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SOME REMARKS ON LIFTING OF ISOMORPHIC PROPERTIES TO INJECTIVE AND PROJECTIVE TENSOR PRODUCTS*

G. Emmanuele

In this short note we want to present two remarks improving two results contained in the papers [C] and [O], the first one about the Dunford–Pettis property and the second one about the containment of complemented copies of l_1 in certain tensor products of Banach spaces. Before starting we remark that our notations are taken from the book [DF].

The first result is about the well-known Dunford–Pettis property.

Definition. We shall say that a Banch space X has the Dunford–Pettis property if any weakly compact operator defined on it is a Dunford–Pettis operator, i.e. it maps weakly null sequences into norm null sequences.

It is well known that this property does not necessarily lift from two Banach spaces E, F to $E\widetilde{\otimes}_\pi F$, as Talagrand proved in his paper [T]. However, in [C] some positive results were got; here we improve these last results, presenting a Proposition that follows from the next

Lemma 1. *Let E, F be Banach spaces such that E^{**} or F has the Bounded Approximation Property. Then $E^{**}\widetilde{\otimes}_\pi F$ is isomorphic to a closed subspace of $(E\widetilde{\otimes}_\pi F)^{**}$.*

Proof: The canonical map $\Phi\colon\ E^{**}\widetilde{\otimes}_\pi F \to (E\widetilde{\otimes}_\pi F)^{**} = (L(E,F^*))^*$ is given by the "trace duality"

$$\langle \Phi(z), \phi\rangle := \langle \hat{\ } \phi, z\rangle$$

(see [DF], p. 161) and has norm 1. On the other hand, it follows from [DF] (p. 179, 182, 60) that the canonical map $I\colon\ E^{**}\widetilde{\otimes}_\pi F \to (E^*\widetilde{\otimes}_\varepsilon F^*)^* = (K(E,F^*))^*$ is an

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isomorphic embedding since E^{**} or F has the B.A.P.. If $\rho : (L(E, F^*))^* \to (K(E, F^*))^*$ denotes the restriction map, it is easy to see that $\rho\,\Phi = I$, hence Φ is an isomorphic embedding as well. We are done. ∎

Remark 1. It is clear from the above proof and the results in [DF], p. 291, that the above Lemma 1 holds for all accessible tensornorms α; if α is totally accessible, it also holds true for all E and F.

Remark 2. It is also clear from the above proof and the result in [DF], p. 60, that if E^{**} or F has the Metric Approximation Property then $E^{**}\widetilde{\otimes}_\pi F$ is isometrically isomorphic to a closed subspace of $(E\widetilde{\otimes}_\pi F)^{**}$.

Proposition 1. *Let E be a $\mathcal{L}_1$-space and F be a Banach space such that $L_1(\mu, F)$ has the Dunford–Pettis Property for any measure μ. Then $E\widetilde{\otimes}_\pi F$ has the Dunford–Pettis Property.*

Proof: Let us consider a weakly compact operator T defined on $E\widetilde{\otimes}_\pi F$; T^{**} defined on $(E\widetilde{\otimes}_\pi F)^{**}$ is also weakly compact as well as its restriction $\widetilde{T}$ to the isomorphic copy of $E^{**}\widetilde{\otimes}_\pi F$ which existence is guaranteed by Lemma 1. But E^{**} is isomorphic to a complemented subspace of some $L_1(\mu)$ space and so $E^{**}\widetilde{\otimes}_\pi F$ is isomorphic to a complemented subspace of $L_1(\mu, F)$ that has the Dunford–Pettis Property by assumptions; hence $E^{**}\widetilde{\otimes}_\pi F$, and its copy inside of $(E\widetilde{\otimes}_\pi F)^{**}$, too, have the same property; this gives that $\widetilde{T}$ is a Dunford–Pettis operator. Clearly its restriction to $E\widetilde{\otimes}_\pi F$ is just T that must be so a Dunford–Pettis operator. We are done. ∎

Remark 3. In [C] it is proved that $\mathcal{L}_1$-spaces and $\mathcal{L}_\infty$-spaces F satisfy the hypothesis of Proposition 1.

The second and last result is about injective tensor products and improves an old result by Oja about the existence of complemented copies of l_1; it shows that an application of a result due to Heinrich and Manckiewicz can be used to drop a separability assumption considered in the paper [O].

Proposition 2. *Let E be a Banach space such that E^* has the Radon–Nikodym Property and the Approximation Property. If F is another Banach space, then $E \otimes_\varepsilon F$ contains a complemented copy of l_1 if and only if F does the same.*

Proof: Clearly we have to show just the "only if" part. By a result in [HM] (Proposition 3.4) there is a separable closed subspace E_0 of E such that $E_0 \widetilde{\otimes}_\varepsilon F$ contains the complemented copy of l_1 living inside $E \widetilde{\otimes}_\varepsilon F$ and E_0^* is isometrically isomorphic to a norm one complemented subspace of E^*; clearly also $E_0 \widetilde{\otimes}_\varepsilon F$ contains a complemented copy of l_1 (actually the same copy as $E \widetilde{\otimes}_\varepsilon F$) and moreover E_0^* is separable, since E^* has the Radon–Nikodym Property (see [DU], p. 198), and it has the Approximation Property. A result in [O] allows us to conclude that F must contain a complemented copy of l_1. ∎

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G. Emmanuele,
Department of Mathematics, University of Catania,
Viale A. Doria 6, 95125 Catania – ITALY
E-mail: Emmanuele@Dipmat.Unict.It