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# COMPARISON RESULTS FOR IMPULSIVE DELAY DIFFERENTIAL INEQUALITIES AND EQUATIONS *

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**Abstract:** In this paper, we are dealing with first order impulsive delay differential equations and inequalities and establish comparison theorems of existence of positive solutions of impulsive delay differential equations and inequalities.

## 1 – Introduction

The theory of impulsive differential equations is emerging as an important area of investigation, since it is a lot richer than the corresponding theory of differential equations without impulse effects. Moreover, such equations may exhibit several real world phenomenon in physics, biology, engineering, etc. In the last few years the theory of the impulsive ordinary differential equations has been studied by many authors. For example, see [5]. However, not much has been developed in the direction of impulsive delay differential equations, see [1] and [2]. In this paper we are dealing with first order impulsive delay differential equations and inequalities and give some comparison results. The results relative to the delay differential equations have been obtained, see [3] and [6].

Let $0 \le t_0 < t_1 < t_2 < ...$, with $\lim_{k\to\infty} t_k = \infty$. Consider the impulsive delay differential equation

$$
(1) \qquad \begin{aligned}
&x'(t) + \sum_{i=1}^{n} p_i(t)\, x(t-\tau_i) = 0\,, \qquad t \neq t_k\,,\\
&x(t_k^+) - x(t_k) = b_k\, x(t_k)\,, \qquad k = 1, 2, 3, ...\,,
\end{aligned}
$$

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where

$$p_i(t) \geq 0 \ (i = 1, 2, ..., n) \text{ are continuous functions on } [0, \infty),\ \tau_i \ (i = 1, 2, ..., n) \text{ are positive constants and } b_k \ (k = 1, 2, ...) \text{ are nonpositive constants,} \tag{2}$$

and the corresponding inequalities

$$\begin{aligned} & x'(t) + \sum_{i=1}^{n} p_i(t)\, x(t - \tau_i) \leq 0\,, \qquad t \neq t_k\,, \\ & x(t_k^+) - x(t_k) \leq b_k\, x(t_k)\,, \qquad k = 1, 2, ...\,. \end{aligned} \tag{3}$$

Let $\tau = \max_{1 \leq i \leq n} \tau_i$ and $\sigma = \min_{1 \leq i \leq n} \tau_i$. Denote by $PC([\,\overline{t}, \infty), R)$ the set of functions $u(t)\colon [\,\overline{t}, \infty) \to R$ which are continuous for $t \geq \overline{t}$ and $t \neq t_k$ $(k{=}1, 2, ...)$ and may have discontinuities of the first kind at $t_k$ $(k = 1, 2, ...)$ at which they are continuous from the left.

By a solution of (1) (resp. (3)) we mean that a real valued function $x \in PC([t_0, \infty), R)$ satisfies (1) (resp. (3)), that is

**i**) for any $t \in [t_0, \infty)$ and $t \neq t_k$, $t \neq t_k + \tau_i$ $(k = 1, 2, ...,\ i = 1, 2, ..., n)$, $x(t)$ is continuous differentiable and satisfies (1) (resp. (3));

**ii**) for every $k$, $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)$ and $x(t_k^+) - x(t_k) = b_k\, x(t_k)$ (resp. $x(t_k^+) - x(t_k) \leq b_k\, x(t_k)$);

**iii**) if $t = t_k + \tau_i$, $t \neq t_k$ $(k = 1, 2, ...,\ i = 1, 2, ..., n)$, $x(t)$ is continuous and $x'(t_k^+)$ and $x'(t_k^-)$ exist.

**Definition 1.** A solution $x(t)$ of (1) or (3) on $[t_0, \infty]$ is called positive (resp. negative) if $x(t) > 0$ (resp. $x(t) < 0$) for all $t \geq t_0 - \tau$.

**Definition 2.** A solution of (1) is called oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory.

**Remark 1.** From Definition 1, a solution $x(t)$ of (1) is oscillatory if and only if it satisfies at least one of the following conditions:

**a**) $x(t)$ has arbitrarily large zeros, that is, for any sufficiently large $T \geq t_0$, there exists $t^* \geq t_0$ such that $x(t^*) = 0$;

**b**) For any sufficiently large integer $K$, there exists an integer $k \geq K$ such that $x(t_k^+)\, x(t_k) = (1 + b_k)\, x^2(t_k) \leq 0$, that is, $b_k \leq -1$, $k = 1, 2, ...\,.$

## 2 – Main results

In order to prove our main results, we need the following fixed point theorem of Knaster and Tarski. See [3] and [4].

**Lemma 1** (Knaster–Tarski Fixed Point Theorem). *Let $E$ be a partially ordered set with ordering $\leq$. Assume that $\inf E \in E$ and that every nonempty subset of $E$ has a supremum (which belongs to $E$). Let $S : E \to E$ be an increasing mapping (that is, $x \leq y$ implies $Sx \leq Sy$). Then $S$ has a fixed point in $E$.* ∎

**Lemma 2.** *Assume that (2) holds and $x(t)$ is a positive solution on $[t_0, \infty]$ of (3). Then*

**i)** *$x(t)$ is nonincreasing on $[t_0 + \tau, \infty)$;*

**ii)** *$-\sum_{k=k_1}^{\infty} b_k\, x(t_k) \leq x(t_{k_1})$, $k_1 \geq t_0 + \tau$.*

**Proof:** **i)** As $x(t-\tau_i) > 0$ $(i = 1, 2, ..., n)$ on $[t_0, \infty]$, from (2)

$$x'(t) \leq -\sum_{i=1}^{n} p_i(t)\, x(t-\tau_i) \leq 0 \quad \text{for } \ t \geq t_0 \ \text{ and } \ t \neq t_k \ ,$$

which implies that $x(t)$ is nonincreasing on $(t_k, t_{k+1}]$, $t_k \geq t_0$. Also, we observe that $x(t_k^+) - x(t_k) \leq b_k\, x(t_k) \leq 0$. Thus $x(t)$ is nonincreasing on $[t_0, \infty)$.

**ii)** From (2), $x'(t) \leq 0$, for $t \in (t_k, t_{k+1}]$ and $t_k \geq t_0$. This implies that $x(t_k^+) \geq x(t_{k+1}) > 0$, $t_k \geq t_0$. Thus, by (2) there exists a $k_1 \geq 1$ and for any integral $m \geq k_1$ such that

$$-\sum_{k=k_1}^{m} b_k\, x(t_k) \leq \sum_{k=k_1}^{m} \Big(x(t_k) - x(t_k^+)\Big) \leq \sum_{k=k_1}^{m} \Big(x(t_k) - x(t_{k+1})\Big) \leq x(t_{k_1})$$

which proves ii). ∎

**Theorem 1.** *Assume that*

$$\sum_{i=1}^{n} p_i(T_0+\tau) > 0 \ \text{ and } \ \sum_{i=1}^{n} p_i(t) \text{ is not identically zero on any interval of the form } [\,\bar{t}, \bar{t}+\sigma],\ \bar{t} \geq T_0+\tau,\ T_0 \geq 0. \tag{4}$$

*Let $y(t)$ be a positive solution on an interval $[T_0, \infty)$, $T_0 \geq t_0$, of (3). Then there exists a nonincreasing positive solution $x(t)$ on $[T_0 + \tau, \infty)$ of (1) with $\lim_{t\to\infty} x(t) = 0$ and such that*

$$0 \leq x(t) \leq y(t) \quad \text{for every } \ t \geq T_0 \ .$$

**Proof:** First, from Lemma 2, $y(t)$ is nonincreasing on $[T_0, \infty)$ and we obtain for all $\bar{t}$, $t$ with $\bar{t} \geq t \geq T_0$

$$y(t^+) \geq y(\bar{t}) + \int_t^{\bar{t}} \sum_{i=1}^n p_i(s)\, y(s-\tau_i)\, ds - \sum_{t<t_k<\bar{t}} b_k\, y(t_k) \ .$$

Thus, letting $\bar{t} \to \infty$ and noting $y(t) \geq y(t^+)$ we obtain

$$y(t) \geq \int_t^{\infty} \sum_{i=1}^n p_i(s)\, y(s-\tau_i)\, ds - \sum_{t<t_k<\infty} b_k\, y(t_k) \ . \tag{5}$$

Let $X$ be the set of all nonincreasing functions $x \in PC([T_0, \infty), R)$ with $0 \leq x(t) \leq y(t)$ for every $t \geq T_0$. The set $X$ is considered to be endowed with the usual pointwise ordering $\leq$, that is, if $x_1$ and $x_2$ belong to $X$. We will say that $x_1 \leq x_2$ if and only if $x_1(t) \leq x_2(t)$ for $t \geq T_0$. Clearly $X$ is a paritially ordered set. Set $T_1 = T_0 + \tau$ and define the mapping $S$ on $X$ as follows:

$$(Sx)(t) = \begin{cases} \displaystyle\int_t^{\infty} \sum_{i=1}^n p_i(s)\, x(s-\tau_i)\, ds - \sum_{t<t_k<\infty} b_k\, y(t_k), & \text{if } t \geq T_1, \\ \displaystyle\int_{T_1}^{\infty} \sum_{i=1}^n p_i(s)\, x(s-\tau_i)\, ds - \sum_{T_1<t_k<\infty} b_k\, y(t_k) + & \\ \qquad \displaystyle + \int_t^{T_1} \sum_{i=1}^n p_i(s)\, y(s-\tau_i)\, ds, & \text{if } T_0 \leq t < T_1 \,. \end{cases} \tag{6}$$

Then, by (5), we can easily verify that the formula (6) makes sense for any $x$ in $X$ and that this formula defines a mapping $S$ on $X$ into itself. If $x_1, x_2 \in X$ and $x_1(t) \leq x_2(t)$ for $t \geq T_0$, then we also have $(Sx_1)(t) \leq (Sx_2)(t)$, for $t \geq T_0$, i.e. $S$ is an increasing mapping.

Finally, $\inf X \in X$, and every nonempty subset of $X$ has a supremum that belongs to $X$. Hence, all the hypotheses of Lemma 1 are satisfied, and so $T$ has a fixed point $x \in X$. That is

$$(Sx)(t) = x(t) = \begin{cases} \displaystyle\int_t^{\infty} \sum_{i=1}^n p_i(t)\, x(t-\tau_i)\, ds - \sum_{t<t_k<\infty} b_k\, x(t_k), & \text{if } t \geq T_1, \\ \displaystyle\int_{T_1}^{\infty} \sum_{i=1}^n p_i(s)\, x(s-\tau_i)\, ds - \sum_{T_1<t_k<\infty} b_k\, x(t_k) + & \\ \qquad \displaystyle + \int_t^{T_1} \sum_{i=1}^n p_i(s)\, y(s-\tau_i)\, ds, & \text{if } T_0 \leq t < T_1 \,. \end{cases} \tag{7}$$

From (7) it follows that

$$\begin{aligned} &x'(t) = -\sum_{i=1}^{n} p_i(t)\, x(t-\tau_i), \qquad \text{for all} \quad t \ge T_1 \quad \text{and} \quad t \ne t_k \ , \\ &x(t_k^+) - x(t_k) = b_k\, x(t_k) \qquad \text{for} \quad t_k \ge T_1 \ , \end{aligned}$$

and hence $x$ is a solution of (1) on $[T_1,\infty)$. Also, from (7), we have that $\lim_{t\to\infty} x(t) = 0$. Moreover, it is clear that $x(t) \le y(t)$ for $t \ge T_0$. So, it remains to establish that the solution $x$ is positive on the interval $[T_0,\infty)$. By taking into account the fact that $y$ is positive on the interval $[T_0-\tau, T_1-\sigma]$ and that $\sum_{i=1}^n p_i(T_1) > 0$, from (7) we have for each $t \in [T_0, T_1)$

$$x(t) \ge \int_t^{T_1} \sum_{i=1}^{n} p_i(s)\, y(s-\tau_i)\, ds \ge \Big[\min_{1\le i\le n} \min_{T_0\le s\le T_1} y(s-\tau_i)\Big] \int_t^{T_1} \sum_{i=1}^{n} p_i(s)\, ds > 0 \ .$$

So, $x$ is positive on $[T_0,T_1]$. Next, we will show that $x$ is also positive on $[T_1,\infty)$. Assume that $\widehat{t} > T_1$ is the first point satisfying $x(\widehat{t}) \le 0$ of $x$ to the right of $T_0$. That is $x(t) > 0$ for $T_0 \le t < \widehat{t}$. Then (7) gives

$$\begin{aligned} 0 \ge x(\widehat{t}) &= \int_{\widehat{t}}^{\infty} \sum_{i=1}^{n} p_i(s)\, x(s-\tau_i)\, ds - \sum_{\widehat{t}<t_k<\infty} b_k\, x(t_k) \\ &\ge \int_{\widehat{t}}^{\infty} \sum_{i=1}^{n} p_i(s)\, x(s-\tau_i)\, ds \ , \end{aligned}$$

which implies that $\sum_{i=1}^n p_i(s)\, x(s-\tau_i) = 0$ for all $s \ge \widehat{t}$. Since the function $\sum_{i=1}^n p_i(t)$ is not identically zero on $[\widehat{t}, \widehat{t}+\sigma]$, we can choose a point $t^*$ with $\widehat{t} < t^* < \widehat{t}+\sigma$ such that $\sum_{i=1}^n p_i(t^*) > 0$. Hence, by taking into account the fact that $x$ is positive on $[\widehat{t}-\tau, \widehat{t}\,)$ we obtain

$$0 = \sum_{i=1}^{n} p_i(t^*)\, x(t^*-\tau_i) \ge \Big[\min_{1\le i\le n} x(t^*-\tau_i)\Big] \sum_{i=1}^{n} p_i(t^*) > 0 \ .$$

But, this is impossible. The proof of Theorem 1 is complete. ∎

Consider the impulsive delay differential equation

$$(1') \qquad \begin{aligned} &x'(t) + \sum_{i=1}^{n} \overline{p}_i(t)\, x(t-\overline{\tau}_i) = 0 \ , \\ &x(t_k^+) - x(t_k) = \overline{b}_k\, x(t_k)\,, \qquad k = 1,2,\dots \ , \end{aligned}$$

where

(2′) $\overline{p}_i \geq 0$ $(i = 1, 2, ..., n)$ are continuous functions on $[t_0, \infty)$, $\overline{\tau}_i > 0$ $(i = 1, 2, ..., n)$ and $\overline{b}_k \leq 0$, $k = 1, 2, ...$, are constants.

We establish the following comparison result.

**Theorem 2.** *Assume that (2) and (2′) hold and*

(4′) $$\sum_{i=1}^{n} \overline{p}_i(T_0 + \tau) > 0 \quad \text{and} \quad \sum_{i=1}^{n} \overline{p}_i(t) \text{ is not identically zero}$$
*on any interval of the form* $[\widehat{t}, \widehat{t} + \sigma]$, $\widehat{t} > T_0 + \tau$, $T_0 \geq 0$,

*and*

(8) $$\overline{p}_i \leq p_i \ \text{ on } [T_0, \infty)\,, \quad \overline{\tau}_i \leq \tau_i, \quad i = 1, 2, ..., n\,, \qquad \overline{b}_k \leq b_k\,, \quad k = 1, 2, ...\,.$$

*If (1) has a positive solution on* $[T_0, \infty)$*, then (1′) has also a positive solution on* $[T_0 + \tau, \infty)$.

**Proof:** Suppose that (1) has a positive solution $x(t)$ on an interval $[T_0, \infty)$. Then by Lemma 2 $x(t)$ is nonincreasing on $[T_0, \infty)$. It follows from (1) and (8) that

(3′) $$x(t) + \sum_{i=1}^{n} \overline{p}_i(t)\, x(t - \overline{\tau}_i) \leq 0\,, \qquad t \neq t_k\,,$$
$$x(t_k^+) - x(t_k) \leq \overline{b}_k\, x(t_k)\,, \qquad k = 1, 2, ...\,.$$

By using Theorem 1 we can conclude that (1′) has also a positive solution on $[T_0 + \tau, \infty)$. The proof of Theorem 2 is complete. ∎

The following corollary is an immediate consequence of Theorem 2.

**Corollary 1.** *Assume that (2), (2′) and (8) hold and* $\sum_{i=1}^{n} \overline{p}_i(t) > 0$*, for* $t \geq t_0$*. If all solutions of (1′) are oscillatory, then all solutions of (1) are also oscillatory.* ∎

If we compare (1) with the delay differential equation

(9) $$x'(t) + \sum_{i=1}^{n} p_i(t)\, x(t - \tau_i) = 0\,,$$

from Theorem 2 the following result can easily be obtained.

**Corollary 2.** *Assume that (2) and (4) hold and (9) has a positive solution on* $[T_0, \infty)$*, than (1) has a positive solution on* $[T_0 + \tau, \infty)$. ∎

**Remark 2.** In case $b_k > 0$ $(k = 1, 2, ...)$, the related results of Corollary 2 have been established, see [1] and [2].

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