PORTUGALIAE MATHEMATICA
Vol. 60 Fasc. 3 – 2003
Nova Série

ON THE INTEGRAL TRANSFORMATION ASSOCIATED WITH THE PRODUCT OF GAMMA-FUNCTIONS

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Abstract: We introduce the following integral transformation

$$\Phi(z) \;=\; 2^{z-2} \int_{\mathbb{R}_+} f(\tau)\, \Gamma\left(\frac{z+i\tau}{2}\right) \Gamma\left(\frac{z-i\tau}{2}\right) d\tau\ ,$$

where $z = x+iy,\ x>0,\ y\in\mathbb{R},\ \Gamma(z)$ is Euler's Gamma-function. Boundedness and analytic properties are investigated. The Bochner representation theorem is proved for functions $f \in L^*(\mathbb{R}_+)$, whose Fourier cosine transforms lie in $L_1(\mathbb{R}_+)$. It is shown, that this transform is an analytic function in the right half-plane and belongs to the Hardy space $\mathbb{H}_2$. When $x \to 0$ it has boundary values from $L_2(\mathbb{R})$. Plancherel type theorem is established by using its relationships with the Mellin and Kontorovich–Lebedev transforms.

1 – Introduction and preliminary results

The aim of this paper is to prove classical theorems of the Bochner and Plancherel type [1] for the following integral transformation

$$\Phi(x+iy) \;=\; 2^{x+iy-2} \int_0^\infty f(\tau)\, \Gamma\left(\frac{x+i(y+\tau)}{2}\right) \Gamma\left(\frac{x+i(y-\tau)}{2}\right) d\tau\ , \tag{1.1}$$

where $x>0,\ y\in\mathbb{R}$ and $\Gamma(z)$ is the Euler Gamma-function [2, Vol. I], which is an analytic function in the right half-plane $\operatorname{Re} z>0$. This transformation gives an

Received: May 9, 2002; *Revised*: July 25, 2002.
AMS Subject Classification: 44A20, 44A15, 42B20, 33B15.
Keywords: Gamma-function; Hardy spaces; Plancherel theorem; Mellin transform; Hilbert transform; Kontorovich–Lebedev transform; Fourier transform; Parseval equality; singular integral.

interesting example in the general composition theory of integral transformations of different structure (cf., for instance, in [3], [5], [6]). In fact, we establish its composition in terms of the Mellin transform [4]

$$f^{\mathcal{M}}(s) = \int_0^\infty f(x)\, x^{s-1}\, dx \,, \tag{1.2}$$

and the following integral transformation due to M.I. Kontorovich and N.N. Lebedev [5]

$$(KLf)(x) = \int_0^\infty K_{i\tau}(x)\, f(\tau)\, d\tau \,, \qquad x > 0 \,, \tag{1.3}$$

which involves as the kernel the modified Bessel function of the second kind or the Macdonald function $K_{i\tau}(x)$ [2, Vol. II]. As we observe the integration in the latter integral is with respect to an index (a parameter) of the Macdonald function. This circumstance differs such a type of integral operators from classical convolution transformations of the Fourier and Mellin types [4]. We will prove mapping and composition properties of the transform (1.1) in Lebesgue's and Hardy's spaces by using the corresponding theorems for operators (1.2), (1.3).

Note that the kernel of the transformation (1.1) consists of the product of two Gamma-functions. It can be represented through the Mellin and Fourier integrals by formulas (cf. [2, Vol. II], [5])

$$2^{x+iy-2}\,\Gamma\Big(\frac{x+i(y+\tau)}{2}\Big)\,\Gamma\Big(\frac{x+i(y-\tau)}{2}\Big) = \int_0^\infty K_{i\tau}(t)\, t^{x+iy-1} dt \,, \quad x > 0 \,, \tag{1.4}$$

$$2^{x+iy-2}\,\Gamma\Big(\frac{x+i(y+\tau)}{2}\Big)\,\Gamma\Big(\frac{x+i(y-\tau)}{2}\Big) = \Gamma(x+iy) \int_0^\infty \frac{\cos(t\tau)\, dt}{\cosh^{x+iy} t} \,. \tag{1.5}$$

Let us introduce for our further purposes the space $L^*(\mathbb{R}_+)$ (see, for instance, in [1]).

Definition 1. The space $L^*(\mathbb{R}_+)$ contains functions f whose Fourier cosine transforms

$$(F_c f)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos(xt)\, dt \tag{1.6}$$

belong to $L_1(\mathbb{R}_+)$. This is a Banach space with the norm

$$\|f\|_{L^*} = \int_0^\infty |(F_c f)(x)|\, dx \,. \ \square$$

In order to prove the corresponding theorems for the transform (1.1) we draw a parallel with the results for the Kontorovich–Lebedev transform (1.3). Therefore first in this preliminary section we consider some properties of the transformation (1.3).

Lemma 1. *The Kontorovich–Lebedev operator (1.3) $KLf\colon L^*(\mathbb{R}_+)\to L^*(\mathbb{R}_+)$ is bounded and we have*

$$\|KLf\|_{L^*} \le \frac{\pi}{2}\,\|f\|_{L^*}\ .$$

Proof: According to Definition 1 we find that

$$f(\tau) = \sqrt{\frac{2}{\pi}}\int_0^\infty (F_c\, f)(t)\cos(t\tau)\,dt\ . \tag{1.7}$$

Consequently, after substitution (1.7) into (1.3) we change the order of integration via the Fubini theorem. Then we calculate the integral with respect to τ invoking the formula [5]

$$\int_0^\infty K_{i\tau}(x)\cos(t\tau)\,d\tau = \frac{\pi}{2}\,e^{-x\cosh t}\,, \qquad x>0\ .$$

Thus we arrive at the following relation

$$(KLf)(x) = \sqrt{\frac{\pi}{2}}\int_0^\infty e^{-x\cosh t}(F_c\, f)(t)\,dt\,, \qquad x>0\ . \tag{1.8}$$

Hence we can calculate the Fourier transform (1.5) of $(KLf)(x)$. Indeed, we substitute $(KLf)(x)$ given by (1.8) into the integral (1.6) and interchange the order of integration via Fubinis theorem. Evaluating an elementary integral we obtain

$$\sqrt{\frac{2}{\pi}}\int_0^\infty (KLf)(x)\cos(x\tau)\,dx \ = \int_0^\infty \frac{(F_c\, f)(t)\cosh t}{\cosh^2 t+\tau^2}\,dt\ .$$

Consequently,

$$\begin{aligned}\|KLf\|_{L^*} \ &\le \int_0^\infty |(F_c\, f)(t)|\cosh t\,dt \int_0^\infty \frac{d\tau}{\cosh^2 t+\tau^2}\\ &= \int_0^\infty |(F_c\, f)(t)|\,dt \int_0^\infty \frac{d\tau}{1+\tau^2} \ = \ \frac{\pi}{2}\,\|f\|_{L^*}\ .\end{aligned}$$

This completes the proof of Lemma 1. ∎

The next theorem states representation properties of an arbitrary function $f\in L^*(\mathbb{R}_+)$ in terms of the Kontorovich–Lebedev operator (1.3).

Theorem 1. *For any $f \in L^*(\mathbb{R}_+)$ and for all $\tau \in \mathbb{R}_+$ the following representation is valid*

$$f(\tau) = \frac{2}{\pi^2}\,\tau\sinh(\pi\tau)\lim_{\varepsilon\to 0+}\int_0^\infty x^{\varepsilon-1}K_{i\tau}(x)\,(KLf)(x)\,dx\ . \tag{1.9}$$

Proof: By using formula (1.8) for $(KLf)(x)$ we substitute it in (1.9) and in view of the absolute convergence of the iterated integral we invert the order of integration. The inner integral can be calculated by using the following representation of the Gauss hypergeometric function [2, Vol. I] in terms of the Laplace integral of the Macdonald function

$$\frac{x^{i\tau-\alpha}}{2^\alpha}\sqrt{\pi}\,\frac{|\Gamma(\alpha+i\tau)|^2}{\Gamma(\alpha+1/2)}\,{}_2F_1\Big(\frac{\alpha-i\tau}{2},\,\frac{\alpha-i\tau+1}{2};\,\alpha+\frac{1}{2};\,1-\frac{1}{x^2}\Big) = \\ = \int_0^\infty t^{\alpha-1}\,e^{-xt}\,K_{i\tau}(t)\,dt\,, \qquad x,\alpha>0\ .$$

Hence the right-hand side of (1.9), which we denote by $I(\tau,\varepsilon)$ is represented as follows

$$I(\tau,\varepsilon) = \frac{2^{1/2-\varepsilon}}{\pi}\,\tau\sinh(\pi\tau)\,\frac{|\Gamma(\varepsilon+i\tau)|^2}{\Gamma(\varepsilon+1/2)} \\ \cdot\int_0^\infty(\cosh t)^{i\tau-\varepsilon}\,{}_2F_1\Big(\frac{\varepsilon-i\tau}{2},\,\frac{\varepsilon-i\tau+1}{2};\,\varepsilon+\frac{1}{2};\,\tanh^2 t\Big)\,(F_c f)(t)\,dt\ . \tag{1.10}$$

In order to deduce (1.9) we only must motivate the passage to the limit under the sign of integral (1.10). Indeed, due to the Boltz formula for the Gauss hypergeometric function (see [2], Vol. I) we have the equality

$$(\cosh t)^{i\tau-\varepsilon}\,{}_2F_1\Big(\frac{\varepsilon-i\tau}{2},\,\frac{\varepsilon-i\tau+1}{2};\,\varepsilon+\frac{1}{2};\,\tanh^2 t\Big) = \\ = {}_2F_1\Big(\frac{\varepsilon-i\tau}{2},\,\frac{\varepsilon+i\tau}{2};\,\varepsilon+\frac{1}{2};\,-\sinh^2 t\Big)\ . \tag{1.11}$$

Hence owing to properties of the Gauss function and its analytic continuation [2, Vol. I] we find that the right-hand side of (1.11) is equal to

$${}_2F_1\Big(\frac{\varepsilon-i\tau}{2},\,\frac{\varepsilon+i\tau}{2};\,\varepsilon+\frac{1}{2};\,-\sinh^2 t\Big) = \\ = \frac{\Gamma(\varepsilon+1/2)}{|\Gamma((\varepsilon+i\tau)/2)|^2}\sum_{n=0}^\infty\frac{\Gamma((\varepsilon-i\tau)/2+n)\,\Gamma((\varepsilon+i\tau)/2+n)}{\Gamma(\varepsilon+1/2+n)}\,\frac{(-1)^n\sinh^{2n}t}{n!}\ , \tag{1.12}$$

when $0 < \sinh t \le 1$ and for $\sinh t > 1$ we obtain

$$
\begin{aligned}
&{}_2F_1\left(\frac{\varepsilon - i\tau}{2}, \frac{\varepsilon + i\tau}{2}; \varepsilon + \frac{1}{2}; -\sinh^2 t\right) = \\
&\qquad = \frac{\Gamma(\varepsilon+1/2)\,\Gamma(i\tau)\,(\sinh t)^{i\tau-\varepsilon}}{|\Gamma((\varepsilon+i\tau)/2)|^2\,\Gamma((1+\varepsilon+i\tau)/2)}\,\frac{\Gamma(1-i\tau)}{\Gamma((1-\varepsilon-i\tau)/2)} \\
&\qquad\quad \cdot \sum_{n=0}^{\infty} \frac{\Gamma((\varepsilon-i\tau)/2+n)\,\Gamma((1-\varepsilon-i\tau)/2+n)}{\Gamma(1-i\tau+n)\sinh^{2n} t}\,\frac{(-1)^n}{n!} \\
&\qquad + \frac{\Gamma(\varepsilon+1/2)\,\Gamma(-i\tau)\,(\sinh t)^{-i\tau-\varepsilon}}{|\Gamma((\varepsilon+i\tau)/2)|^2\,\Gamma((1+\varepsilon-i\tau)/2)}\,\frac{\Gamma(1+i\tau)}{\Gamma((1-\varepsilon+i\tau)/2)} \\
&\qquad\quad \cdot \sum_{n=0}^{\infty} \frac{\Gamma((\varepsilon+i\tau)/2+n)\,\Gamma((1-\varepsilon+i\tau)/2+n)}{\Gamma(1+i\tau+n)\sinh^{2n} t}\,\frac{(-1)^n}{n!}\,.
\end{aligned}
\tag{1.13}
$$

Therefore, via the Stirling formula of the asymptotic of Gamma-function (see [2, Vol. 1]), an elementary inequality $|\Gamma(z)| \le \Gamma(\operatorname{Re} z)$, $\operatorname{Re} z > 0$ and the convergence of the hypergeometric series one can majorize functions in (1.12), (1.13) uniformly for all $t, \tau \in \mathbb{R}$ and $0 \le \varepsilon \le \varepsilon_0 < 1/2$ as follows

$$
\begin{aligned}
\left|{}_2F_1\left(\frac{\varepsilon-i\tau}{2}, \frac{\varepsilon+i\tau}{2}; \varepsilon+\frac{1}{2}; -\sinh^2 t\right)\right| &\le 1 + \frac{\Gamma(\varepsilon_0+1/2)}{|\Gamma((\varepsilon+i\tau)/2)|^2} \sum_{n=1}^{\infty} \frac{[\Gamma(\varepsilon_0/2+n)]^2}{\Gamma(1/2+n)}\,\frac{1}{n!} \\
&\le 1 + \frac{1}{|\Gamma((\varepsilon+i\tau)/2)|^2}\, O\left(\sum_{n=1}^{\infty} \frac{1}{n^{3/2-\varepsilon_0}}\right),
\end{aligned}
$$

where $0 < \sinh t \le 1$;

$$
\begin{aligned}
&\left|{}_2F_1\left(\frac{\varepsilon-i\tau}{2}, \frac{\varepsilon+i\tau}{2}; \varepsilon+\frac{1}{2}; -\sinh^2 t\right)\right| \le \\
&\le \frac{\Gamma(\varepsilon_0+1/2)\,|\Gamma(i\tau)|\,(\sinh t)^{-\varepsilon}}{|\Gamma((\varepsilon+i\tau)/2)\,\Gamma((1+\varepsilon+i\tau)/2)|} \\
&\quad \cdot \left[1 + \frac{1}{|\Gamma((\varepsilon-i\tau)/2)\,\Gamma((1-\varepsilon-i\tau)/2)|} \sum_{n=1}^{\infty} \frac{\Gamma(\varepsilon/2+n)\,\Gamma((1-\varepsilon)/2+n)}{|\Gamma(1-i\tau+n)|\,n!}\right] \\
&\quad + \frac{\Gamma(\varepsilon_0+1/2)\,|\Gamma(-i\tau)|\,(\sinh t)^{-\varepsilon}}{|\Gamma((\varepsilon-i\tau)/2)\,\Gamma((1+\varepsilon-i\tau)/2)|} \\
&\quad \cdot \left[1 + \frac{1}{|\Gamma((\varepsilon+i\tau)/2)\,\Gamma((1-\varepsilon+i\tau)/2)|} \sum_{n=1}^{\infty} \frac{\Gamma(\varepsilon/2+n)\,\Gamma((1-\varepsilon)/2+n)}{|\Gamma(1+i\tau+n)|\,n!}\right] \le
\end{aligned}
$$

$$\leq \frac{|\Gamma(i\tau)|}{|\Gamma((\varepsilon+i\tau)/2)\ \Gamma((1+\varepsilon+i\tau)/2)|}$$
$$\cdot \left[1+\frac{1}{|\Gamma((\varepsilon-i\tau)/2)\ \Gamma((1-\varepsilon-i\tau)/2)|}\ O\left(\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}\right)\right]$$
$$+ \frac{|\Gamma(-i\tau)|}{|\Gamma((\varepsilon-i\tau)/2)\ \Gamma((1+\varepsilon-i\tau)/2)|}$$
$$\cdot \left[1+\frac{1}{|\Gamma((\varepsilon+i\tau)/2)\ \Gamma((1-\varepsilon+i\tau)/2)|}\ O\left(\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}\right)\right] ,$$

where $\sinh t > 1$. The obtained estimates imply that the integrand in (1.10) is uniformly bounded over the interval $(0, \varepsilon_0)$ and it is majorized by $C_\tau |(F_c f)(t)|$, with $C_\tau > 0$ is a constant depending only on τ. Furthermore, since in view of the formula in [2, Vol. I]

$$\lim_{\varepsilon\to 0+} {}_2F_1\left(\frac{\varepsilon-i\tau}{2}, \frac{\varepsilon+i\tau}{2}; \varepsilon+\frac{1}{2}; -\sinh^2 t\right) = \cos(\tau t) ,$$

then one can appeal to the Lebesgue dominated convergence theorem. Passing to the limit in (1.10) when $\varepsilon \to 0+$ by using the relation [2, Vol. I]

$$|\Gamma(i\tau)|^2 = \frac{\pi}{\tau \sinh(\pi\tau)} , \tag{1.14}$$

we obtain the desired representation (1.9). Theorem 1 is proved. ∎

Finally in this section we exhibit the Plancherel type theorem for the Kontorovich–Lebedev transform (cf. [5], [6]) as a bounded operator from the space $L_2(\mathbb{R}_+; [\tau \sinh(\pi\tau)]^{-1} d\tau)$ onto the space $L_2(\mathbb{R}_+; x^{-1} dx)$, which will be used in Section 3 to establish the corresponding theorem for the transformation (1.1).

Theorem 2. *Let $f \in L_2(\mathbb{R}_+; [\tau \sinh(\pi\tau)]^{-1} d\tau)$. Then the formula (1.3) for the Kontorovich–Lebedev transform holds in the sense that, as $N \to \infty$, the integral*

$$(KLf)_N(x) = \int_{1/N}^{N} K_{i\tau}(x)\, f(\tau)\, d\tau \tag{1.15}$$

converges in mean to $(KLf)(x)$ with respect to the norm of the space $L_2(\mathbb{R}_+; x^{-1} dx)$; and

$$f_N(\tau) = \frac{2}{\pi^2}\ \tau \sinh(\pi\tau) \int_{1/N}^{N} K_{i\tau}(x)\, (KLf)(x)\, \frac{dx}{x}$$

converges in mean to $f(\tau)$ *with respect to the norm of the space* $L_2(\mathbb{R}_+; [\tau\sinh(\pi\tau)]^{-1}d\tau)$*. Moreover, the following Parseval equality is true*

$$\frac{\pi^2}{2}\int_0^\infty f(\tau)\,\overline{g(\tau)}\,\frac{d\tau}{\tau\sinh(\pi\tau)} \;=\; \int_0^\infty (KLf)(x)\,\overline{(KLg)(x)}\,\frac{dx}{x}\;,$$

where $f,g \in L_2(\mathbb{R}_+; [\tau\sinh(\pi\tau)]^{-1}d\tau)$*. In particular,*

$$\frac{\pi^2}{2}\int_0^\infty |f(\tau)|^2\,\frac{d\tau}{\tau\sinh(\pi\tau)} \;=\; \int_0^\infty |(KLf)(x)|^2\,\frac{dx}{x}\;. \tag{1.16}$$

2 – Representation theorem for transformation (1.1) in $L^*(\mathbb{R}_+)$

We begin to study the boundedness and analytic properties of the transformation (1.1) in the space $L^*(\mathbb{R}_+)$. First we apply the Parseval equality for Fourier transforms [4] to the right-hand side of (1.1) and in view of Definition 1 and representation (1.5) it becomes

$$\Phi(x+iy) \;=\; \Gamma(x+iy)\,\sqrt{\frac{\pi}{2}}\int_0^\infty \frac{(F_c\,f)(t)}{\cosh^{x+iy} t}\,dt\,, \qquad x>0,\quad y\in\mathbb{R}\;. \tag{2.1}$$

Hence we have the following estimate

$$|\Phi(x+iy)| \;\le\; |\Gamma(x+iy)|\,\sqrt{\frac{\pi}{2}}\int_0^\infty |(F_c\,f)(t)|\,dt\;. \tag{2.2}$$

It is clear that the integral (2.1) converges uniformly for $\operatorname{Re} z = x > 0$. It implies that $\Phi(x+iy)$ is an analytic function in the right-half plane. Moreover, we obtain

$$\sup_{x>0}\left|\frac{\Phi(x+iy)}{\Gamma(x+iy)}\right| = \sqrt{\frac{\pi}{2}}\,\|f\|_{L^*}\;.$$

The main theorem of this section is based on the composition representation of the transform (1.1) through the Kontorovich–Lebedev operator (1.3) and the operator of the Mellin transform (1.2). We give here also its inversion formula by

$$f(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f^{\mathcal{M}}(s)\,x^{-s}\,ds\,, \qquad s=\gamma+it,\quad x>0\;, \tag{2.3}$$

where integrals (1.2), (2.3) exist as Lebesgue integrals or converge in mean by the norm of spaces $L_2(\gamma-i\infty,\gamma+i\infty)$ and $L_2(\mathbb{R}_+; x^{2\gamma-1})$, respectively. In addition, in the latter case the Parseval equality holds

$$\int_0^\infty |f(x)|^2\,x^{2\gamma-1}\,dx \;=\; \frac{1}{2\pi}\int_{-\infty}^\infty |f^{\mathcal{M}}(\gamma+it)|^2\,dt\;. \tag{2.4}$$

Theorem 3. *Let $f \in L^*(\mathbb{R}_+)$. Then for all $\tau \in \mathbb{R}$ the following representation holds*

$$
\begin{aligned}
f(\tau) \;=\; & \frac{1}{4\pi^3}\,\tau\,\sinh(\pi\tau)\lim_{x\to 0+}\int_{-\infty}^{\infty} 2^{x-iy} \\
& \cdot\;\Phi(x+iy)\,\Gamma\Big(\frac{x-i(y+\tau)}{2}\Big)\,\Gamma\Big(\frac{x-i(y-\tau)}{2}\Big)dy\;.
\end{aligned}
\tag{2.5}
$$

Proof: As it follows from (2.2) for each $x > 0$ and $f \in L^*(\mathbb{R}_+)$ the transform $\Phi(x+iy) \in L_1(\mathbb{R})$. Then in view of representation (1.4) and asymptotic of the Macdonald function we can substitute it into (2.5) and invert the order of integration via Fubini's theorem. Hence we obtain the equality

$$
\begin{aligned}
& \frac{1}{\pi^3}\,\tau\,\sinh(\pi\tau)\int_{-\infty}^{\infty} 2^{x-iy-2}\,\Phi(x+iy)\,\Gamma\Big(\frac{x-i(y+\tau)}{2}\Big)\,\Gamma\Big(\frac{x-i(y-\tau)}{2}\Big)\,dy \;= \\
& \quad = \;\frac{1}{\pi^3}\,\tau\sinh(\pi\tau)\int_0^{\infty} K_{i\tau}(t)\,t^{x-1}\,dt\int_{-\infty}^{\infty}\Phi(x+iy)\,t^{-iy}\,dy\;.
\end{aligned}
\tag{2.6}
$$

We show now that the latter integral with respect to y may be evaluated by using the Mellin transform formulas (1.2), (2.3) and the Kontorovich–Lebedev operator (1.3). Precisely, we have

$$
\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(x+iy)\,t^{-x-iy}\,dy \;=\; (KLf)(t)\;. \tag{2.7}
$$

Indeed, the integral of the left-hand side in (2.7) is the inverse Mellin transform of Φ and it exists since $\Phi \in L_1(\mathbb{R})$. On the other hand via condition $f \in L^*(\mathbb{R}_+)$ and the uniform estimate [5] for the Macdonald function

$$
|K_{i\tau}(t)| \;\le\; e^{-\delta|\tau|}K_0(t\cos\delta)\,, \qquad \delta \in (0,\pi/2)\;, \tag{2.8}
$$

where $K_0(z)$ is the Macdonald function of the index zero we easily find that $(KLf)(t) \in L_1(\mathbb{R}_+; t^{x-1}dt)$. This result follows from the estimate

$$
\begin{aligned}
\int_0^{\infty} t^{x-1}|(KLf)(t)|\,dt \;\le\; & \int_0^{\infty} t^{x-1}K_0(t\cos\delta)\,dt\;\int_0^{\infty} e^{-\delta\tau}|f(\tau)|\,d\tau \\
\le\; & C_{x,\delta}\,\|f\|_{L^*}\,, \qquad x>0,\quad \delta\in(0,\pi/2)\;,
\end{aligned}
$$

where

$$
C_{x,\delta} \;=\int_0^{\infty} t^{x-1}K_0(t\cos\delta)\,dt\int_0^{\infty} e^{-\delta\tau}d\tau \;=\; \frac{2^{x-2}}{\delta}\,(\cos\delta)^{-x}\,\Gamma^2\Big(\frac{x}{2}\Big)\;.
$$

Consequently, according to the L_1-theorems for the Mellin transform [4] the composition of operators (1.3), (1.2) $(KLf)^{\mathcal{M}}(x+iy)$ exists. Moreover, it equals $\Phi(x+iy)$ after changing the order of integration in the obtained iterated integral and by using representation (1.4). Therefore we deduce equality (2.7). Hence we substitute its right-hand side in (2.6) and it becomes

$$\frac{1}{\pi^3}\,\tau\sinh(\pi\tau)\int_{-\infty}^{\infty} 2^{x-iy-2}\,\Phi(x+iy)\,\Gamma\Big(\frac{x-i(y+\tau)}{2}\Big)\,\Gamma\Big(\frac{x-i(y-\tau)}{2}\Big)\,dy \;= \\ =\; \frac{2}{\pi^2}\,\tau\sinh(\pi\tau)\int_0^{\infty} K_{i\tau}(t)\,t^{2x-1}(KLf)(t)\,dt\;.$$

The desired result now is an immediate consequence of Theorem 1. Theorem 3 is proved. ∎

3 – Plancherel type theorem

Let us consider transformation (1.1) in L_2-spaces and establish the Plancherel theorem for this transform. As we could see in the previous section for $f\in L^*(\mathbb{R}_+)$ $\Phi(z)$ is an analytic function in the right half-plane. We will prove here that it remains true also for $f\in L_2(\mathbb{R}_+;[\tau\sinh(\pi\tau)]^{-1}d\tau)$ and

$$\Phi(x+iy) \;=\; \operatorname*{l.i.m.}_{N\to\infty}\int_{1/N}^{N} 2^{x+iy-2} f(\tau)\,\Gamma\Big(\frac{x+i(y+\tau)}{2}\Big)\,\Gamma\Big(\frac{x+i(y-\tau)}{2}\Big)\,d\tau\;, \tag{3.1}$$

where the latter limit is in mean square sense. Moreover, it will follow that Φ is an element of the Hardy space $\mathbb{H}_2^{(x_1,x_2)}(\mathbb{R})$, i.e.

$$\sup_{0<x_1\le x\le x_2<\infty}\;\frac{1}{2\pi}\int_{-\infty}^{\infty}|\Phi(x+iy)|^2\,dy \;<\; \infty\;. \tag{3.2}$$

As it is known from the theory of $\mathbb{H}$-spaces (cf. in [4]), $\Phi(x+iy)\to\Phi(iy)$ for almost all y when $x\to 0$ and $\Phi(iy)\in L_2(\mathbb{R})$. Taking into account possible singularities of Gamma-functions for $x=0$ the corresponding integral (3.1) converges in the principal value sense. Indeed, with the reduction formula [2, Vol. I] for Gamma-functions $\Gamma(z+1)=z\,\Gamma(z)$ the transform $\Phi(iy)$ has the form (cf. (3.1))

$$\Phi(iy) \;=\; \operatorname*{l.i.m.}_{N\to\infty}\;\text{P.V.}\int_{1/N}^{N} 2^{iy} f(\tau)\,\frac{\Gamma\Big(\frac{i(y+\tau)}{2}+1\Big)\,\Gamma\Big(\frac{i(y-\tau)}{2}+1\Big)}{\tau^2-y^2}\,d\tau\;. \tag{3.3}$$

Theorem 4. *Integral transform (3.1) is a bounded operator* $\Phi : L_2(\mathbb{R}_+; |\Gamma(2x+i\tau)|^2 d\tau) \to \mathbb{H}_2^{(x_1,x_2)}(\mathbb{R})$ *and for* $x \in [x_1, x_2]$ *we have the inequality*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |\Phi(x+iy)|^2\, dy \le \frac{\pi^{3/2}\, 2^{-2x-1}}{\Gamma(2x+1/2)} \int_0^{\infty} |\Gamma(2x+i\tau)|^2\, |f(\tau)|^2\, d\tau \ . \tag{3.4}$$

In particular, when $x \to 0+$ *it becomes*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |\Phi(iy)|^2\, dy \le \frac{\pi^2}{2}\int_0^{\infty} |f(\tau)|^2 \frac{d\tau}{\tau \sinh(\pi\tau)} \ . \tag{3.5}$$

Proof: First as a consequence of Stirlings formula of the asymptotic at infinity of Gamma-functions and relation (1.14) we see that the variety of spaces $L_2(\mathbb{R}_+; |\Gamma(2x+i\tau)|^2 d\tau)$ is well-ordered. This gives the following embedding

$$L_2\Big(\mathbb{R}_+; |\Gamma(2x_2+i\tau)|^2 d\tau\Big) \subseteq L_2\Big(\mathbb{R}_+; |\Gamma(2x_1+i\tau)|^2 d\tau\Big)\,, \qquad x_2 \ge x_1 > 0 \ .$$

Denoting by $\Phi_N(x+iy)$ the integral (3.1) we have that under conditions of the theorem for each $N > 0$ it exists as the Lebesgue integral. Further, by using representation (1.4) we substitute it in (3.1) and invert the order of integration via the absolute convergence of the corresponding iterated integral. Hence we obtain

$$\Phi_N(x+iy) = \int_0^{\infty} t^{x+iy-1}\, dt \int_{1/N}^{N} K_{i\tau}(t)\, f(\tau)\, d\tau \ . \tag{3.6}$$

The latter integral (3.6) is the composition of the Mellin transform (1.2) and the Kontorovich–Lebedev operator (1.3) of the function f, which is zero outside of the interval $(1/N, N)$. Appealing to inequality (2.8) it is not difficult to verify that the Kontorovich–Lebedev operator in (3.6) is a function also from $L_2(\mathbb{R}_+; t^{2x-1}dt)$. Then as a consequence of the Parseval equality (2.4) we immediately find that

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} |\Phi_N(x+iy)|^2\, dy &= \int_0^{\infty} t^{2x-1}\left|\int_{1/N}^{N} K_{i\tau}(t)\, f(\tau)\, d\tau\right|^2 dt \\ &= \int_0^{\infty} t^{2x-1}\, dt \int_{1/N}^{N} K_{i\tau}(t)\, f(\tau)\, d\tau \int_{1/N}^{N} K_{iu}(t)\, \overline{f(u)}\, du \\ &= \int_{1/N}^{N}\int_{1/N}^{N} f(\tau)\, \overline{f(u)}\, d\tau\, du \int_0^{\infty} t^{2x-1} K_{i\tau}(t)\, K_{iu}(t)\, dt \ .\end{aligned} \tag{3.7}$$

By virtue of the formula (cf. [5])

$$\int_0^{\infty} t^{2x-1} K_{i\tau}(t)\, K_{iu}(t)\, dt = \frac{2^{2x-3}}{\Gamma(2x)}\left|\Gamma\Big(x+\frac{i(u+\tau)}{2}\Big)\, \Gamma\Big(x+\frac{i(\tau-u)}{2}\Big)\right|^2$$

we therefore have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|\Phi_N(x+iy)|^2\,dy \ =$$

$$= \ \frac{2^{2x-3}}{\Gamma(2x)}\int_{1/N}^{N}\int_{1/N}^{N} f(\tau)\,\overline{f(u)}\left|\Gamma\Big(x+\frac{i(u+\tau)}{2}\Big)\,\Gamma\Big(x+\frac{i(\tau-u)}{2}\Big)\right|^2 d\tau\,du\ .$$

Hence via Schwarz inequality for double integrals we deduce

$$\frac{2^{2x-3}}{\Gamma(2x)}\int_{1/N}^{N}\int_{1/N}^{N} f(\tau)\,\overline{f(u)}\left|\Gamma\Big(x+\frac{i(u+\tau)}{2}\Big)\,\Gamma\Big(x+\frac{i(\tau-u)}{2}\Big)\right|^2 d\tau\,du \ \le$$

$$\le \ \frac{2^{2x-3}}{\Gamma(2x)}\int_{1/N}^{N}|f(\tau)|^2\,d\tau\int_{1/N}^{N}\left|\Gamma\Big(x+\frac{i(u+\tau)}{2}\Big)\,\Gamma\Big(x+\frac{i(\tau-u)}{2}\Big)\right|^2 du$$

$$\le \ \frac{\pi^{3/2}\,2^{-2x-1}}{\Gamma(2x+1/2)}\int_{1/N}^{N}|\Gamma(2x+i\tau)|^2\,|f(\tau)|^2\,d\tau\ ,$$

where we have used the value of the following integral [5], [6]

$$\int_0^{\infty}\left|\Gamma\Big(x+\frac{i(u+\tau)}{2}\Big)\,\Gamma\Big(x+\frac{i(\tau-u)}{2}\Big)\right|^2 du \ = \ \frac{\pi^{3/2}\,2^{2-4x}\,\Gamma(2x)}{\Gamma(2x+1/2)}\,|\Gamma(2x+i\tau)|^2, \quad x>0\,.$$

Combining with (3.10) we obtain that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|\Phi_N(x+iy)|^2\,dy \ \le \ \frac{\pi^{3/2}\,2^{-2x-1}}{\Gamma(2x+1/2)}\int_{1/N}^{N}|\Gamma(2x+i\tau)|^2\,|f(\tau)|^2\,d\tau\ .$$

Consequently it is easily seen that if $f\in L_2(\mathbb{R}_+;|\Gamma(2x+i\tau)|^2d\tau)$ then limit (3.1) exists and equals $\Phi(x+iy)$. Furthermore, the corresponding transformation (3.1) belongs to the Hardy space (3.2) and inequality (3.4) holds. Passing to the limit through (3.4) when $x\to 0+$ we get (3.5). Theorem 4 is proved. ∎

Finally we are ready to prove the Plancherel theorem for the Φ-transform (3.3).

Theorem 5. *Let $f\in L_2(\mathbb{R}_+;[\tau\sinh(\pi\tau)]^{-1}d\tau)$. Then integral (3.3) converges in mean with respect to the norm of $L_2(\mathbb{R})$ to $\Phi(iy)$. Conversely, the integral*

$$f_N(\tau) \ = \ \frac{1}{4\pi^3}\,\tau\sinh(\pi\tau)\ \text{P.V.}\int_{-N}^{N}2^{-iy}\,\Phi(iy)\ \overline{\Gamma\Big(\frac{i(y+\tau)}{2}\Big)\,\Gamma\Big(\frac{i(y-\tau)}{2}\Big)}\,dy \tag{3.8}$$

converges in mean to $f(\tau)$ with respect to the norm of the space $L_2(\mathbb{R}_+;[\tau\sinh(\pi\tau)]^{-1}d\tau)$. Moreover, inequality (3.5) takes the form of the Parseval equality for the Φ-transformation (3.3)

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|\Phi(iy)|^2\,dy \ = \ \frac{\pi^2}{2}\int_0^{\infty}|f(\tau)|^2\,\frac{d\tau}{\tau\sinh(\pi\tau)}\ . \tag{3.9}$$

Proof: As it follows from Theorem 2 the latter equality (3.9) holds true due to Parseval equality (1.16). In fact, if $f \in L_2(\mathbb{R}_+; [\tau \sinh(\pi\tau)]^{-1} d\tau)$ and vanishes outside of the interval $(1/N, N)$ then the integral in the right-hand side of the first equality in (3.7) with x replaced by 0, is convergent via Fatou's lemma. Further, by virtue of estimate (2.8) we have that integral (1.15) belongs to $L_2(\mathbb{R}_+)$. Meanwhile, for all x, $0 \le x \le \frac{1}{2}$,

$$\int_0^\infty t^{2x-1} |(KLf)_N(t)|^2 \, dt \le \int_0^\infty \Big[\chi_{[0,1]}(t) + t\,\chi_{[1,\infty)}(t)\Big] \, |(KLf)_N(t)|^2 \, \frac{dt}{t} < \infty \, ,$$

where $\chi_{(a,b)}(t)$ is the characteristic function of the corresponding interval. Hence the limit $x \to 0$ in the first equality of (3.7) can be taken under the integral sign by virtue of Theorem 4 and the Lebesgue dominated convergence theorem. Therefore via equality (1.17) we obtain

$$\frac{1}{2\pi} \int_{-\infty}^\infty |\Phi_N(iy)|^2 \, dy = \frac{\pi^2}{2} \int_{1/N}^N |f(\tau)|^2 \, \frac{d\tau}{\tau \, \sinh(\pi\tau)} \, .$$

Now for the difference $\Phi_N(iy) - \Phi_M(iy)$ we easily majorize as

$$\frac{1}{2\pi} \int_{-\infty}^\infty |\Phi_N(iy) - \Phi_M(iy)|^2 \, dy = \frac{\pi^2}{2} \left[\int_{1/M}^{1/N} + \int_N^M \right] |f(\tau)|^2 \, \frac{d\tau}{\tau \, \sinh(\pi\tau)} \, . \tag{3.10}$$

Since the right-hand side of (3.10) tends to zero as $M \to \infty$, $N \to \infty$, so does the left-hand side. That is, $\Phi_N(iy)$ converges in mean to a function, $\Phi(iy)$ say, of the class $L_2(\mathbb{R})$.

Further, let us establish that $f_N(\tau) \in L_2(\mathbb{R}_+; [\tau \sinh(\pi\tau)]^{-1} d\tau)$. This implies that the corresponding square of norm

$$\begin{aligned} &\|f_N\|^2_{L_2(\mathbb{R}_+; [\tau \sinh(\pi\tau)]^{-1} d\tau)} = \\ &= \frac{1}{4\pi^3} \int_0^\infty \tau \, \sinh(\pi\tau) \left| \int_{-N}^N 2^{-iy} \, \Phi(iy) \, \overline{\Gamma\Big(\frac{i(y+\tau)}{2}\Big) \Gamma\Big(\frac{i(y-\tau)}{2}\Big)} \, dy \right|^2 d\tau \\ &< \infty \, . \end{aligned} \tag{3.11}$$

By taking sufficiently large $X > 0$ we split up the latter integral with respect to τ on two integrals over $[0, X]$ and $[X, \infty)$ which we denote correspondingly by $I_1(X)$ and $I_2(X)$. Keeping X fixed we observe that the integral $I_1(X)$ is convergent. Indeed, first we see that the inner integral with respect to y is uniformly convergent when $|y \pm \tau| \ge \delta > 0$. Secondly it can be represented through the Hilbert type integrals (cf. in [4] and (3.3)) in domains of integration $|y \pm \tau| < \delta$ which are

$L_2(0,X)$-functions. Therefore the integrand (3.11) is a summable function over $[0,X],\ X>0$.

Concerning the convergence of the integral $I_2(X)$ let us apply the Stirling formula of the asymptotic behavior of Gamma-functions [2] when $\tau\to+\infty$ and $y\in[-N,N]$. In fact, this gives the following asymptotic relation for the kernel (3.8)

$$\Gamma\Big(\frac{-i(y+\tau)}{2}\Big)\,\Gamma\Big(\frac{i(\tau-y)}{2}\Big) = \frac{2\pi}{\tau}\exp\Big[-\frac{\pi}{2}\tau - iy\log\tau - \frac{iy^2}{\tau} + O\Big(\frac{(1+iy)\,y^2}{8\,\tau^2}\Big)\Big] \cdot\Big(1+O\Big(\frac{y}{y^2-\tau^2}\Big)\Big),\qquad \tau\to+\infty,\ \ y\in[-N,N]\ .$$

Consequently, by using the substitution $u=\log\tau$ and the Parseval equality for Fourier transform we obtain

$$\begin{aligned} I_2(X) &= \frac{1}{4\pi^3}\int_X^\infty \tau\sinh(\pi\tau)\left|\int_{-N}^N 2^{-iy}\,\Phi(iy)\,\overline{\Gamma\Big(\frac{i(y+\tau)}{2}\Big)\,\Gamma\Big(\frac{i(y-\tau)}{2}\Big)}\,dy\right|^2 d\tau\\ &= O\left(\int_X^\infty \frac{d\tau}{\tau}\left|\int_{-N}^N 2^{-iy}\,\Phi(iy)\,e^{-iy\log\tau}\,dy\right|^2\right)\\ &= O\left(\int_{\log X}^\infty du\left|\int_{-N}^N 2^{-iy}\,\Phi(iy)\,e^{-iyu}\,dy\right|^2\right)\\ &\le \int_{-N}^N |\Phi(iy)|^2\,dy \ <\ \infty\ . \end{aligned}$$

Thus $f_N(\tau)\in L_2(\mathbb{R}_+;[\tau\sinh(\pi\tau)]^{-1}d\tau)$ and from (3.9) we have that there exists $\mathrm{l.i.m.}_{N\to\infty} f_N(\tau)=\varphi(\tau)$. We may prove now that $\varphi(\tau)=f(\tau)$ almost everywhere on $\mathbb{R}_+$. Indeed, since both functions are from the space $L_2(\mathbb{R}_+;[\tau\sinh(\pi\tau)]^{-1}d\tau)$ it is easily seen that functions $\varphi(\tau)[\sinh(\pi\tau)]^{-1},\ f(\tau)[\sinh(\pi\tau)]^{-1}\in L_1(\mathbb{R}_+)$. Then it is sufficient to show that

$$\int_0^\xi \frac{\varphi(\tau)}{\sinh(\pi\tau)}\,d\tau = \int_0^\xi \frac{f(\tau)}{\sinh(\pi\tau)}\,d\tau \tag{3.12}$$

for all values of $\xi>0$. If we take two functions $f(\tau),g(\tau)$ and their transformations (3.3) $\Phi(iy),G(iy)$ respectively, then the Parseval formula (3.9) gives

$$\frac{1}{2\pi}\int_{-\infty}^\infty \Phi(iy)\,\overline{G(iy)}\,dy = \frac{\pi^2}{2}\int_0^\infty f(\tau)\,\overline{g(\tau)}\,\frac{d\tau}{\tau\sinh(\pi\tau)}\ .$$

Let $g(\tau)=\tau\ (0\le\tau\le\xi),\ \ g(\tau)=0\ (\tau>\xi)$. Then

$$\int_0^\xi \frac{f(\tau)}{\sinh(\pi\tau)}\,d\tau = \frac{1}{4\pi^3}\int_{-\infty}^\infty 2^{-iy}\,\Phi(iy)\int_0^\xi \tau\Gamma\Big(\frac{-i(y+\tau)}{2}\Big)\,\Gamma\Big(\frac{i(\tau-y)}{2}\Big)\,d\tau\,dy\,. \tag{3.13}$$

The latter integral exists as the Lebesgue integral since for each $\xi > 0$

$$\int_0^\xi \tau\Gamma\left(\frac{-i(y+\tau)}{2}\right)\Gamma\left(\frac{i(\tau-y)}{2}\right)d\tau \ \in \ L_2(\mathbb{R}) \ .$$

This fact can be established similar to the above discussions. On the other hand integrating through in equality (3.8) and inverting the order of integration we obtain that

$$\int_0^\xi \frac{f_N(\tau)}{\sinh(\pi\tau)}\,d\tau = \frac{1}{4\pi^3}\int_{-N}^N 2^{-iy}\,\Phi(iy)\int_0^\xi \tau\,\overline{\Gamma\left(\frac{i(y+\tau)}{2}\right)\Gamma\left(\frac{i(y-\tau)}{2}\right)}\,d\tau\,dy\,. \tag{3.14}$$

We motivate this interchange by splitting up integral (3.8) on four integrals. As the result we obtain

$$\begin{aligned}\int_0^\xi \frac{f_N(\tau)}{\sinh(\pi\tau)}\,d\tau \;&=\; \frac{1}{4\pi^3}\int_0^\xi \tau\,d\tau\left[\int_{|y\pm\tau|\ge\delta}+\int_{|y\pm\tau|<\delta}\right]2^{-iy}\,\Phi(iy)\\ &\qquad\cdot\,\overline{\Gamma\left(\frac{i(y+\tau)}{2}\right)\Gamma\left(\frac{i(y-\tau)}{2}\right)}\,dy\\ &=\; J_1+J_2+J_3+J_4\ .\end{aligned}$$

The inner integrals in J_1, J_2 converge uniformly by τ. This means that we can integrate with respect to τ. Meanwhile, in the case of integrals J_3, J_4 the order of integration may be inverted due to the simple case of the Poincaré–Bertrand formula [3]. Precisely, here we have two repeated integrals, which are compositions of singular integrals (see above) and absolutely convergent integrals.

Finally, passing to the limit in (3.14) when $N\to\infty$ and taking into account (3.13) we prove (3.12). Then by the differentiation with respect to ξ in (3.12) we justify the equality $\varphi(\tau) = f(\tau)$ for almost all $\tau\in\mathbb{R}_+$. Theorem 5 is proved. ∎

ACKNOWLEDGEMENT – The present investigation was supported, in part, by the "Centro de Matemática" of the University of Porto. The author sincerely indebted to the referee for the valued remarks, which rather improved the presentation of the paper.

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