ substituted for F. Thus, for every n, there is an rcbv function $u_n\colon I \to \mathbb{R}^d$ and a function $z_n \in L^\infty(I,dt;\mathbb{R}^d)$ such that $u_n(0) = u_0$; $u_n(t) \in C(t)$, $\forall\, t \in I$; $z_n(t) \in \widetilde{F}_n(t,u_n(t))$ dt-a.e. and

$$-du_n - z_n\,dt \;\in\; N_{C(t)}(u_n(t)) \;,$$

in the sense explained above (Definition 3.1). From Theorem 3.2, we also know that, for every n, $\|z_n(t)\| \leq M$ (dt-a.e.); $\|\frac{du_n+z_n\,dt}{d\nu}(t)\| \leq M+1$, $d\nu$-a.e. and also $|du_n| \leq c\,d\nu$, where $c = 2M+1$. The already mentioned compactness result ([13, Theorem 0.3.4]) allows the extraction of a subsequence still denoted (u_n) which pointwisely converges to an rcbv function $u\colon I \to \mathbb{R}^d$ with $|du| \leq c\,d\nu$. And simultaneously we may extract a subsequence, still denoted (z_n), which weakly-$*$ converges to a function $z \in L^\infty(I,dt;\mathbb{R}^d)$.

Clearly $u(0) = u_0$ and $u(t) \in C(t)$, $\forall\, t$.

We show that (3.9) holds. By construction, there exist Lebesgue null sets N_n such that $\forall\, t \in J_n\backslash N_n$:

$$z_n(t) \in F_0(t,u_n(t)) \;. \tag{3.51}$$

Let $N_0 := (I\backslash \bigcup_n J_n) \cup \bigcup_n N_n$, which has zero measure. If $t \notin N_0$, then there is $p = p(t)$ such that $t \notin J_n\backslash N_n$ for all $n \geq p$. Hence, $z_n(t) \in F_0(t,u_n(t))$, for all $n \geq p$. Since F_0 is upper semicontinuous in $J_p \times \mathbb{R}^d$ and $u_n(t) \to u(t)$, it follows that

$$\forall\, x' \in (\mathbb{R}^d)' = \mathbb{R}^d\,, \qquad \limsup_n \delta^*(x' \,|\, F_0(t, u_n(t))) \le \delta^*(x' \,|\, F_0(t, u(t)))\ ;$$

here $\delta^*(x' \,|\, A) = \sup\{\langle x', x\rangle :\ x \in A\}$ is the support function of a set A and $\langle \cdot, \cdot\rangle$ is the scalar and duality product of $\mathbb{R}^d$. For $t \notin N_0$ and $n \ge p = p(t)$, we have

$$\langle x', z_n(t)\rangle \le \delta^*(x' \,|\, F_0(t, u_n(t))), \qquad \text{thus} \quad \limsup_n \langle x', z_n(t)\rangle \le \delta^*(x' \,|\, F_0(t, u(t)))\ ,$$

where the right-hand side is a measurable function. Since this leaves out only a null set, it follows that, for every measurable set $A \subset I$ and every $x' \in \mathbb{R}^d$,

$$\int_A \langle x', x(t)\rangle\, dt = \lim_n \int_A \langle x', z_n(t)\rangle\, dt \le \int_A \delta^*(x' \,|\, F_0(t, u(t)))\, dt\ ,$$

by Fatou's lemma. This is known to imply that $z(t) \in F_0(t, u(t)) \subset F(t, u(t))$ a.e..

Concerning (3.10), we extract from $\xi_n := -\frac{du_n + z_n\, dt}{d\nu}$ a subsequence — notation unchanged — that converges weakly-$*$ in $L^\infty(I, d\nu; \mathbb{R}^d)$ to $\xi := -\frac{du + z\, dt}{d\nu}$. With ϕ defined in (3.48), we have $\xi_n(t) \in \phi(t, u_n(t))$. Hence, as in the end of the proof of Theorem 3.2, we conclude that $\xi(t) \in \phi(t, u(t)) \subset N_{C(t)}(u(t))\ d\nu$-a.e. ∎

The variations of our techniques allow us to obtain several variants without fundamental changes. Let us mention particularly the following one which is an extension of Theorem 4.1 in [6].

Corollary 3.5. *Let $I = [0, T]$, $T > 0$. Let $K\colon\ I \to \mathrm{ck}(\mathbb{R}^d)$ be a multifunction with compact convex values in $\mathbb{R}^d$, having nonempty interior, and which is lipschitzean with respect to Hausdorff distance: $h(K(t), K(s)) \le k_1\, |t - s|$. The complement of the interior of $K(t)$ is denoted $C(t) = \mathbb{R}^d \setminus \mathrm{Int}\, K(t)$. Let $X = \mathbb{R}^d$ and $Y = k_2\, \overline{B}$, where $\overline{B}$ is the closed unit ball of $\mathbb{R}^d$. Let $F\colon\ I \times X \to \mathrm{ck}(Y)$ (nonempty compact convex subsets of Y) be a multifunction that satisfies the following hypotheses:*

(**i**) $\forall\, t \in I$, $\mathrm{graph}\, F_t = \{(x, y) \in X \times Y \mid y \in F(t, x)\}$ *is closed in* $X \times Y$,

(**ii**) $\forall\, x \in X$, *the multifunction* $t \mapsto F(t, x)$ *admits a measurable selection.*

Then, for every $a \in C(0)$ there is a k-lipschitzean function $x\colon\ I \to \mathbb{R}^d$ such that $x(0) = a$, $x(t) \in C(t)$, $\forall\, t$, and

$$-\dot{x}(t) \in N_{C(t)}(x(t)) + F(t, x(t)), \qquad \textit{almost everywhere in}\ \ I\ , \tag{3.52}$$

where $N_C(x)$ denotes the Clarke normal cone at x. Moreover, we have

$$\|x(t)-x(s)\| \le \sqrt{k_1^2+k_2^2}\,|t-s| = k\,|t-s| \;. \tag{3.53}$$

Proof: Theorem 3.4 applies with $\mu = k_1\,dt$ and $M = k_2$: there is an rcbv solution u, relabeled as x, to (3.7)–(3.10). Moreover, by (3.41), $|dx| \le (2\,k_2+1)\,d\nu$, where $d\nu = d\mu + dt = (k_1+1)\,dt$. Hence x is k^*-lipschitzean, with $k^* = (2\,k_2+1)\cdot$ $\cdot(k_1+1)$, and (3.9)–(3.10) reduce to (3.52). By applying the existence Theorem 4.1 in [6] to the approximate problems (with perturbations $\widetilde{F}_n$) we would obtain the precise lipschitzean constant $k = (k_1^2+k_2^2)^{1/2}$ in (3.53). ∎

ACKNOWLEDGEMENT – The authors gratefully acknowledge the support of Fundação Calouste Gulbenkian (Portugal) and of Project JNICT – STRDA/C/CEN/531/92.

REFERENCES

[1] Benabdellah, H. – *Contribution aux problèmes de convergence fort-faible, à la géométrie des espaces de Banach et aux inclusions différentielles*, Thèse, Montpellier, 1991.

[2] Benabdellah, H., Castaing, C. and Gamal Ibrahim, M.A. – *BV* solutions of multivalued differential equations on closed moving sets in Banach spaces, *Séminaire d'Analyse Convexe Montpellier,* 22, exposé nº 10 (1992), 48 pages; to appear in *Proceedings of the Banach Center*, 1995.

[3] Benabdellah, H., Castaing, C. and Salvadori, A. – Compactness and discretization methods for differential inclusions and evolution problems, *C. Rend. Acad. Sci. Paris Série I,* 320 (1995), 769–774.

[4] Benabdellah, H., Castaing, C. and Salvadori, A. – *Compactness and discretization methods for differential inclusions and evolution problems*, Rapporto tecnico no. 13/1994, Dipartimento di Matematica, Perugia (33 pages); to appear in *Atti Sem. Mat. Univ. Modena*, 1995.

[5] Bothe, D. – Multivalued differential equations on graphs, *Nonlinear Analysis, Theory, Methods and Applications,* 18 (1992), 245–252.

[6] Castaing, C., Duc Ha, T.X. and Valadier, M. – Evolution equations governed by the sweeping process, *Set-Valued Analysis,* 1 (1993), 109–139.

[7] Castaing, C. and Monteiro Marques, M.D.P. – A multivalued version of Scorza–Dragoni's theorem with an application to normal integrands, *Bull. Pol. Acad. Sci.-Mathematics,* 42 (1994), 133–140.

[8] Deimling, K. – Extremal solutions of multivalued differential equations II, *Results in Math.,* 15 (1989), 197–201.

[9] Dugundji, J. – An extension of Tietze's theorem, *Pacific J. Math.,* 1 (1951), 353–367.

[10] Frankowska, H., Plaskacz, S. and Rzezuchowski, T. – *Measurable viability theorems and Hamilton–Jacobi–Bellman equation*, preprint Instytut Matematyki Politechnika Warszawska, 1992.

[11] Jarnik, J. and Kurzweil, J. – On conditions on right hand sides of differential relations, *Cas. Pest. Mat.,* 102 (1977), 334–349.

[12] Monteiro Marques, M.D.P. – Perturbations convexes semi-continues supérieurement de problèmes d'évolution dans les espaces de Hilbert, *Sém. Anal. Convexe Montpellier,* 14, exposé nº 2 (1984).

[13] Monteiro Marques, M.D.P. – *Differential inclusions in nonsmooth mechanical problems — shocks and dry friction*, Birkhäuser Verlag, 1993.

[14] Moreau, J.J. – Evolution problem associated with a moving convex set in a Hilbert space, *J. Diff. Eqs.,* 26 (1977), 347–374.

[15] Rzezuchowski, T. – Scorza–Dragoni type theorem for upper semicontinuous multivalued functions, *Bull. Pol. Acad. Sci.,* 28 (1980), 61–66.

[16] Valadier, M. – Entraînement unilatéral, lignes de descente, fonctions lipschitziennes non-pathologiques, *C. Rend. Acad. Sci. Paris Série I,* 308 (1989), 241–244.

C. Castaing,
Département de Mathématiques, Case 051,
Université de Montpellier II, F-34095 Montpellier Cédex 5 – FRANCE

and

Manuel D.P. Monteiro Marques,
C.M.A.F. and Faculdade de Ciências da Universidade de Lisboa,
Av. Prof. Gama Pinto, 2, P-1699 Lisboa – PORTUGAL