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# From extreme values of i.i.d. random fields to extreme eigenvalues of finite-volume Anderson Hamiltonian

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**Abstract:** The aim of this paper is to study asymptotic geometric properties almost surely or/and in probability of extreme order statistics of an i.i.d. random field (potential) indexed by sites of multidimensional lattice cube, the volume of which unboundedly increases. We discuss the following topics: (I) high level exceedances, in particular, clustering of exceedances; (II) decay rate of spacings in comparison with increasing rate of extreme order statistics; (III) minimum of spacings of successive order statistics; (IV) asymptotic behavior of values neighboring to extremes and so on. The conditions of the results are formulated in terms of regular variation (RV) of the cumulative hazard function and its inverse. A relationship between RV classes of the present paper as well as their links to the well-known RV classes (including domains of attraction of max-stable distributions) are discussed.

The asymptotic behavior of functionals (I)–(IV) determines the asymptotic structure of the top eigenvalues and the corresponding eigenfunctions of the large-volume discrete Schrödinger operators with an i.i.d. potential (Anderson Hamiltonian). Thus, another aim of the present paper is to review and comment a recent progress on the extreme value theory for eigenvalues of random Schrödinger operators as well as to provide a clear and rigorous understanding of the relationship between the top eigenvalues and extreme values of i.i.d. random potentials. We also discuss their links to the long-time intermittent behavior of the parabolic problems associated with the Anderson Hamiltonian via spectral representation of solutions.

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## Contents

## 1. Introduction

### 1.1. Extremes of i.i.d. random fields

In this paper, we assume that $\xi(x)$, $x \in \mathbb{Z}^\nu$, are independent identically distributed (i.i.d.) random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, indexed by sites of the $\nu$-dimensional integer lattice $\mathbb{Z}^\nu$, with a distribution function $\mathbb{P}(\xi(0) \leqslant t) =: 1 - \mathrm{e}^{-Q(t)}$, $t \in \mathbb{R}$; here $Q$ denotes the cumulative hazard function of distribution. Define $V = [-n; n]^\nu \cap \mathbb{Z}^\nu$, the cubes in $\mathbb{Z}^\nu$. Let $|V|$ denote the number of sites in $V$. We write $|x| = \sum_{i=1}^{\nu} |x^i|$ for the lattice $l^1$-distance between $x = (x^1, \ldots, x^\nu) \in \mathbb{Z}^\nu$ and $0 \in \mathbb{Z}^\nu$.

We consider the variational series (order statistics)

$$\xi_{1,V} := \xi(z_{1,V}) \geqslant \xi_{2,V} := \xi(z_{2,V}) \geqslant \ldots \geqslant \xi_{|V|,V} := \xi(z_{|V|,V}) \tag{1.1}$$

based on the sample $\xi_V := \{\xi(x)\colon\ x \in V\}$; here $V = \{z_{k,V}\colon\ 1 \leqslant k \leqslant |V|\}$. The first $|V|^\varepsilon$ $(0 < \varepsilon < 1)$ terms of the variational series (1.1) are referred to as $\xi_V$-extremes or $\xi_V$-peaks. The coordinate $z_{k,V} \in V$ stands for a location of the $k$th extreme value of $\xi_V$; $1 \leqslant k \leqslant |V|$.

In this paper, letting $|V| \to \infty$, we study the asymptotic geometric properties of $\xi_V$-extremes almost surely and/or in probability. We are interested in the following functionals of order statistics (1.1):

- **(EX)** *Exceedances* of the sample $\xi_V$ over high levels $L_V$, in particular, clustering of exceedances (Theorem 3.1).
- **(SP)** *The decay rate of the spacings* $\xi_{K,V} - \xi_{K+1,V}$ and $\xi_{[|V|^\varepsilon],V} - \xi_{[|V|^\theta],V}$ in comparison with increasing rate of $\xi_{K,V}$ for fixed natural $K \in \mathbb{N}$ and $0 \leqslant \varepsilon < \theta < 1$ (Theorems 4.3–4.7).
- **(MIN)** *Minimum of the spacings* $\xi_{l,V} - \xi_{l+1,V}$, $1 \leqslant l \leqslant |V|^\varepsilon$, for each $0 < \varepsilon < 1$ (Theorems 4.8 and 4.9).
- **(NEI)** *$\xi_V$-values neighboring to $\xi_V$-extremes*, in particular, $\xi(z_{l,V} + y)$ for $1 \leqslant l \leqslant |V|^\varepsilon$ and for fixed $y \neq 0$ (Lemma 5.1 and Theorems 5.3, 5.4).

The conditions of the asymptotic results for (EX), (SP), (MIN) and (NEI) are given in terms of regular variation (RV) of the inverse function of $Q$. In Appendix A and Section 6, we discuss a relationship between RV classes of the present paper as well as their links to the well-known RV classes including domains of attraction of max-stable distributions.

The asymptotic results for (EX), (SP), (MIN) and (NEI) and related RV classes were announced without the proof in (Astrauskas, 2007; 2008; 2012; 2013). In this survey, these results are given in the most general setting with the detailed proof; therefore, they present self-contained topics of probability theory and may be considered of independent interest.

### 1.2. *Extreme value theory for eigenvalues of large-volume Anderson Hamiltonians*

Let us consider the finite-volume Schrödinger operators $\mathcal{H}_V = \kappa\Delta_V + \xi_V$ on $l^2(V)$ with periodic boundary conditions (Anderson Hamiltonian); here $\kappa > 0$ is a diffusion constant; $\Delta\psi(x) := \sum_{|y-x|=1} \psi(y)$ is the lattice Laplacian, and the i.i.d. random field $\xi_V := \{\xi(x)\colon\ x \in V\}$ is the multiplication operator (potential). Denote by $\lambda_{K,V}$ the $K$th largest eigenvalue of the operators $\mathcal{H}_V$, and let $\psi(x; \lambda_{K,V})$ $(x \in V)$ be the corresponding eigenfunction normalized to have unit $l^2$-norm, $\sum_{x\in V} \psi(x; \lambda_{K,V})^2 = 1$. Another aim of this paper is to show in what manner the asymptotic behavior of functionals (EX), (SP), (MIN) and

(NEI) determines the asymptotic structure of the top eigenvalues $\lambda_{K,V}$ and the corresponding eigenfunctions, as $V \uparrow \mathbb{Z}^\nu$ and $K \geqslant 1$ fixed. In Section 2, we give an overview of rigorous statements on this relationship which are proved in the papers by Gärtner and Molchanov (1998) and Astrauskas (2007; 2008; 2012). In Section 6, we review and comment results on the asymptotic expansion formulas and Poisson limit theorems for the largest eigenvalues $\lambda_{K,V}$ as well as localization properties of the corresponding eigenfunctions. These results are proved by Astrauskas and Molchanov (1992), Gärtner and Molchanov (1998), Astrauskas (2007; 2008; 2012; 2013), Germinet and Klopp (2013), Biskup and König (2016) and other mathematicians. These papers are complemented by the present survey on the asymptotic geometric properties of $\xi_V$-extremes and related RV classes of distributions. We here give proof sketches of the results on the extreme value theory for eigenvalues $\lambda_{K,V}$ (Sections 2 and 6) demonstrating their connections to asymptotic properties of $\xi_V$-extremes.

Thus, in this survey, we discuss in detail the following important branches of probability theory: (i) extreme value theory for eigenvalues of the Anderson Hamiltonian $\mathcal{H}_V = \kappa\Delta_V + \xi_V$ which is a particular model of random matrices (Sections 1.4, 2 and 6); (ii) asymptotic geometric properties of random i.i.d. fields (Sections 3–5) and (iii) regular variation of distribution functions (Appendix A). On the other hand, we briefly comment the links of the extreme value theory for eigenvalues to the following important topics of statistical physics: (iv) Anderson localization for the random Schrödinger operators $\mathcal{H} = \kappa\Delta + \xi(\cdot)$ in the whole lattice $\mathbb{Z}^\nu$ (Section 1.3), and (v) long-time intermittent behavior of solutions $u$ of the parabolic problems associated with the Anderson Hamiltonian (PAM) via spectral representation of $u$ (Section 7).

Of course, asymptotic results for the Anderson models (time-dependent or -independent) are heavily necessitated by the asymptotic structure of high $\xi(\cdot)$-values, which in turn is determined by conditions on the regularity and tail decay of the distribution $\mathbb{P}(\xi(0) > t) = \mathrm{e}^{-Q(t)}$ at its right endpoint $t_Q := \mathit{esssup}\ \xi(0)$. In the present survey, we focus on the case of unbounded from above i.i.d. potentials, i.e. $t_Q := \infty$, with distributional tails heavier than double exponential, i.e., $\mathbb{P}(\xi(0) > t) = \exp\{-\mathrm{e}^{\mathrm{o}(t)}\}$ as $t \to \infty$; cf. Sections 1.2, 2.2–2.3 and 6.1. For such distributions satisfying additional RV and continuity conditions, we will show that with probability one the $\xi_V$-peaks are spatially separated and differ in height as $V \uparrow \mathbb{Z}^\nu$; therefore, the top eigenvalue $\lambda_{K,V}$ of the operator $\mathcal{H}_V = \kappa\Delta_V + \xi_V$ is approximated by an isolated $\xi_V$-peak, say $\xi(z_{\tau(K),V})$, plus some corrections of order $\mathrm{o}(1)$ involving neighboring $\xi_V$-values. Moreover, the $K$th eigenfunction $\psi(\cdot\,; \lambda_{K,V})$ is asymptotically delta like function at the site $z_{\tau(K),V} \in V$, the localization center. (In this case, we will say that the $K$th eigenvalue is associated with the site $z_{\tau(K),V}$, viz. $\lambda_{K,V} \leftrightarrow z_{\tau(K),V}$). Therefore, we are able to apply the standard extreme value theory to prove Poisson limit theorems for the normalized extreme eigenvalues and their localization centers. From these Poisson limit theorems one obtains the limiting joint (max-stable) distribution for the normalized largest eigenvalues and their spacings, limiting uniform distribution for the normalized localization centers and other important limiting distributions for eigenvalue statistics (Section 6.1). Eigenvalue statistics in turn play a crucial role in studying the intermittent behavior of the parabolic Anderson model, PAM (Section 7).

For the lighter upper tails including the double exponential $\mathbb{P}(\xi(0) > t) = \exp\{-\mathrm{e}^t\}$ and bounded tails i.e. $t_Q < \infty$, we will prove the rough asymptotic expansion formulas for the largest eigenvalues. For such distributional tails, it will turn out that all $\xi_V$-extremes are of comparable amplitude; therefore, the $K$th largest eigenvalue is associated with a large island of higher $\xi_V$-values of a particular preferred shape, rather than an isolated $\xi_V$-peak.

To illustrate the relationship between $\xi_V$-extremes and the largest eigenvalues $\lambda_{K,V}$ $(V \uparrow \mathbb{Z}^\nu)$ more precisely, we now formulate Propositions 1.1–1.3 which are "typical" examples of the statements given in Sections 2–6. The first proposition tells us that, if the peaks of *deterministic (nonrandom) functions* $\xi_V =: \xi_V(\cdot)$ are extremely sharp and widely spaced, then the $K$th largest eigenvalue $\lambda_{K,V}$ is approximated by the $K$th largest value of $\xi_V$ with sufficiently small error.

**Proposition 1.1** (see Theorem 2.2(ii) in Section 2.2)**.** *Fix constants $K \in \mathbb{N}$ and $0 < \theta < 1/2$, and assume that the deterministic functions $\xi_V$ satisfy the following conditions as $V \uparrow \mathbb{Z}^\nu$:*

$$\min_{1\leqslant l\leqslant K} \xi_{l+1,V}(\xi_{l,V} - \xi_{l+1,V}) \to \infty \quad \textit{(distinct height of peaks)}, \tag{1.2}$$

$$\frac{1}{\log|V|} \min_{1\leqslant k<n\leqslant|V|^\theta} \left|z_{k,V} - z_{n,V}\right| \to \infty \quad \textit{(sparseness of peaks)} \tag{1.3}$$

*and, finally,*

$$\xi_{[|V|^\theta],V}/\xi_{K,V} < \mathrm{const}\,(\theta) \tag{1.4}$$

*for some $0 < \mathrm{const}\,(\theta) < 1$ (negligibility of the lower peaks). Then*

$$\lambda_{l,V} = \xi_{l,V} + \mathrm{O}(1/\xi_{l,V}) \quad \textit{for all} \quad 1 \leqslant l \leqslant K.$$

We now give an example of i.i.d. random field $\xi(\cdot)$ with sufficiently "heavy tails" possessing extremes like those in Proposition 1.1.

**Proposition 1.2** (see Theorem 4.3(i) with $p = 1$, Theorem 3.1 with $R = 0$ and Theorem 4.5)**.** *If $\xi(0)$ has the Weibull distribution*

$$\mathbb{P}(\xi(0) > t) = \mathrm{e}^{-Q(t)} = \mathrm{e}^{-t^\alpha} \quad (t \geqslant 0) \tag{1.5}$$

*with $\alpha < 2$, then the i.i.d. sample $\xi_V$ $(V \uparrow \mathbb{Z}^\nu)$ satisfies (1.2)–(1.4) with probability $1 + \mathrm{o}(1)$.*

Since, by Propositions 1.1 and 1.2, the eigenvalues $\lambda_{K,V}$ are very close to $\xi_{K,V}$ as $V \uparrow \mathbb{Z}^\nu$, it turns out that Poisson limit theorems (and the corresponding renormalization constants) for the largest eigenvalues are the same as that for $\xi_V$-extremes according to the following proposition.

**Proposition 1.3** (see Theorem 6.9 and Astrauskas (2012))**.** *Assume that* $Q(t) = t^{\alpha}$ *with* $\alpha < 2$*, and write* $b_V := (\log |V|)^{1/\alpha}$*. Define the point process* $\mathcal{N}_V^{\lambda}$ *on* $[-1/2; 1/2]^{\nu} \times \mathbb{R}$ *by*

$$\mathcal{N}_V^{\lambda} := \sum_{k=1}^{|V|} \delta_{\Lambda_V(k)} \quad \textit{with} \quad \Lambda_V(k) := \left( \frac{z_{k,V}}{|V|^{1/\nu}}, \frac{\lambda_{k,V} - b_V}{\alpha^{-1} b_V^{1-\alpha}} \right),$$

*where* $\delta_X$ *denotes the Dirac measure at* $X \in [-1/2; 1/2]^{\nu} \times \mathbb{R}$*. Then* $\mathcal{N}_V^{\lambda}$ *converges weakly to the Poisson process on* $[-1/2; 1/2]^{\nu} \times \mathbb{R}$ *with the intensity measure* $\mathrm{d}\, x \times \mathrm{e}^{-t} \mathrm{d}\, t$*.*

*Moreover, for fixed* $K \geqslant 1$*, the eigenfunction* $\psi(\cdot; \lambda_{K,V})$ *is exponentially localised at the site* $z_{K,V}$*:*

$$\limsup_{V} \max_{x \neq z_{K,V}} \frac{\log \left| \psi(x; \lambda_{K,V}) \right|}{|x - z_{K,V}| \log b_V} \leqslant -1 \tag{1.6}$$

*in probability*

According to Proposition 1.3 the $K$th largest eigenvalue $\lambda_{K,V}$ is associated with the $K$th largest value of $\xi_V$, viz., $\lambda_{K,V} \leftrightarrow z_{K,V}$. For the lighter tails, say, Weibull distributions (1.5) with $\alpha \geqslant 2$, the landscape of $\xi_V$ gets "smoother", in particular, (1.2) fails. Therefore, $\lambda_{K,V}$ is associated with a lower and "slightly supported" $\xi_V$-peak, viz., $\lambda_{K,V} \leftrightarrow z_{\tau(K),V}$, where for $\alpha > 3$, the index $\tau(K) = \tau_V(K)$ tends to infinity as $|V| \to \infty$. This in turn implies that further terms in expansion for $\lambda_{K,V}$ become essential; see (2.22) and Examples 6.12–6.13. Let us distinguish three classes (J)–(JJJ) of light tailed distributions (i.e., *universality classes*), which ensure a different asymptotic behavior of the eigenvalues $\lambda_{K,V}$.

**(J)** *Distribution tails heavier than the double exponential function.* Assume that

$$\log Q(t) = \mathrm{o}(t) \tag{1.7}$$

and $Q$ satisfies additional regularity and continuity conditions as $t \to \infty$. This class is presented by Weibull distributions (1.5) for arbitrary $\alpha > 0$ and those with fractional-double-exponential tails

$$\mathbb{P}(\xi(0) > t) = \mathrm{e}^{-Q(t)} = \exp\{- \mathrm{e}^{t^{\gamma}}\} \quad (t \geqslant t_0) \tag{1.8}$$

for $\gamma < 1$. For such distributions, $\xi_V$-extremes possess a *strongly pronounced geometric structure* which can be described as follows:

For arbitrary sufficiently small constants $0 < \varepsilon < \theta$, there exist constants $c_1 > c_2 > 0$ and (large) $C > 0$ such that almost surely

$$\min_{1 \leqslant l < n \leqslant |V|^{\theta}} \left( \xi_{l,V} - \xi_{n,V} \right) \geqslant \mathrm{e}^{-|V|^{c_2}} \quad \text{(distinct height of peaks)}, \tag{1.9}$$

$$\min_{1 \leqslant l < n \leqslant |V|^{\theta}} \left| z_{l,V} - z_{n,V} \right| \geqslant |V|^{c_1} \quad \text{(sparseness of peaks)} \tag{1.10}$$

and, finally,

$$\xi_{[|V|^{\varepsilon}],V} - \xi_{[|V|^{\theta}],V} \geqslant C \quad \text{(negligibility of the lower peaks)} \tag{1.11}$$

for each large $V$; see Theorem 4.8 with $\kappa = 0$, Theorem 3.1 with $R = 0$ and Theorem 4.6 with $\rho = \infty$. By the standard finite-rank perturbation arguments in (Astrauskas and Molchanov, 1992) and (Astrauskas, 2008), these properties of $\xi_V$ yield that there is no resonance between $\xi_V$-peaks in the Anderson model for large $V$; therefore, the eigenvalues associated with a block of peaks can be determined by the local eigenvalues associated with separate peaks (i.e., *"relevant single peak" approximation*). More precisely, for fixed natural $K$ and $V \uparrow \mathbb{Z}^\nu$, almost surely the eigenvalue $\lambda_{K,V}$ of $\mathcal{H}_V = \kappa\Delta_V + \xi_V$ is approximated by the principal (i.e., the first largest) eigenvalue of the "single peak" Hamiltonian $\kappa\Delta_V + \widetilde{\xi}(\cdot) + \xi(z_{\tau(K),V})\delta_{z_{\tau(K),V}}$ where $\log \tau(K) = \mathrm{o}(\log |V|)$. Here $\widetilde{\xi}(\cdot)$ is the "noise" potential; the site $z_{\tau(K),V} \in V$ is a localization center of the $K$th eigenfunction $\psi(\cdot; \lambda_{K,V})$ of $\mathcal{H}_V$. Thus, Poisson limit theorems for the eigenvalues $\lambda_{K,V}$ of $\mathcal{H}_V$ are reduced to those for the principal eigenvalues of the "single peak" Hamiltonians, which in turn are expanded into certain (nonlinear) series in $\xi(x)$ $(x \in V)$; cf. formulas (2.20)–(2.22), Theorems 2.3, 6.2 and discussions in Section 6.4. We finally notice that the $K$th eigenfunction $\psi(\cdot; \lambda_{K,V})$ is exponentially well localized, i.e., there exist non-random constants (decay rates) $C > 0$ and $0 < M_V \to \infty$ such that with probability one

$$|\psi(x; \lambda_{K,V})| \leqslant C \exp\{-M_V |x - z_{\tau(K),V}|\} \quad (x \in V) \tag{1.12}$$

for all $V$ large enough. Consequently, $\psi(\cdot; \lambda_{K,V})$ is asymptotically delta-like function at $z_{\tau(K),V}$ (Astrauskas, 2008; 2013). This refers to the correspondence $\lambda_{K,V} \leftrightarrow z_{\tau(K),V}$. Under assumption (1.7), asymptotic expansion formulas, Poisson limit theorems and localization theorems for the largest eigenvalues are derived by Astrauskas and Molchanov (1992), Astrauskas (2007; 2008; 2012; 2013). See also Grenkova et al. (1983) and Grenkova et al. (1990) for the case of Weibull distribution (1.5) with $\alpha < 2$.

**(JJ)** *Distribution tails lighter than the double exponential function.* Assume that

$$t^{-1} \log Q(t) \to \infty \tag{1.13}$$

and $Q$ satisfies additional regularity conditions as $t$ tends to $t_Q$ (= the right endpoint of $Q$). This class of potentials contains the important case of $\xi(\cdot)$ which is bounded from above ($t_Q < \infty$) and those with fractional-double-exponential tails (1.8) for $\gamma > 1$. For such $\xi(\cdot)$, it turns out that $\xi_V$-peaks possess a *weakly pronounced geometric structure*. In particular, almost surely $\xi_{[|V|^{\varepsilon}],V} - \xi_{[|V|^{\theta}],V} \to 0$ as $|V| \to \infty$, for all $0 \leqslant \varepsilon < \theta < 1$, so that the height of all $\xi_V$-extremes is of the same order $\xi_{1,V} + \mathrm{o}(1)$ (see Theorem 4.6 with $\rho = 0$ and Theorem 3.1(i) with arbitrary $R \geqslant 1$ and $\theta(\cdot) \equiv \theta = \text{const}$). In this case, the eigenvalue $\lambda_{K,V}$ $(K \geqslant 1$ fixed) does not longer correspond to an isolated potential peak, but to a flat extremely large "relevant island" of high $\xi_V$-values. More precisely, almost surely the top eigenvalue $\lambda_{K,V}$ of the Hamiltonian $\mathcal{H}_V$ approaches the local principal eigenvalue of the Hamiltonian restricted to a random connected region $\mathbb{A}^K_{V;\mathrm{opt}} \subset V$ with the following features: The diameter of $\mathbb{A}^K_{V;\mathrm{opt}}$ unboundedly increases, and $\xi(\cdot)$ possesses in $\mathbb{A}^K_{V;\mathrm{opt}}$ values of the order $\xi_{1,V} + \mathrm{o}(1)$ as $V \uparrow \mathbb{Z}^\nu$, i.e., relevant island of potential values. Moreover, the $K$th eigenfunction is expected to be highly concentrated in the neighborhood of the region $\mathbb{A}^K_{V;\mathrm{opt}}$. In this case, we will say that the $K$th eigenvalue is associated with $\mathbb{A}^K_{V;\mathrm{opt}}$, viz. $\lambda_{K,V} \leftrightarrow \mathbb{A}^K_{V;\mathrm{opt}}$. For more explanations, see Theorem 6.14 and the proof of Theorem 2.6, where the second order expansion formula for $\lambda_{K,V}$ is obtained.

For the Bernoulli i.i.d. random variables $\xi(x)$ with $t_Q = 1$ (which is the particular case of (1.13)), Bishop and Wehr (2012) derived a more accurate expansion formula for the principal eigenvalue $\lambda_{1,V}$ of the one-dimensional Hamiltonian $\mathcal{H}_V$ ($\nu = 1$). They have showed that $\lambda_{1,V}$ is associated with the longest consecutive sequence of sites $x \in V$ with $\xi(x) = 1$, i.e., the "relevant island" of $\xi_V$-extremes, the length of which unboundedly increases as $V \uparrow \mathbb{Z}$. See also (Sznitman, 1998) for similar asymptotic results in the case of spatially continuous Schrödinger operators with a bounded Poisson potential of obstacles. Cf. Section 6.2 below.

To the best of our knowledge, Poisson limit theorems for the (unfolded) largest eigenvalues were proved only in the case $\nu = 1$ and bounded $\xi(0)$, provided the distribution $1 - \mathrm{e}^{-Q}$ satisfies additional continuity and tail decay conditions (Germinet and Klopp, 2013); see also Section 6.2 below. For $\nu \geqslant 2$ or general RV conditions on $Q$ satisfying (1.13), the Poissonian convergence of the top eigenvalues still remains an open problem.

**(JJJ)** *Double exponential type tails.* Finally, assume that $t^{-1}\log Q(t)$ tends to a positive finite constant $\rho^{-1}$ as $t \to \infty$, i.e., the double exponential tails

$$\mathbb{P}(\xi(0) > t) = \mathrm{e}^{-Q(t)} = \exp\{-\mathrm{e}^{t(\rho^{-1}+\mathrm{o}(1))}\} \tag{1.14}$$

satisfying additional RV and continuity conditions. This class of distributions presents the intermediate case between (J) and (JJ). For such $\mathrm{e}^{-Q}$, it turns out that all $\xi_V$-extremes are of comparable amplitude; i.e., almost surely $\xi_{[|V|^\varepsilon],V} - \xi_{[|V|^\theta],V} = \mathrm{O}(1)$ as $|V| \to \infty$, for any $0 \leqslant \varepsilon < \theta < 1$. Therefore, with probability one the top eigenvalue $\lambda_{K,V}$ ($K \geqslant 1$ fixed) of the Hamiltonian $\mathcal{H}_V$ is approximated by the local principal eigenvalue of the Hamiltonian restricted to a random connected region $\mathbb{A}^K_{V;\mathrm{opt}} \subset V$ of bounded diameter, where $\xi_V$ possesses high values of the optimal shape (so that $\lambda_{K,V} \leftrightarrow \mathbb{A}^K_{V;\mathrm{opt}}$). The optimal shape of $\xi_V$-values in $\mathbb{A}^K_{V;\mathrm{opt}} \subset V$ is specified by deterministic variational principles. These considerations are referred to as the *"relevant island" approximation*; see Theorems 2.7 and 6.19 for the second order expansion formulas for the first largest eigenvalue $\lambda_{1,V}$, which have been originally derived by Gärtner and Molchanov (1998). Moreover, the $K$th eigenfunction $\psi(\cdot\,;\lambda_{K,V})$ is highly concentrated in the neighborhood of the region $\mathbb{A}^K_{V;\mathrm{opt}}$, as proved by Astrauskas (2008; 2013) for $\rho$ large enough, and by Biskup and König (2016) for arbitrary $\rho$; see also Section 6.3 of the present survey.

Rigorous results on Poisson limit theorems and further localization properties for the largest eigenvalues (in the case of double exponential tails) have been proved by Astrauskas (2007; 2008; 2013) for $\rho$ large enough, and by Biskup and König (2016) for arbitrary $\rho$; see also the review paper by König (2016) and Sections 6.3–6.4 of the present survey for the discussions on their results and the proofs.

Let us finally summarize the above observations: As the upper tails of potential distribution get lighter, the $\xi_V$-extremes ($V \uparrow \mathbb{Z}^\nu$) get less expressed; therefore, the number of higher $\xi_V$-values contributing to the asymptotic amount of the top eigenvalues gets larger and concentration properties of the corresponding eigenfunctions become weaker. On the other hand, the general theory of Anderson localization (cf. Section 1.3 below) suggests that almost surely the eigenfunctions associated with the upper spectral edge of the Hamiltonian $\mathcal{H}_V$ decay exponentially like in (1.17), for arbitrary potential distribution satisfying certain continuity conditions.

### *1.3. Relations to infinite-volume Anderson Hamiltonians*

**(I)** *Anderson localization.* The Anderson model on the whole lattice $\mathbb{Z}^\nu$ is given by the Hamiltonian

$$\mathcal{H} = \kappa\Delta + \xi(\cdot\,)$$

acting on $l^2(\mathbb{Z}^\nu)$. Here, as above, $\Delta$ is the lattice Laplacian, $\kappa > 0$ is a diffusion constant, and $\xi(x)$ ($x \in \mathbb{Z}^\nu$) are i.i.d. random variables with a common distribution function $1 - \mathrm{e}^{-Q}$. This is a basic model of disordered quantum systems introduced to describe the regions of energy levels (spectrum) of the electron in the random potential modelling electrical conductance regimes of alloys, crystals with impurities and so on (Anderson, 1958). The energy spectrum Spect $(\mathcal{H})$ of the Hamiltonian $\mathcal{H} = \kappa\Delta + \xi(\cdot\,)$ is almost surely nonrandom:

$$\mathrm{Spect}\,(\mathcal{H}) = \mathrm{Spect}\,(\kappa\Delta) + \mathrm{Spect}\,(\xi(\cdot\,)) = [-2\nu\kappa; 2\nu\kappa] + \mathrm{supp}\,(1 - e^{-Q}),$$

where supp $(F)$ is the support of a probability measure generated by the distribution function $F$, and "+" denotes the algebraic sum of subsets of real line. Therefore, with probability one, the spectrum consists of spectral bands situated in the interval $[L_{\min}; L_{\max}]$, where $L_{\min}$ and $L_{\max}$ are respectively the infimum (i.e. bottom) and the supremum (i.e. upper edge) of the spectrum. The most important property of the Hamiltonian $\mathcal{H}$ on $l^2(\mathbb{Z}^\nu)$ is the presence of pure point spectrum in the neighborhood of edges of spectral bands for any $\kappa > 0$ and any $\nu \geqslant 1$. In particular, there exist (nonrandom) real constants $L_i = L_i(\kappa, \nu, Q)$, $L_1 < L_2$, such that with probability one

$$\text{the spectrum in } \;(L_{\min}; L_1) \cup (L_2; L_{\max})\; \text{ is dense purely point,}$$

say $\{\lambda_k\}$, and the corresponding eigenfunctions $\psi(\cdot; \lambda_k)$ decay exponentially:

$$|\psi(x; \lambda_k)| \leqslant C_k \exp\{-M|x - z_k|\} \quad (x \in \mathbb{Z}^\nu)$$

for some random $C_k > 0$, $M > 0$ and $z_{_k} \in \mathbb{Z}^\nu$ (the localization center), provided $\mathrm{e}^{-Q}$ is Hölder continuous and $\xi(0)$ has some finite statistical moments. Moreover, for small $\kappa$ or the one-dimensional case $\nu = 1$, the whole spectrum $\mathrm{Spect}\,(\mathcal{H})$ is dense purely point with a complete set of eigenfunctions in $l^2(\mathbb{Z}^\nu)$ that decay exponentially with probability 1. This phenomenon is known as *Anderson localization* for disordered systems; see, e.g., (Fröhlich and Spencer, 1983; Fröhlich et al., 1985; Simon and Wolff, 1986; Carmona et al., 1987; Aizenman and Molchanov, 1993; Aizenman et al., 2001) for the proof of the above assertions under various conditions on potential distributions. Recall that, for the periodic $\xi(\cdot)$, all the spectrum $\mathrm{Spect}\,(\mathcal{H})$ is absolutely continuous in arbitrary dimension $\nu \geqslant 1$; and this is quite a contrast to the Anderson localization in the case of random potential. See the monographs (Pastur and Figotin, 1992; Stolz, 2011; Kirsch, 2008) and references therein for more discussions on the subject. In the present survey as well as in (Astrauskas, 2007; 2008; 2012; 2013) and (Biskup and König, 2016), the phenomenon of Anderson localization is illustrated for the top eigenvalues and eigenfunctions of finite-volume models. We here emphasize the relationship between asymptotic geometric properties of $\xi_V$-extremes and localization properties of the leading eigenfunctions of the operator $\mathcal{H}_V$ regarding their localization strength, localization centers, etc., as $V \uparrow \mathbb{Z}^\nu$.

Let us discuss briefly the basic ideas and methods explored in the study of the Anderson localization phenomenon in the multidimensional case $\nu \geqslant 1$. Fix an open bounded interval $I \subset \mathbb{R}$ which is covered almost surely by the spectrum $\mathrm{Spect}\,(\mathcal{H})$. The *proof of Anderson localization* in the spectral intervals $I$ relies heavily on the study of the resolvents $\mathcal{G}(\lambda+i\varepsilon) := (\lambda+i\varepsilon-\mathcal{H})^{-1}$ and the corresponding Green functions $\mathcal{G}(\lambda+i\varepsilon; x, y) := \mathcal{G}(\lambda+i\varepsilon)\delta_y(x)$ $(x, y \in \mathbb{Z}^\nu)$ in the complex domain $\varepsilon > 0$, $\lambda \in I$, where $i = \sqrt{-1}$. Alternatively, their finite-volume versions $\mathcal{G}_{_V}(\lambda + i\varepsilon) := (\lambda + i\varepsilon - \mathcal{H}_{_V})^{-1}$ and $\mathcal{G}_{_V}(\lambda + i\varepsilon; x, y) := \mathcal{G}_{_V}(\lambda + i\varepsilon)\delta_y(x)$ $(x, y \in V)$ are also explored. The main task here is to prove that, for $\lambda \in I$, the Green functions $\mathcal{G}(\lambda + i\varepsilon; x, y)$ or $\mathcal{G}_{_V}(\lambda + i\varepsilon; x, y)$ decay exponentially in $|x - y|$ uniformly in $\varepsilon > 0$, provided $I$ is chosen close to the spectral edge or the diffusion constant $\kappa$ is small. For simplicity, assume throughout that the potential distribution has a bounded density $p(\cdot)$ with bounded support.

Fröhlich and Spencer (1983) developed the *multiscale method*. They constructed inductively a sequence of relevant cubes $V'$, $V' \uparrow \mathbb{Z}^\nu$ with the following properties: For any fixed $\lambda \in I$ and $V' \uparrow \mathbb{Z}^\nu$, with high probability the Green function $\mathcal{G}_{_{V'}}(\lambda + i\varepsilon; x, y)$ decays exponentially in $|x - y|$ for all $y \in \partial V'$, the "boundary" of $V'$, and all $x \in V'$ far away from $\partial V'$, and this estimate holds uniformly in $\varepsilon \geqslant 0$. By finite-rank perturbation formulas, this estimate implies an absence of absolutely continuous spectrum in $I$ with probability one. Even the stronger form of the construction of the relevant cubes $V'$ ("uniformity" in $\lambda \in I$) is applied to prove the exponential decay of the eigenfunctions $\psi(\cdot; \lambda)$ associated with (generalized) spectral values in $I$ and, consequently, the presence of the pure point spectrum in $I$ with probability one (Fröhlich et al., 1985). See also the survey by Kirsch (2008) for a detailed discussion on the multiscale analysis.

In the above considerations, one should apply the *Wegner estimate*. Loosely speaking, this estimate states that the mean number of eigenvalues of $\mathcal{H}_V$ in the interval $I$ does not exceed $|I||V|C$, provided the density $p(\cdot)$ of potential distribution satisfies $p(\cdot) \leqslant C$. In particular, the latter guarantees the bound of probability to find at least one eigenvalue $\lambda_{\iota,V}$ in a small spectral interval. The Wegner estimate and its modifications are the basic probabilistic tools in the proof of Anderson localization; see, e.g., (Kirsch, 2008).

Simon and Wolff (1986) applied the so-called spectral averaging methods in multiscale analysis to obtain the effective condition for the Anderson localization in the interval $I$, for the distributional density as above. Recall the *Simon-Wolff criterion*: If for each $x \in \mathbb{Z}^\nu$ and for Lebesgue-almost every $\lambda \in I$ with probability one

$$\lim_{\varepsilon \downarrow 0} \sum_{y \in \mathbb{Z}^\nu} |\mathcal{G}(\lambda + i\varepsilon; x, y)|^2 < \infty, \tag{1.15}$$

then the operator $\mathcal{H}$ has only pure point spectrum in $I$ with probability one. If in (1.15) one claims a square-summability of the Green function with weights $\mathrm{e}^{m|x-y|}$ for some nonrandom $m > 0$, then the corresponding eigenfunctions decay exponentially with probability one.

Aizenman and Molchanov (1993) and Aizenman et al. (2001) developed the *fractional moment method* to prove the Anderson localization in the spectral intervals $I$. The task here is to obtain (and explore) the exponential decay of the averaged Green functions, where the average is taken over the random potential. This method enables to avoid a complicated dependency of the previous constructions on individual potential configurations in the almost sure setting. Under certain continuity conditions on potential distribution, the key statement in (Aizenman and Molchanov, 1993) is the following *fractional-moment criterion*: If for fixed $0 < \sigma < 1$, there are constants $C_1 > 0$, $M_1 > 0$ such that

$$\mathbb{E}\left(|\mathcal{G}(\lambda + i\varepsilon; x, y)|^\sigma\right) \leqslant C_1\, \mathrm{e}^{-M_1|x-y|} \quad \text{for all} \quad x,\ y \in \mathbb{Z}^\nu, \tag{1.16}$$

for all $\lambda \in I$ and uniformly in $\varepsilon > 0$, then one has, with probability one, the Anderson localization in the interval $I$ for the operator $\mathcal{H}$.

This implication can be proved by using the Simon-Wolff criterion like in (1.15). On the other hand, $\sigma < 1$ is chosen to depend only on continuity assumptions for potential distribution. In particular, the Hölder continuity implies that the left-hand side of (1.16) is finite.

Eq. (1.16) can be proved by using suitable finite rank perturbation arguments, i.e., Krein formulas. These formulas imply that the kernel $\mathcal{G}(\lambda+i\varepsilon; x, y)$ is equal to a simple rational function in the variable $\xi(x)$ with coefficients depending on potential values outside $x$. Now one can estimate the left-hand side of (1.16) as an integral of rational function. These estimates are shown to form a certain iteration procedure, from which one deduces (1.16). When applying the iteration scheme, a crucial fact is the assumption that the diffusion constant $\kappa$ is small or the interval $I$ is near the upper edge of the spectrum.

Applying similar arguments, Aizenman et al. (2001) deduced a *fractional-moment finite-volume criteria* for the Anderson localization in the interval $I$. Roughly speaking, these criteria state that, if for some $0 < \sigma < 1$ and some $V$, the expectation

$$\begin{aligned}&\mathbb{E}\left(|\mathcal{G}_V(\lambda+i\varepsilon;0,y)|^{\sigma}\right) \quad \text{is sufficiently small}\\ &\text{for all} \quad y \in \partial V, \text{ uniformly in } (\lambda;\varepsilon) \in I \times \mathbb{R}_+,\end{aligned}$$

then the exponential decay (1.16) holds true in $I$. The finite-volume criteria and their implications are shown to hold under continuity assumptions on potential distribution mentioned above.

The criteria of (Aizenman et al., 2001) also imply the exponential decay of eigenfunctions of the operators $\mathcal{H}_V$, associated with the eigenvalues in (compact) spectral intervals $I$ of Anderson localization, in particular, for $I$ at the upper edge of the spectrum Spect$\,(\mathcal{H})$. Under the continuity conditions on potential distribution as above, we are able to formulate this result in a precise form: There are nonrandom constants $c > 0$ and $M > 0$ such that with probability one

$$|\psi(x;\lambda_{k,V})| \leqslant |V|^c \exp\{-M|x - z_{\tau(k),V}|\} \quad (x \in V) \tag{1.17}$$

for some $z_{\tau(k),V} \in V$ (localization center), for all $\lambda_{k,V} \in I$ and all $V$ large enough; cf. (Klopp, 2011). See also the surveys (Hundertmark, 2008; Stolz, 2011) for detailed discussions on the fractional-moment methods and their applications.

**(II)** *Local fluctuations of eigenvalues in the spectral regions of Anderson localization.* We denote by $I_{\rm pp} := (a;b)$ an open interval of real axis such that a certain fractional-moment finite-volume criterion is fulfilled in $I_{\rm pp}$. Therefore, the spectrum in the whole of $I_{\rm pp}$ is purely point, so Spect$\,(\mathcal{H}) \cap I_{\rm pp} \subset$ Spect$_{\rm pp}(\mathcal{H})$ with probability one (i.e., Anderson localization in $I_{\rm pp}$). The spectral intervals $I_{\rm pp}$ are distinguished by the *Poissonian asymptotic behavior* of eigenvalues $\lambda_{\iota,V}$, close to a fixed $\lambda^0 \in I_{\rm pp}$, of the finite-volume model $\mathcal{H}_V$ as $V \uparrow \mathbb{Z}^\nu$. Here and to the end of this section the potential distribution is again assumed to have a bounded smooth density with bounded support. These limit theorems are formulated in terms of *the integrated density of states*, viz. $N(\lambda)$, and *the density of states*, viz. $n(\lambda) := N'(\lambda)$ $(\lambda \in \mathbb{R}^\nu)$. Recall that $N(\cdot)$ is the nonrandom distribution function of eigenvalues defined as the almost sure limit of the empirical distribution function $N_V(\lambda) := \#\{k\colon\ \lambda_{k,V} \leqslant \lambda\}/|V|$ as $|V| \to \infty$. Moreover, the Wegner estimate implies that $N(\cdot)$ is an absolutely continuous function with bounded density $n(\cdot)$, provided the potential distribution has a bounded density. The support of $n(\cdot)$ coincides almost surely with the spectrum Spect$\,(\mathcal{H})$; therefore, $N(\lambda) \to 0$ (resp., $N(\lambda) \to 1$) as $\lambda$ approaches the bottom (resp., the upper edge) of the spectrum. See, e.g., (Kirsch, 2008) for the definition of functions $N(\cdot)$, $n(\cdot)$ and their properties.

Now pick a number $\lambda^0$ from the interval $I_{\rm pp}$ such that $n(\lambda^0) > 0$. We consider the normalized eigenvalues

$$\Lambda^0_{k,V} := |V| n(\lambda^0) \left(\lambda_{k,V} - \lambda^0\right) \quad (1 \leqslant k \leqslant |V|), \tag{1.18}$$

and define the corresponding point process $\mathcal{M}_V^0$ on $\mathbb{R}$ by

$$\mathcal{M}_V^0 := \sum_{k=1}^{|V|} \delta_{\Lambda_{k,V}^0}.$$

With these assumptions and abbreviations, one needs to show that the point process $\mathcal{M}_V^0$ converges weakly, as $V \uparrow \mathbb{Z}^\nu$, to the Poisson point process on $\mathbb{R}$ with intensity measure $\mathrm{d}\,\lambda$, i.e. the Lebesgue measure. The first result in this direction was proved by Molchanov (1981), who considered the one-dimensional spatially continuous random Schrödinger operators. In the case of the Anderson Hamiltonians in $\mathbb{Z}^\nu$ with arbitrary $\nu \geqslant 1$, this Poisson limit theorem was established by Minami (1996). Killip and Nakano (2007) proved the Poisson convergence of both the normalized spectral values (1.18) and localization centers of the corresponding eigenfunctions, extending Minami's result. The proof in the multidimensional case $\nu \geqslant 1$ relies on applications of the fractional-moment criteria for the Anderson localization, combined with the Wegner and Minami estimates that control the probability of finding, respectively, at least one and two eigenvalues of $\mathcal{H}_V$ in a small interval. In particular, these estimates imply the upper and lower bounds of order $|V|^{-1}$ for the gap between successive eigenvalues close to a fixed $\lambda^0$ as above. From Poisson limit theorems for (1.18), one can extract some information on the statistical properties of eigenvalues in the spectral regions of Anderson localization. For example, one can obtain the limiting distribution for the normalized spacings of eigenvalues, the limiting joint distribution for the normalized eigenvalue and its localization center, and other important limiting distributions for eigenvalue statistics.

Recently, Germinet and Klopp (2013; 2014) presented a comprehensive study on limit theorems for eigenvalues of large-volume Hamiltonians in the spectral regions $I_{\mathrm{pp}}$ of Anderson localization. For fixed $\lambda^0 \in I_{\mathrm{pp}}$ as above, the authors considered the *unfolded eigenvalues*

$$\Upsilon_{k,V}^0 := |V|(N(\lambda_{k,V}) - N(\lambda^0)) \quad (1 \leqslant k \leqslant |V|), \tag{1.19}$$

i.e., the eigenvalues under nonlinear renormalization. They proved the following limit theorems for eigenvalue statistics:

1) Poisson limit theorem for the point process based on the unfolded eigenvalues (1.19), where the limiting Poisson process coincides with that for (1.18);
2) Poisson limit theorem for the point process based on both the unfolded eigenvalues (1.19) and the normalized localization centers of the corresponding eigenfunctions;
3) Limit theorems for the empirical distribution function of the normalized spacings of eigenvalues close to $\lambda^0$;
4) Limit theorems for the normalized distance between localization centers of the corresponding eigenfunctions;

and other important limit theorems for various statistics related to the eigenvalues and eigenfunctions in the intervals $I_{\mathrm{pp}}$. Moreover, Germinet and Klopp

(2013) considered more general random Hamiltonians $\mathcal{H}$ with convolution-type long-range kinetic operators instead of the Laplacian: For such Hamiltonians, the Poissonian asymptotic results were shown to hold also for the eigenvalues at the spectral edges (in particular, for the $K$th largest eigenvalues with fixed $K \geqslant 1$). At the spectral edges of the Schrödinger operators, this result was shown to hold for the one-dimensional case $\nu = 1$; cf. Section 6.2 below.

The proof of the limit theorems in (Germinet and Klopp, 2013; 2014) relies heavily on the techniques of the general theory of Anderson localization, including applications of the fractional-moment criteria in the large-volume setting, as well as various versions of the Wegner and Minami estimates.

In the proof of the above results, the following crucial observation is related to localization properties of the corresponding eigenfunctions of the operator $\mathcal{H}_V$ (and is also quite close to the context of our paper treating the top eigenvalues and eigenfunctions of $\mathcal{H}_V$): Namely, the eigenvalues $\lambda_{k,V}$ near $\lambda^0$ of the operator $\mathcal{H}_V$ can be approximated, with a "good" error, by independent local eigenvalues of the operators restricted to much smaller disjoint cubes $\widetilde{V}(z) \subset V$ centered at $z$. This approximation is feasible due to the facts that the localization centers of the corresponding eigenfunctions of $\mathcal{H}_V$ are located far away from each other (so the probability of having at least two centers in cubes $\widetilde{V}(z)$ is asymptotically negligible), and the eigenvalues of $\mathcal{H}_V$ "live" on potential values in small spatial neighborhoods of the corresponding localization centers because of the exponential decay of eigenfunctions. One needs to consider the small spectral interval $I_V \subset I_{\mathrm{pp}}$, centered at $\lambda^0$, such that the number of eigenvalues $\lambda_{k,V}$ in $I_V$ unboundedly increases as $V \uparrow \mathbb{Z}^\nu$. The latter is fulfilled if, say, $|I_V| \asymp |V|^{-c}$ for some $0 < c < 1$. The study of the structure of eigenvalues $\lambda_{k,V}$ inside $I_V$ is based on the Wegner and Minami estimates. In particular, using the eigenvalue approximation described above, one has with high probability that the number of eigenvalues in $I_V$ is roughly approximated by $N(I_V)|V|$ (large deviation principle); here $N(I)$ is the probability measure associated with the integrated density of states. Also, note that the above continuity conditions on potential distribution imply another important bound:

$$N(I_V) \geqslant \mathrm{const}\, |I_V|^{1+\vartheta} \tag{1.20}$$

for some $\vartheta \geqslant 0$ and for all $V$ large enough. These bounds are crucial in estimating the probabilities of the occurrence of a single or several eigenvalues in small spectral intervals for operators over cubes $V$ and $\widetilde{V}(z) \subset V$ introduced above.

In (Germinet and Klopp, 2013), the results on local fluctuations for eigenvalues $\lambda_{k,V}$ are extended up to spectral edges. Here the proofs rely on the improved versions of Wegner and Minami estimates which ensure the more explicit control of eigenvalues $\lambda_{k,V}$ in small spectral intervals $I_V$, since the amount $N(I_V)$ is now allowed to be exponentially small in $|I_V|^{-1}$ instead of (1.20). Thus, this case includes situations, where the measure $N(\cdot)$ is extremely small, for example, at the edges of spectral bands where the phenomenon of "Lifshits tails" occurs.

We finally notice that the Poisson limit theorems for the lower eigenvalues (1.18) or (1.19) (in the spectral regions $I_{\mathrm{pp}}$ of Anderson localization) agree with the results of the present paper and our earlier papers on the extreme value theory for the largest eigenvalues $\lambda_{K,V}$ of $\mathcal{H}_V$, with $K \geqslant 1$ fixed. However, in limit theorems for the largest eigenvalues, the choice of normalizing constants depends strongly on the regularity and tail decay conditions of the potential distribution, in contrast to limit theorems for eigenvalues near $\lambda^0 \in I_{\mathrm{pp}}$, where the normalizing constants are simply expressed in terms of the (integrated) density of states. In particular, the spacings of the top eigenvalues have an asymptotic order, which is much larger than $|V|^{-1}$ = the order of the gaps between successive eigenvalues close to $\lambda^0 \in I_{\mathrm{pp}}$. It is worth mentioning that the proof of Poisson limit theorems for eigenvalues in $I_{\mathrm{pp}}$ relies heavily on the methods and ideas of the general theory of Anderson localization; and neither the extreme value theory nor links to the asymptotic geometric properties of random potential are explored.

We finally mention the following open problems regarding the *Anderson transition phenomenon* for the infinite-volume Hamiltonians $\mathcal{H} = \kappa\Delta + \xi(\cdot)$: The first conjecture is that for $\nu \geqslant 3$, the spectral bands outside some neighborhood of the spectral edges consist of purely absolutely continuous spectrum and the corresponding eigenfunctions are delocalized. The second conjecture is that the eigenvalues of $\mathcal{H}_V$ in the spectral intervals of delocalization obey non-Poissonian asymptotic behavior as $V \uparrow \mathbb{Z}^\nu$, rather the limit theorems like in the theory of Wigner random matrices with light-tailed entries; cf. Section 1.4 below. The first problem is partially solved for the very special models on the Bethe lattice as well as for the Schrödinger operators with sparse potential (e.g., Kirsch, 2008; Molchanov and Vainberg, 1998, 2000).

### *1.4. Relations to random matrices*

**(I)** *Wigner random matrices.* Another important model of disordered quantum systems (in particular, heavy nuclei atoms) is presented by real symmetric random matrices

$$H_N = \big(h_{i,j}\big)_{1\leqslant i,j\leqslant N}$$

with i.i.d. centered entries $h_{i,j}$ $(i \leqslant j)$ and $N \to \infty$; i.e., large *Wigner matrices* (Mehta, 2004; Anderson et al., 2010). The extreme eigenvalues (i.e., high energy levels) $\lambda_{K,N}$ and the corresponding $l^2$-normalized eigenvectors of $H_N$ are here interpreted as the basic states of quantum systems.

Recently, there has been much progress toward the extreme value theory for the eigenvalues $\lambda_{K,N}$ of Wigner matrices $H_N$ as $N \to \infty$ and $K \geqslant 1$ fixed. It has been turned out that there are two different regimes of asymptotic behavior of the largest eigenvalues, depending on the tail decay rate of the entries in absolute value:

$(\mathrm{I}_1)$ For polynomially decaying distributions

$$\mathbb{P}\big(\big|h_{i,j}\big| > t\big) = t^{-\beta}(1 + \mathrm{o}(1)) \quad \text{as} \quad t \to \infty \tag{1.21}$$

with $\beta < 4$ (very heavy tails), Auffinger et al. (2009) proved that with high probability, the $K$th largest eigenvalue $\lambda_{K,N}$ of Wigner matrices $H_N$ is approximately equal to the $K$th largest value among $|h_{i,j}|$ $(1 \leqslant i \leqslant j \leqslant N)$, $K \geqslant 1$ fixed. This in turn implies Poisson limit theorems for the normalized eigenvalues $\lambda_{K,N} A_N$, where the normalizing constants $A_N > 0$ are chosen the same as in the corresponding limit theorems for extremes of $|h_{i,j}|$ $(1 \leqslant i \leqslant j \leqslant N)$. Recall that distributional tails (1.21) are in the domain of attraction of the max-stable Fréchet law $G_\beta$, therefore, the eigenvalue $\lambda_{K,N}$ is of the order $A_N^{-1} = N^{2/\beta}(\text{const} + \text{o}(1))$; cf. Example 6.11 below. Moreover, with high probability, the $K$th eigenvector is asymptotically concentrated on two coordinates, i.e., it behaves like a superposition of two delta functions in limit as $N \to \infty$. See also (Soshnikov, 2004) for the case $\beta < 2$. To prove these assertions, one first observes that, under condition (1.21) with $\beta < 4$, the largest entries (in absolute value) are extremely sparse and strongly pronounced in comparison to other entries in $H_N$. Thus, the standard perturbation theory for symmetric matrices is applied to conclude that the top eigenvalues and eigenvectors of the former matrix $H_N$ are approximated by the corresponding eigenvalues and eigenvectors of a very sparse (symmetric $N \times N$) matrix whose entries are the extremes of $h_{i,j}$'s in absolute value. This fact in turn enables us to apply the extreme value theory for the random variables $|h_{i,j}|$ and, as a consequence, to prove Poisson limit theorems for the eigenvalues of Wigner matrices $H_N$.

($\mathrm{I}_2$) Assume that the $|h_{i,j}|$'s have lighter tails (including (1.21) with $\beta > 4$ and Bernoulli entries), or the $h_{i,j}$'s have some finite statistical moments of higher order and satisfy additional conditions on a distributional symmetry. Then the largest eigenvalues $\lambda_{K,N}$ of Wigner matrices $H_N$ are distinguished by non-Poissonian asymptotic behavior, rather the Tracy-Widom limit law; see, e.g., (Soshnikov, 1999; Lee and Yin, 2014; Bourgade et al., 2014). In particular, the normalized top eigenvalues $\lambda_{K,N}/\sqrt{N}$ tend almost surely to the nonrandom constant $2(\mathbb{E}h_{1,2}^2)^{1/2}$, i.e., the right endpoint of the support of their limiting spectral distribution density; cf. (Bai and Yin, 1988). Moreover, the corresponding $l^2$-normalized eigenvectors are completely delocalized; i.e., with high probability their sup-norm does not exceed $N^{-1/2}(\log N)^{\text{const}}$. See, e.g., (Tao and Vu, 2010; Erdős et al., 2013a; Vu and Wang, 2015; Götze et al., 2015), where this delocalization property is extended to all the eigenvectors of $H_N$ provided the tails of $|h_{i,j}|$ are lighter than the exponential, or the $h_{i,j}$'s have a large enough number of moments. It is worth mentioning that limiting distributions for eigenvalues or eigenvectors (in particular, the Tracy-Widom limit law for the largest eigenvalues) can be explicitly computed for Wigner matrices with Gaussian entries; see, e.g., (Anderson et al., 2010). Thus, the usual comparison methods (four moments theorem, Green function comparison method, etc.) can be used to extend the asymptotic results for the Gaussian case to general Wigner matrices; e.g., (Tao and Vu, 2014).

The transition from Poisson limit theorems to Tracy-Widom asymptotics for the top eigenvalues of Wigner random matrices was discussed in detail by Biroli et al. (2007). The value $\beta = 4$ in (1.21) or, roughly speaking, the fourth statistical moment indicates here the threshold separating these two different regimes of asymptotic behavior. Note that the Wigner matrix model with light-tailed entries reflects the global or mean-field interaction; thus, the asymptotic geometric properties of entries do not play any role in limit behavior of eigenvalues and eigenvectors. The latter is in a sharp contrast to the one-dimensional Anderson Hamiltonian in $V \subset \mathbb{Z}$, which is a random band (tridiagonal) matrix reflecting local interaction: the diagonal elements are i.i.d. random variables and the deterministic off-diagonal elements are given by the Laplacian. In view of discussions on the Anderson model (Section 1.2 above), it turns out that the diagonal operator necessitates concentration properties of eigenvectors; meanwhile, the Laplacian forces these properties to be less expressed (thus, the geometric features of the model play here a crucial role).

**(II)** *Random band matrices.* Recently, there has been a considerable attention drawn to symmetric *random band matrices* $H_N^{(W)}$ of size $N \to \infty$, where the matrix entries $h_{i,j}$ vanish if $|i-j|$ exceeds $W$, and other entries (above the diagonal) are i.i.d. centered random variables; here $0 \leqslant W \leqslant N$ is a band width. Random band matrices are natural interpolations between Anderson Hamiltonians ($\nu = 1$) and Wigner matrices. For band models, the asymptotic behavior of the top eigenvalues and eigenvectors depends strongly on the growth rate of the band width $W = W_N \to \infty$ as well. Benaych-Georges and Péché (2014) considered the random band matrices $H_N^{(W)}$, whose entries have polynomially decaying distributions (1.21) with arbitrary $\beta > 0$ and band width $W = N^\mu$ with $0 < \mu \leqslant 1$. They established that the band model exhibits a phase transition depending on $\mu$ and $\beta$, with $\beta = 2(1+\mu^{-1})$ as the threshold separating two different regimes of asymptotic behavior of the largest eigenvalues $\lambda_{K,N}$ and the corresponding eigenvectors ($K \geqslant 1$ fixed):

($\mathrm{II}_1$) Assume (1.21) and $W = N^\mu$ such that $\beta < 2(1+\mu^{-1})$, i.e., either the distributional tails are sufficiently heavy or the band of matrix is sufficiently narrow. This case includes heavy-tailed Wigner matrices, i.e., $\mu = 1$ and $\beta < 4$, considered in ($\mathrm{I}_1$) above. For $0 < \mu < 1$ and $\beta < 2(1+\mu^{-1})$, the asymptotic results are similar to that in ($\mathrm{I}_1$). I.e., with probability $1+\mathrm{o}(1)$, the $K$th largest eigenvalue of $H_N^{(W)}$ is approximately equal to the $K$th largest value of the sample $|h_{i,j}|$ ($1 \leqslant i \leqslant N$, $0 \leqslant j-i \leqslant W$). Therefore, Poisson limit theorems for the normalized eigenvalues $\lambda_{K,N} A_N$ hold true, where $A_N = N^{-(1+\mu)/\beta}(\mathrm{const}' + \mathrm{o}(1))$ (cf. Example 6.11 below), and the $K$th eigenvector is asymptotically localized on two coordinates. The proof of these assertions is again heavily based on techniques of the extreme value theory, in particular, describing asymptotic geometric properties of entries of band matrices. The latter is combined with the perturbation theory for matrices to derive simple asymptotic formulas for the largest eigenvalues and eigenfunctions of $H_N^{(W)}$.

($\mathrm{II}_2$) Assume the $h_{i,j}$'s are symmetrically distributed with tails (1.21) and let $W = N^\mu$ with $\beta > 2(1+\mu^{-1})$, i.e., either the distributional tails are sufficiently light or the band of matrix is sufficiently wide. In this case, the band models $H_N^{(W)}$ posses the mean-field features, like in the light-tailed Wigner models considered in ($\mathrm{I}_2$) above. Thus, each eigenvector associated with the upper spectral edge is asymptotically delocalized in the sense that its $l^2$ mass is more or less uniformly spread over $N$ coordinates. Moreover, the extreme eigenvalues do not longer obey Poissonian asymptotic behavior; in particular, they tend to a nonrandom positive constant when divided by $N^{\mu/2}$.

Earlier, Sodin (2010) considered the band matrices $H_N^{(W)}$ whose entries are symmetrically distributed with tails lighter than the Gaussian tails, including Bernoulli entries. He studied the transition from localization to delocalization for eigenvectors at the upper spectral edge (as well as the corresponding limit theorems for eigenvalues) with varying degrees of strength and generality. In particular, the results of this paper suggest that the eigenvectors associated with the largest eigenvalues are delocalized, provided $W = N^\mu$ with $\mu > \mu_{\rm cr} = 5/6$. See also (Erdős et al., 2013b) for similar delocalization results for eigenvectors associated with the inner part of the spectrum Spect $(H_N^{(W)})$.

In view of these observations on the localization properties at the upper spectral edge, we distinguish two paradigmatic models in the theory of random matrices. First, the general theory of Anderson localization suggests that, with probability 1, the one-dimensional Schrödinger operators $\mathcal{H} = \kappa\Delta + \xi$ (i.e., the particular model of the band matrices with $W = 1$) have exponentially localized eigenvectors at the spectral edges, provided the potential distribution is arbitrary satisfying very mild continuity conditions (Carmona et al., 1987). On the other hand, the large Wigner matrices $H_N$ (i.e., the band matrices with $W = N$) have localized eigenvectors associated with the largest eigenvalues if only entries are very heavy-tailed like in (1.21) with $\beta < 4$; meanwhile, for the light tails, eigenvectors of $H_N$ are typically delocalized. See also (Spencer, 2011) for a discussion on Anderson-type models ($M = \mathrm{O}(1)$), band matrices ($M = \mathrm{o}(N)$) and Wigner matrices ($M = N$).

Finally, it is worth emphasizing that the above asymptotic results for eigenvalues remain valid for the corresponding complex Hermitian random matrices with i.i.d. entries, instead of the real symmetric matrices.

### *1.5. The earlier literature on extremes of i.i.d. random fields*

As already mentioned, most statements of the present paper on the $\xi_V$-extremes and the corresponding RV classes were announced in (Astrauskas, 2007; 2008; 2012; 2013). We now provide a brief overview of the earlier literature on the related asymptotic results for extreme order statistics of i.i.d. random sequences and fields.

High-level exceedances consisting of single rare $\xi_V$-peaks were studied in (Astrauskas, 2001). Related asymptotic results (in particular, the so-called longest head runs in coin tossing) for Bernoulli distributed i.i.d. random variables $\xi(x)$, $x \in \mathbb{Z}$, were discussed, e.g., in (Binswanger and Embrechts, 1994).

In the case of exponentially distributed $\eta(0)$, strong limit theorems for the spacings $\eta_{K,V} - \eta_{K+1,V}$ ($K$ fixed) were proved by Astrauskas (2006). Devroye (1982) derived strong and weak limit theorems for $\min_{1\leqslant k\leqslant|V|}(\zeta_{k,V} - \zeta_{k+1,V})$ where $\zeta(0)$ is uniformly distributed.

In the case of $\xi(0)$ with arbitrary distribution, strong asymptotic bounds for $\xi_{K,V}$ are given in (Shorack and Wellner, 1986) where $K$ is fixed, and in (Deheuvels, 1986) where $K = K_V \to \infty$. Wellner (1978) derived strong asymptotic bounds for the uniform $k$th order statistics $\zeta_{k,V}$ (thus, for $\eta_{k,V}$) uniformly in $k \geqslant 1$.

For the Gaussian random fields $\{\xi(x) \colon x \in \mathbb{Z}^\nu\}$ with correlated values, Astrauskas (2003) studied some asymptotic geometric properties of $\xi_V$-extremes almost surely, in particular, high level exceedances and minimum of spacings. The geometry of high level excursion sets of smooth Gaussian random fields in $\mathbb{R}^\nu$ was investigated in the monograph by Adler and Taylor (2007). See also (Gärtner et al., 2000) for some geometric aspects of high peaks of smooth Gaussian random potentials related to the long-time asymptotics for the spatially continuous parabolic Anderson models.

For the extreme value theory for random variables, in particular, characterization of the domains of attraction of max-stable distributions, we refer to the monographs by Resnick (1987), de Haan and Ferreira (2006), Leadbetter et al. (1983), Embrechts et al. (1997). See also the monograph by Shorack and Wellner (1986) for a detailed account of strong and weak limit theorems for order statistics and their functions related to mathematical statistics. Finally, the monograph by Bingham et al. (1987) provides a detailed account of the theory of regularly varying functions.

In the proof of a number of our statements on $\xi_V$-extremes, we explore the representation $\xi_{k,V} = f(\eta_{k,V})$, where $f := Q^{\leftarrow}$ is the generalized inverse function of $Q$ and, as above, $\eta_{k,V}$ stands for the $k$th extreme value among independent exponentially distributed random variables $\eta(x)$ $(x \in V)$ with mean 1. Due to the nice properties of $\eta_{k,V}$ (for instance, $\eta_{k,V}$ is a sum of independent exponentially distributed random variables), we first obtain the asymptotic results for $\eta_{k,V}$, which are then transferred to $\xi_{k,V}$ under appropriate conditions on $f$. These conditions are formulated in terms of regular variation (RV) of $f(s)$ as $s \to \infty$. We further give a characterization of RV classes, in particular, their links to continuity and tail decay of the distribution $1 - \mathrm{e}^{-Q}$ at the right endpoint; see Appendix A. They are also compared with the well-known RV classes including the domains of attraction of max-stable laws, O-regular variation, asymptotically balanced, etc; see Appendix A.

An interesting further problem is an extension of the present asymptotic results for functionals (EX), (SP), (MIN), (NEI) to other classes of random fields $\xi(\cdot)$ including: 1) independent non-identically distributed random variables; 2) random fields with correlated values, in particular, Gaussian fields (Astrauskas, 2003) and moving average fields defined as a linear combination of i.i.d. random variables with nonrandom real coefficients. See, e.g., the review papers by Elgart et al. (2012), Tautenhahn and Veselić (2015) for a detailed background of the random alloy type models $\kappa\Delta + \xi(\cdot)$ with the moving average potential $\xi(\cdot)$.

### 1.6. Notation. Representation of i.i.d. random fields

Let us introduce the further notation and remarks we use throughout the paper. We denote by $\mathbb{R}_+$ the positive half-axis and by $\mathbb{N}$ positive integers. Let $\log_j$ stand for the $j$times iterated natural logarithm. For real $a$, $b$, we write $a \vee b := \max(a,b)$ and $a \wedge b := \min(a,b)$, and $[a]$ for the integer part of $a$. Given a subset $U \subset \mathbb{Z}^\nu$, we write $|U|$ for the number of its elements. Let $\operatorname{dist}(U,U')$ stand for the lattice $l^1$-distance between subsets $U,U' \subset \mathbb{Z}^\nu$. The summation over $x \in V\colon\ a \leqslant |x| \leqslant b$ is abbreviated to $\sum_{a\leqslant|x|\leqslant b}$. By $t_0$, $|V_0|$, etc. we denote various large numbers, values of which may change from one appearance to the next. Similarly, const , const$'$ etc. stand for various positive constants. We write $1/0+ = \infty$, $\log(0+) = -\infty$ and $1/\infty = 0$. Let $g \circ h = g(h(\cdot))$ stand for a composition of real functions $g$ and $h$. Also, for $g > 0$ and $h > 0$, we write $g(t) \asymp h(t)$ as $t \to \infty$, if the ratio $g(t)/h(t)$ is bounded away from zero and from above for all large $t$.

By $\mathcal{G}_V(\lambda;\zeta_V;x,y)$ $(x \in V,\ y \in V)$ we denote the Green function of the Hamiltonian $\kappa\Delta_V + \zeta_V$ in $l^2(V)$, viz.

$$\mathcal{G}_V(\lambda;\zeta_V;x,y) := \mathcal{G}_V(\lambda;\zeta_V)\delta_y(x) := (\lambda - \kappa\Delta_V - \zeta_V)^{-1}\delta_y(x).$$

Here $\delta_y(\cdot)$ is the Kronecker delta function, i.e., $\delta_y(x) := 1$ if $x = y$, and $\delta_y(x) := 0$ if $x \neq y$.

Throughout the paper we suppose that all random variables are defined on a common probability space $(\Omega,\mathcal{F},\mathbb{P})$. Let $\mathbb{E}$ stand for the expectation with respect to $\mathbb{P}$. Recall $Q(t) := -\log\mathbb{P}(\xi(0) > t)$ is the cumulative hazard function of an i.i.d. random field $\xi(x) = \xi^{(\omega)}(x)$ $(\omega \in \Omega;\ x \in \mathbb{Z}^\nu)$, and let $t_Q$ denote its right endpoint $t_Q := \sup\{t\colon\ Q(t) < \infty\}$. Without loss of generality, we shall assume throughout that $0 < t_Q \leqslant \infty$. Clearly $Q : (-\infty;t_Q) \to \mathbb{R}_+ \cup \{0\}$ is a right-continuous nondecreasing function such that $Q(-\infty) = 0$ and $Q(t_Q) = \infty$. Most of the conditions of our results are formulated in terms of the inverse of the cumulative hazard function defined by

$$f(s) := Q^{\leftarrow}(s) := \inf\{t\colon\ Q(t) \geqslant s\} \quad (s \in \mathbb{R}_+) \tag{1.22}$$

(thus $f : \mathbb{R}_+ \to (-\infty;t_Q)$ is a left-continuous nondecreasing function such that $f(s)$ tends to $t_Q$ as $s \to \infty$). The reason for this is the following useful *representation of order statistics* $\xi_{k,V}$:

$$\xi_{1,V} := f(\eta_{1,V}) \geqslant \xi_{2,V} := f(\eta_{2,V}) \geqslant \ldots \geqslant \xi_{|V|,V} := f(\eta_{|V|,V}), \tag{1.23}$$

where

$$\eta_{1,V} := \eta(z_{1,V}) > \eta_{2,V} := \eta(z_{2,V}) > \ldots > \eta_{|V|,V} := \eta(z_{|V|,V}) \tag{1.24}$$

is the variational series based on the sample $\eta_V := \{\eta(x)\colon\ x \in V\}$ of exponential i.i.d. random variables with mean 1.

### 1.7. Outline

In Section 2, we collect conditions on deterministic functions $\xi_V$ in terms of functionals (EX), (SP), (MIN), and (NEI), which yield expansion formulas for the largest eigenvalues $\lambda_{_{K,V}}$ of the discrete Schrödinger operator $\mathcal{H}_V = \kappa\Delta_V + \xi_V$ on $l^2(V)$ as $V \uparrow \mathbb{Z}^\nu$. Section 2.1 provides rough bounds for $\lambda_{_{K,V}}$. We then study $\lambda_{_{K,V}}$ in the cases of $\xi_V$ with extremely sharp peaks (Section 2.2), dominating single peaks (Section 2.3), dominating large islands of high $\xi_V$-values the diameter of which unboundedly increases (Section 2.4) and, finally, dominating islands of high $\xi_V$-values the diameter of which is bounded (Section 2.5). The results of Sections 2.2–2.5 follow simply from the more general statements of (Astrauskas, 2008; 2012) and Section 2.4 in (Gärtner and Molchanov, 1998).

Sections 3–5 contain the main results of the paper dealing with asymptotic behavior of extremes of the i.i.d. random field $\xi(\cdot)$ with the distribution function satisfying certain RV and continuity conditions at the right endpoint. Functionals (EX), (SP), (MIN) and (NEI) are studied in Sections 3, 4.1–4.3 and 5, respectively.

Section 6 provides an overview of current results on extreme value theory for the spectrum of the Anderson Hamiltonian $\mathcal{H}_V = \kappa\Delta_V + \xi_V$, $V \uparrow \mathbb{Z}^\nu$, with an i.i.d. potential $\xi(\cdot)$. The issues under discussion include the asymptotic expansion formulas and Poisson limit theorems for the largest eigenvalues and their localization centers. We consider separately three cases of the distribution tails $\mathrm{e}^{-Q}$ of $\xi(0)$: the tails are heavier than the double exponential function (Section 6.1); the tails are lighter than the double exponential function (Section 6.2), and the double exponential tails (Section 6.3). As already mentioned, we give proof sketches of most theorems of this section demonstrating their connections to the results of Sections 3–5 on $\xi_V$-extremes. In Section 6.4, we comment and compare the proofs of Poisson limit theorems stated in Sections 6.1 and 6.3 and proved in the earlier papers by Astrauskas and Molchanov (1992), Astrauskas (2007; 2008; 2012; 2013) and Biskup and König (2016).

In Section 7, we discuss the long-time intermittent behavior of the solutions to the parabolic problems associated with the Anderson Hamiltonian $\mathcal{H} = \kappa\Delta + \xi(\cdot)$. We focus on the representation of the solutions in the spectral terms of the operators $\mathcal{H}_V = \kappa\Delta_V + \xi_V$. In view of this representation, we discuss some techniques of the extreme value theory for eigenvalues of $\mathcal{H}_V$, that can be applied to study the intermittency properties of time-dependent Anderson models.

Finally, in Appendix A, we characterize and compare the RV classes of distributions introduced in Sections 3–6.

## 2. Asymptotic expansion formulas for the largest eigenvalues of deterministic Hamiltonians

Let $V = [-n; n]^\nu \cap \mathbb{Z}^\nu$ $(n \in \mathbb{N})$ be a sequence of cubes. By introducing the periodic norm $|x| := |x|_n := \min_{y \in (2n+1)\mathbb{Z}^\nu} |x - y|$, $V$ may be considered as a sequence of tori tending to $\mathbb{Z}^\nu$. We are interested in the finite-volume Schrödinger operators $\mathcal{H}_V = \kappa\Delta_V + \xi_V$ on $l^2(V)$ with periodic boundary conditions. Recall that $\kappa > 0$ is a diffusion constant, $\Delta_V$ denotes the lattice Laplacian on $l^2(V)$ (i.e., a restriction of the operator $\Delta\psi(x) := \sum_{|y-x|=1} \psi(y)$ to torus $V$) and $\xi_V := \{\xi_V(x):\ x \in V\} \in [-\infty;\infty)^{|V|}$ are deterministic functions, i.e., potential. The values $-\infty$ of $\xi_V$ (i.e., "hard obstacles") are allowed to include the cases which are interesting from a physical point of view; see, e.g., (Biskup and König, 2001; König, 2016). Write $V_b := \{x \in V:\ \xi_V(x) > -\infty\}$. Then $\mathcal{H}_V$ is interpreted as an operator on $l^2(V)$ with zero boundary conditions outside $V_b$. The spectral problem

$$\mathcal{H}_V\psi = \lambda\psi \qquad (\lambda \in \mathbb{R};\ \psi \in l^2(V)) \tag{2.1}$$

has $|V_b|$ solutions $\lambda_{1,V} \geqslant \lambda_{2,V} \geqslant \ldots \geqslant \lambda_{|V_b|,V}$, i.e., the ordered eigenvalues of the operator $\mathcal{H}_V$.

In this section, we provide asymptotic expansion formulas for the first $K$ largest eigenvalues $\lambda_{k,V}$ under conditions on the first terms of the variational series $\xi_{1,V} \geqslant \xi_{2,V} \geqslant \ldots \geqslant \xi_{|V|,V}$ of the sample $\xi_V$ and their coordinates $z_{k,V} \in V$ defined by $\xi_{k,V} = \xi_V(z_{k,V})$ $(1 \leqslant k \leqslant |V|)$; here $V = \{z_{k,V}:\ 1 \leqslant k \leqslant |V|\}$. The results of this section follow simply from more general results of (Astrauskas, 2008; 2012) and Section 2.4 of (Gärtner and Molchanov, 1998), where one finds more discussions on the relationship between $\xi_V$-extremes and the top eigenvalues of $\mathcal{H}_V$.

In this section, the proof of the statements relies on deterministic spectral arguments. It is worth mentioning that the (probabilistic) Feynman-Kac representations of the Green function and the principal eigenfunction of Schrödinger operators $\mathcal{H}_V$ as well as the related path decomposition techniques present powerful probabilistic tools for deriving the explicit upper bounds for the principal eigenvalue and eigenfunction of $\mathcal{H}_V$ (Gärtner and Molchanov, 1998; Gärtner et al., 2007). However, this method is not explained in the present section; see Section 7 below for some aspects of these techniques related to the parabolic Anderson models.

### *2.1. Preliminaries: Rough bounds*

We start with the following simple bounds for eigenvalues $\lambda_{k,V}$, provided $|V_b| \geqslant 2$.

**Theorem 2.1.** (i) *For any $V$ and any $\xi_V$,*

$$\xi_{1,V} \leqslant \lambda_{1,V} \leqslant \xi_{1,V} + 2\nu\kappa \quad \textit{and} \quad |\lambda_{l,V} - \xi_{l,V}| \leqslant 2\nu\kappa \quad (2 \leqslant l \leqslant |V_b|). \tag{2.2}$$

(ii) *For any $V$, any $K \leqslant |V_b|$ and $\xi_V$ such that $\min_{1\leqslant k<l\leqslant K} |z_{k,V} - z_{l,V}| \geqslant 2$, we have that*

$$\xi_{l,V} \leqslant \lambda_{l,V} \leqslant \xi_{l,V} + 2\nu\kappa \quad \textit{for all} \quad 1 \leqslant l \leqslant K. \tag{2.3}$$

*Proof.* We repeatedly use the fact that the $K$th eigenvalue $\lambda_{K,V} = \lambda_{K,V}(\xi_V)$ of the operator $\kappa\Delta_V + \xi_V$ is a nondecreasing function in each variable $\xi_V(x)$ tending to infinity ($K \geqslant 1$, $x \in V$); i.e., the monotonicity property of eigenvalues (Lankaster, 1969, Theorem 3.6.3).

(i) To estimate $\lambda_{1,V}$, we abbreviate $\xi'(x) := \xi_{1,V}$ if $x = z_{1,V}$, and $\xi'(x) := -\infty$, otherwise. Note that $\xi'(\cdot) \leqslant \xi_V(\cdot) \leqslant \xi_{1,V}$ in $V$. Therefore, $\lambda_{1,V}$ is bounded from below by $\xi_{1,V}$, i.e., the principal eigenvalue of the operator $\kappa\Delta_V + \xi'_V$. Moreover, $\lambda_{1,V}$ is bounded from above by the principal eigenvalue of the operator $\kappa\Delta_V + \xi_{1,V}$ on $l^2(V)$, which in turn does not exceed $2\nu\kappa + \xi_{1,V}$, since the norm of the Laplacian $\kappa\Delta_V$ is less than $2\nu\kappa$. Similarly, since each eigenvalue $\lambda_{l,V}$ is bounded from above (resp., from below) by the $l$th eigenvalue of the diagonal operator $\xi_V + 2\nu\kappa$ on $l^2(V)$ (resp., $\xi_V - 2\nu\kappa$), we obtain (2.2) for $l \geqslant 2$.

(ii) We need to show the lower bound in (2.3). Without loss of generality, we assume that $\xi_{K,V} > 0$ (this may be achieved by shift transform of $\xi_V$ and $\lambda$ in the spectral problem (2.1)). Write $\mathcal{E}_V^K := \{z_{1,V}, \dots, z_{K,V}\}$. We introduce the following functions: $\zeta(x) := \xi_V(x)$ if $x \in \mathcal{E}_V^K$, and is zero, otherwise; and further on, $\widetilde{\zeta}(x) := 0$ if $x \in \mathcal{E}_V^K$, and $\widetilde{\zeta}(x) := -\infty$, otherwise. Then $\xi_V(\cdot) \geqslant \zeta(\cdot) + \widetilde{\zeta}(\cdot)$ in $V$, therefore, each eigenvalue $\lambda_{l,V}$ is bounded from below by the corresponding eigenvalue $\underline{\lambda}_{l,V}$ of the operator $\kappa\Delta_V + \zeta_V + \widetilde{\zeta}_V$; here $1 \leqslant l \leqslant K$. To estimate $\underline{\lambda}_{l,V}$, we rewrite the corresponding spectral problem in the form:

$$(\lambda - \kappa\Delta_V - \widetilde{\zeta}_V)\psi = \zeta_V\psi \qquad (\lambda > 0, \psi \in l^2(V)) \tag{2.4}$$

and apply the resolvent operator $\mathcal{G}_V(\lambda; \widetilde{\zeta}_V) := (\lambda - \kappa\Delta_V - \widetilde{\zeta}_V)^{-1}$ to both sides of (2.4). Since $\mathcal{G}_V(\lambda; \widetilde{\zeta}_V)\delta_z = \lambda^{-1}\delta_z$ for $z \in \mathcal{E}_V^K$, equation (2.4) is transferred to

$$\psi = \sum_{z \in \mathcal{E}_V^K} \xi_V(z)\psi(z)\lambda^{-1}\delta_z \qquad (\lambda > 0);$$

here $\delta_y(\cdot)$ is the Kronecker delta function. Clearly, for each $1 \leqslant l \leqslant K$, the pair $\underline{\lambda}_{l,V} = \xi_{l,V}$ and $\psi(\cdot\,; \underline{\lambda}_{l,V}) = \delta_{z_{l,V}}(\cdot)$ solves this equation. Summarizing, we have that $\lambda_{l,V} \geqslant \underline{\lambda}_{l,V} = \xi_{l,V}$ ($1 \leqslant l \leqslant K$), as claimed. Theorem 2.1 is proved. □

In Sections 2.2–2.5 below, we consider three classes of functions $\xi_V$:

**(J)** Sparse distinct $\xi_V$-peaks dominate in the landscape of $\xi_V$ as $V \uparrow \mathbb{Z}^\nu$, i.e., $\xi_V$ possess properties like (1.9)–(1.11). Then the $K$th largest eigenvalue $\lambda_{K,V}$ is associated with an isolated peak $\xi_{\tau(K),V}$, so that $\lambda_{K,V} \leftrightarrow z_{\tau(K),V}$ for some $\tau(K) = \tau_V(K) \geqslant 1$ (Section 2.3). In particular, if the functions $\xi_V$ possess extremely sharp peaks like (1.2)–(1.4), then the eigenvalue $\lambda_{K,V}$ is associated with the $K$th largest value of $\xi_V$, viz., $\lambda_{K,V} \leftrightarrow z_{K,V}$ (Section 2.2). In both cases, the lower bounds in (2.3) are achieved as $V \uparrow \mathbb{Z}^\nu$.

**(JJ)** The landscape of $\xi_V$ is dominated by flat islands of large values with an unboundedly increasing diameter. Then the largest eigenvalues are associated with such relevant islands. In this case, the upper bounds in (2.2) are achieved as $V \uparrow \mathbb{Z}^\nu$ (Section 2.4).

**(JJJ)** Similarly as in (JJ), bounded islands of large values prevail in the landscape of $\xi_V$. Then the asymptotic expansion terms of the principal eigenvalue $\lambda_{1,V}$ fill the gap between its lower and upper bounds in (2.2) (Section 2.5).

In the case of (J) we obtain the explicit expansion formulas for eigenvalues in terms of $\xi_V$-values. Meanwhile, for (JJ) and (JJJ) we restrict ourselves to a derivation of the second order expansion formulas for eigenvalues.

### *2.2. Potentials with extremely sharp single peaks*

For $N \geqslant 2$, let us write

$$\mathcal{E}_V^N := \{z_{1,V}, z_{2,V}, \dots, z_{N,V}\} \subset V \tag{2.5}$$

for the subset of coordinates of the first $N$ largest values of $\xi_V$, and

$$r_{N,V} = \min_{1\leqslant l<k\leqslant N} |z_{l,V} - z_{k,V}| = \min_{\substack{x,y\in\mathcal{E}_V^N \\ x\neq y}} |x-y| \tag{2.6}$$

for the minimum distance between sites in $\mathcal{E}_V^N$. For natural $1 \leqslant K = K_V < N = N_V < |V|$ and $p \geqslant 0$, we introduce the following conditions on functions $\xi_V$:

$$\lim_V \min_{1\leqslant l\leqslant K} \xi^p_{(l+1)\wedge K,V}(\xi_{l,V} - \xi_{l+1,V}) = \infty \quad \text{where} \quad \lim_V \xi_{K,V} = \infty, \tag{2.7}$$

$$C := \limsup_V \frac{\xi_{N,V}}{\xi_{K,V}} < 1, \tag{2.8}$$

$$\lim_V \frac{r_{N,V}}{\log N} = \infty, \tag{2.9}$$

$$M := \limsup_V \max_{1\leqslant l\leqslant K} \max_{|x-z_{l,V}|=1} |\xi_V(x)| < \infty \tag{2.10}$$

and, finally,

$$\lim_V \min_{K+1\leqslant l\leqslant N} \xi^2_{K,V}\left(\xi_{K,V} + \frac{2\nu\kappa^2}{\xi_{K,V}} - \xi_{l,V} - \kappa^2 \sum_{|x-z_{l,V}|=1} \frac{1}{\xi_{K,V} - \xi_V(x)}\right) = \infty. \tag{2.11}$$

We write $\xi_{0,V} := \infty$, and $s_V(l) := (\xi_{l-1,V} - \xi_{l,V}) \wedge (\xi_{l,V} - \xi_{l+1,V})$ for $1 \leqslant l \leqslant K$.

**Theorem 2.2.** (i) *Under (2.7) with $p = 0$, we have that*

$$\limsup_V \max_{1\leqslant l\leqslant K} \left|\lambda_{l,V} - \xi_{l,V}\right| s_V(l) \leqslant \text{const}_1(\kappa, \nu).$$

*and*

$$\limsup_V \max_{1\leqslant l\leqslant K} \max_{x\neq z_{l,V}} \frac{\log\left|\psi(x;\lambda_{l,V})\right|}{|x - z_{l,V}| \log s_V(l)} \leqslant -1.$$

(ii) *Under (2.7)–(2.9) with $p = 1$, we have that*

$$\limsup_{V} \max_{1\leqslant l\leqslant K} \bigl|\lambda_{l,V} - \xi_{l,V}\bigr|\xi_{l,V} \leqslant \frac{\text{const}_2(\kappa,\nu)}{1-C}.$$

*and*

$$\limsup_{V} \max_{1\leqslant l\leqslant K} \max_{x\neq z_{l,V}} \frac{\log\bigl|\psi(x;\lambda_{l,V})\bigr|}{|x-z_{l,V}|\log\xi_{l,V}} \leqslant -1. \tag{2.12}$$

(iii) *If $\xi_V$ satisfies (2.7)–(2.11) with $p = 2$, then*

$$\limsup_{V} \max_{1\leqslant l\leqslant K} \left|\lambda_{l,V} - \xi_{l,V} - \frac{2\nu\kappa^2}{\xi_{l,V}}\right|\xi_{l,V}^2 \leqslant \frac{M\cdot\text{const}_3(\kappa,\nu)}{(1-C)^2} + \text{const}_4(\kappa,\nu).$$

*and (2.12) holds true.*

*Proof.* We first note that condition (2.10) implies $\xi_{N,V} > -\infty$ for any $V \supset V_0$. On the other hand, if $\xi_{N,V} = -\infty$ and $r_{N,V} > 1$, then $\lambda_{l,V} = \xi_{l,V}$ for all $1 \leqslant l \leqslant N$. The latter is shown by the same arguments as in the proof of the lower bound in (2.3).

Now, assuming $\xi_{N,V} > -\infty$ and letting $V \uparrow \mathbb{Z}^\nu$, the assertions of Theorem 2.2(i), (ii) and (iii) are derived from Theorem A.1(i), (ii) and (iii), respectively, in (Astrauskas, 2012, Appendix A) with the abbreviations $\Pi := \mathcal{E}_V^N$, $L := \xi_{N,V}$ and $r := r_{N,V}$. □

In the case of conditions (2.7)–(2.9) with $p = 2$ (part (iii) of the theorem), we have imposed additional restrictions (2.10) and (2.11) to control the influence of the lower $\xi_V$-values on the correspondence $\lambda_{K,V} \leftrightarrow z_{K,V}$. If $\xi_{K,V}^2(\xi_{K,V} - \xi_{K+1,V}) = \mathrm{O}(1)$, then the lower $\xi_V$-values may essentially contribute to the expansion of the eigenvalues $\lambda_{K,V}$ and, therefore, the correspondence $\lambda_{K,V} \leftrightarrow z_{K,V}$ fails. See Section 6.1 of the present paper where we consider the case of i.i.d. samples $\xi(\cdot)$ in $V \uparrow \mathbb{Z}^\nu$ with "weakly" pronounced asymptotic peaks.

Let $\xi(\cdot)$ be an i.i.d. random field with the distribution function $1 - \mathrm{e}^{-Q}$. We will show that, if $Q$ satisfies the condition $Q(t) = \mathrm{o}(t^{p+1})$ for $p = 0, 1$ and 2 ("heavy tails" $\mathrm{e}^{-Q}$) and additional RV conditions as $t \to \infty$, then with high probability $\xi_V$ satisfies the assumptions of Theorem 2.2(i), (ii) and (iii), respectively, where $K \in \mathbb{N}$ is fixed and $N = [|V|^\theta]$ for some $0 < \theta < 1/2$; see Theorems 4.3(i), 4.5, 3.1 ($R = 0$), 5.3 and 5.4 with $0 < \varepsilon < \theta$. Therefore, Poisson limit theorems for the largest eigenvalues $\lambda_{K,V}$ are reduced to those for extreme values of i.i.d. random fields $\xi(\cdot)$ or $\xi(\cdot) + 2\nu\kappa^2/(\xi(\cdot) \vee 1)$ (Theorem 6.9).

### *2.3. Potentials with dominating single peaks: The general case*

To simplify the proceedings, we need some notation and remarks. For $N \geqslant 2$ and $\mathcal{E}_V^N$ as in (2.5), we introduce the following function: $\widetilde{\xi}_V(x) := 0$ if $x \in \mathcal{E}_V^N$, and $\widetilde{\xi}_V(x) := \xi_V(x)$ if $x \in V \setminus \mathcal{E}_V^N$. Then

$$\xi_V = \sum_{z\in\mathcal{E}_V^N} \xi_V(z)\delta_z + \widetilde{\xi}_V,$$

i.e., $\xi_V$ is a superposition of $\xi_V$-peaks and the noise component $\widetilde{\xi}_V$. To exclude the trivialities, we assume that $\xi_{N,V} > -\infty$ for each $V$ (for the case when $\xi_{N,V} = -\infty$ and $r_{N,V} > 1$, see the proof of Theorem 2.2 above). For each $z \in V$, let $\widetilde{\lambda}_V(z)$ be the principal eigenvalue of the "single peak" Hamiltonian $\kappa\Delta_V + \xi_V(z)\delta_z + \widetilde{\xi}_V(1-\delta_z)$ on $l^2(V)$. We associate the sites $z_{\tau(l),V} \in V$ with the variational series

$$\widetilde{\lambda}_{1,V} := \widetilde{\lambda}(z_{\tau(1),V}) \geqslant \widetilde{\lambda}_{2,V} := \widetilde{\lambda}(z_{\tau(2),V}) \geqslant \ldots \geqslant \widetilde{\lambda}_{|V|,V} := \widetilde{\lambda}(z_{\tau(|V|),V}) \tag{2.13}$$

based on the sample $\widetilde{\lambda}_V$; here $V = \{z_{\tau(l),V} :\ 1 \leqslant l \leqslant |V|\}$.

**Theorem 2.3.** *Assume that there are natural numbers $1 \leqslant K = K_V < N = N_V < |V|$ such that the functions $\xi_V$ satisfy condition (2.9) and the following conditions:*

$$\lim_V \left(\xi_{K,V} - \xi_{N,V}\right) = \infty \tag{2.14}$$

*and*

$$\liminf_V \min_{1\leqslant l\leqslant K} \frac{\log\left(\widetilde{\lambda}_{l,V} - \widetilde{\lambda}_{l+1,V}\right)}{r_{N,V}\log\left(\xi_{(l+1)\wedge K,V} - \xi_{N,V}\right)} \geqslant 0. \tag{2.15}$$

*Then*

$$\limsup_V \max_{1\leqslant l\leqslant K} \frac{\log\left|\lambda_{l,V} - \widetilde{\lambda}_{l,V}\right|}{r_{N,V}\log\left(\xi_{l,V} - \xi_{N,V}\right)} \leqslant -2. \tag{2.16}$$

*and*

$$\limsup_V \max_{1\leqslant l\leqslant K} \max_{x\neq z_{\tau(l),V}} \frac{\log\left|\psi(x;\lambda_{l,V})\right|}{|x - z_{\tau(l),V}|\log(\xi_{l,V} - \xi_{N,V})} \leqslant -1.$$

*Proof.* We write

$$\widetilde{\mathcal{E}}_{h,V} := \{z \in \mathcal{E}_V^N : \widetilde{\lambda}_V(z) \geqslant \xi_{N,V} + 2\nu\kappa + h\} \quad \text{where} \quad h := \frac{\xi_{K,V} - \xi_{N,V}}{2}. \tag{2.17}$$

By the first bound in Theorem 2.1(i), $\widetilde{\lambda}_V(z) \geqslant \xi_V(z)$ for all $z \in \mathcal{E}_V^N$. This combined with (2.14) gives

$$\left|\widetilde{\mathcal{E}}_{h,V}\right| \geqslant K \tag{2.18}$$

for any $V \supset V_0$. Finally, according to (Astrauskas, 2008, Section 2.2 and Appendix B.2),

$$\xi_V(z) \leqslant \widetilde{\lambda}_V(z) \leqslant \xi_V(z) + 2\nu\kappa^2/h \quad \text{for any} \quad z \in \widetilde{\mathcal{E}}_{h,V}. \tag{2.19}$$

In view of (2.18) and (2.19), the assertion of Theorem 2.3 is derived similarly as in the proof of Theorem B.3 in (Astrauskas, 2008, Appendix B) with the abbreviations $L := \xi_{N,V}$, $\Pi := \mathcal{E}_V^N$ (2.5), $\widetilde{\Pi} := \widetilde{\mathcal{E}}_{h,V}$ (2.17) and $r := r_{N,V}$ (2.6). □

Note that conditions (2.14) and (2.15) of Theorem 2.3 are substantially weaker than (2.8) and (2.7), respectively, in Theorem 2.2. According to (2.16) and (2.19) with $h \to \infty$ as in (2.17), we obtain that $\lambda_{l,V} = \xi_{l,V} + \mathrm{o}(1)$ uniformly in $1 \leqslant l \leqslant K$, so that the eigenvalues $\lambda_{l,V}$ achieve their lower bounds in (2.3) as $V \uparrow \mathbb{Z}^\nu$.

On the other hand, from (Astrauskas, 2008, Appendices A and B) we know that, for each $z \in \widetilde{\mathcal{E}}_{h,V}$, the eigenvalue $\widetilde{\lambda}_V(z)$ is the maximal solution to the equation

$$\mathcal{G}_V(\lambda; \widetilde{\xi}_V; z, z) = \frac{1}{\xi_V(z)}; \tag{2.20}$$

here $\mathcal{G}_V(\lambda; \widetilde{\xi}_V; \cdot, \cdot)$ is the Green function of the Hamiltonian $\kappa\Delta_V + \widetilde{\xi}_V$ on $l^2(V)$, so that $\mathcal{G}_V(\lambda; \widetilde{\xi}_V; z, z)$ is expanded over paths:

$$\mathcal{G}_V(\lambda; \widetilde{\xi}_V; z, z) = \sum_{\Gamma} \kappa^{|\Gamma|} \prod_{v \in V} \big(\lambda - \widetilde{\xi}(v)\big)^{-n_v(\Gamma)}, \tag{2.21}$$

where the sum $\sum_\Gamma$ is taken over all paths $\Gamma : v_0 := z \to v_1 \to \cdots \to v_m := z$ in $V$ such that $|v_i - v_{i-1}| = 1$ for each $1 \leqslant i \leqslant m$ and each $m \in \mathbb{N}$, $n_v(\Gamma)$ denotes the number of times the path $\Gamma$ visits the site $v \in V$, $|\Gamma| := \sum_{v \in V} n_v(\Gamma) - 1 \geqslant 0$. Substituting (2.21) to the left-hand side of (2.20) and iterating this with respect to the eigenvalue $\lambda = \widetilde{\lambda}_V(z)$, we obtain the explicit expansion formulas for $\widetilde{\lambda}_V(z)$ ($z \in \widetilde{\mathcal{E}}_{h,V}$) presented as a power series in the variables $\xi_V(z)$ and $\widetilde{\xi}_V(x)$ ($|x - z| \geqslant 1$), in particular,

$$\begin{aligned}\widetilde{\lambda}_V(z) = \xi_V(z) + \kappa^2 \sum_{|x-z|=1} \frac{1}{\xi_V(z) - \widetilde{\xi}_V(x)} + \\ + \mathrm{O}\Bigg( \sum_{\substack{|x-z|=1 \\ |y-z|=1 \\ |u-z|\leqslant 2}} \frac{1}{(\xi_V(z) - \widetilde{\xi}_V(x))(\xi_V(z) - \widetilde{\xi}_V(y))(\xi_V(z) - \widetilde{\xi}_V(u))} \Bigg)\end{aligned} \tag{2.22}$$

as $V \uparrow \mathbb{Z}^\nu$.

**Remark 2.4.** With notation at the beginning of Section 2.2, assume that there are natural numbers $1 < N = N_V < |V|$ such that the functions $\xi_V$ satisfy the conditions: $\lim_V r_{N,V} = \infty$ and $\lim_V(\xi_{1,V} - \xi_{N,V}) = \infty$. Then there are constants $\mathrm{const}_i = \mathrm{const}_i(\kappa, \nu) > 0$ such that

$$|\lambda_{1,V} - \xi_{1,V}| \leqslant \frac{\mathrm{const}_1}{\xi_{1,V} - \xi_{N,V}} + \frac{\mathrm{const}_2}{r_{N,V}} = \mathrm{o}(1) \tag{2.23}$$

as $V \uparrow \mathbb{Z}^\nu$.

As shown above, limit (2.23) follows from the assumptions of Theorem 2.3 or Theorem 2.2(ii) with $K = 1$. *To prove (2.23)* under the (weaker) conditions of Remark 2.4, we first observe from Theorem 2.1(i) that $\lambda_{1,V} \geqslant \xi_{1,V}$ for all $V$.

Second, we apply Lemma 2.8 below with $R := \frac{1}{3} r_{N,V} \to \infty$ to see that the eigenvalue $\lambda_{1,V}$ is bounded from above (with accuracy $\mathrm{O}(R^{-1})$) by the maximum of local principal eigenvalues of the single-peak Hamiltonians over all balls in $V$ of radius $R$. Using formulas (2.20)–(2.22) for such local principal eigenvalues, we finally obtain the bound $\lambda_{1,V} \leqslant \xi_{1,V} + \mathrm{const}_1/(\xi_{1,V} - \xi_{N,V}) + \mathrm{const}/R$, as claimed. □

**Remark 2.5.** Theorems 2.2, 2.3 and Remark 2.4 include the condition on asymptotic sparseness of $\xi_V$-peaks: for instance, $r_{N,V} \to \infty$ as $V \uparrow \mathbb{Z}^\nu$. Molchanov and Vainberg (1998; 2000) studied the existence and location of spectral components (pure point, absolutely continuous, etc) of the Schrödinger operators $\kappa\Delta + \xi(\cdot)$ in $l^2(\mathbb{Z}^\nu)$ with sparse deterministic potential $\xi(\cdot) = \sum_{k\geqslant 1} a_k \delta_{z_k}(\cdot)$, where amplitudes $a_k$ are bounded and $\{z_k\}$ is a rare subset of $\mathbb{Z}^\nu$, for example, $\widetilde{r}_n := \min_{l\neq n} |z_l - z_n| \to \infty$ as $n \to \infty$. □

Let $\xi(\cdot)$ be an i.i.d. random field with the distribution function $1 - \mathrm{e}^{-Q}$. We will show that, if the tails $\mathrm{e}^{-Q}$ are heavier than the double exponential function (i.e., $\log Q(t) = \mathrm{o}(t)$) and satisfy additional regularity and continuity conditions at infinity, then with probability one $\xi_V$ satisfies the assumptions of Theorem 2.3, where $K \in \mathbb{N}$ is fixed and $N = [|V|^\theta]$ for some $0 < \theta < 1/2$; see Theorems 3.1 ($R = 0$), 4.6 ($\rho = \infty$) and 4.8. Therefore, Poisson limit theorems for the largest eigenvalues $\lambda_{K,V}$ are reduced to those for extremes of nonlinear functions (2.22) on $\xi_V$ (Theorem 6.2).

## *2.4. Potentials with dominating flat increasing islands of high values*

Let $\mathbb{B}_R(z) := \{y \in V\colon\ |y - z| \leqslant R\}$ denote the ball in $V$ with center $z \in V$ and radius $R \geqslant 0$. The following theorem gives a simple condition on $\xi_V$ which ensures that the largest eigenvalues $\lambda_{K,V}$ achieve their upper bounds in (2.2) as $V \uparrow \mathbb{Z}^\nu$.

**Theorem 2.6.** *If*

$$\lim_{R\to\infty} \limsup_{V} \min_{z\in V} \bigl(\xi_{1,V} - \min_{x\in\mathbb{B}_R(z)} \xi_V(x)\bigr) = 0, \tag{2.24}$$

*then, for arbitrarily fixed $K \in \mathbb{N}$,*

$$\lim_{V} \bigl(\lambda_{K,V} - \xi_{K,V}\bigr) = 2\nu\kappa.$$

*Proof.* Because of Theorem 2.1(i), we only need to show the lower limit bound

$$\liminf_{V} \bigl(\lambda_{K,V} - \xi_{1,V}\bigr) \geqslant 2\nu\kappa. \tag{2.25}$$

From (2.24) we see that there exist a sequence $0 < \varepsilon_R \to 0$ and sites $z_V \in V$ such that $\xi_V(\cdot) \geqslant \xi_{1,V} - \varepsilon_R$ in $\mathbb{B}_R(z_V)$ for any $R \geqslant R_0$ and any $V \supset V_0(R)$. Abbreviate $\xi_V^{(R)}(x) := \xi_V(x)$ if $x \in \mathbb{B}_R(z_V)$, and $\xi_V^{(R)}(x) := -\infty$, otherwise. Since $\xi_V(\cdot) \geqslant$

$\xi_V^{(R)}(\cdot)$ in $V$ and $\xi_V^{(R)}(\cdot) \geqslant \xi_{1,V} - \varepsilon_R$ in $\mathbb{B}_R(z_V)$ for $R$ and $V$ as above, the monotonicity property of eigenvalues implies that $\lambda_{K,V} \geqslant \lambda_{K,V}^{(R)} + \xi_{1,V} - \varepsilon_R$, where $\lambda_{K,V}^{(R)}$ is the $K$th eigenvalue of the operator $\kappa\Delta$ on $l^2(\mathbb{B}_R(z_V))$ with zero boundary conditions. Since $\lambda_{K,V}^{(R)}$ tends to $2\nu\kappa$ letting first $V \uparrow \mathbb{Z}^\nu$ and then $R \to \infty$ (Kirsch, 2008, Section 3.1), this estimate implies (2.25), as claimed. □

Clearly condition (2.24) is fulfilled if and only if there are a sequence $R_V \to \infty$ and sites $z_V \in V$ such that

$$\lim_V \big(\xi_{1,V} - \min_{|x-z_V|\leqslant R_V} \xi_V(x)\big) = 0.$$

From the proof of the theorem we know that the eigenvalues $\lambda_{K,V}$ of the operator $\mathcal{H}_V = \kappa\Delta_V + \xi_V$ in $V$ are approximated by the corresponding local eigenvalues in the regions $\mathbb{B}_{R_V;\mathrm{opt}} := \mathbb{B}_{R_V}(z_V) \subset V$ where $\xi_V(\cdot)$ is close to $\xi_{1,V}$, i.e., relevant regions.

Let $\xi(\cdot)$ be an i.i.d. random field with the distribution function $1 - \mathrm{e}^{-Q}$. We will show that, if the tails $\mathrm{e}^{-Q}$ are lighter than the double exponential function (i.e., $t^{-1}\log Q(t) \to \infty$) and satisfy additional RV conditions at infinity, then with probability one $\xi_V$ satisfies the assumption of Theorem 2.6; see Theorem 4.6 with $\rho = 0$ and Theorem 3.1(i) for any large $R$ and $\theta(\cdot) \equiv \theta = \mathrm{const}$. In this case, we obtain the second order expansion formulas for the largest eigenvalues $\lambda_{K,V}$ (Theorem 6.14).

## *2.5. Potentials with dominating bounded islands of high values*

In this section, we describe a class of deterministic functions (potential) $\xi_V : V \to [-\infty;\infty)$ for which the asymptotic terms for the principal eigenvalue $\lambda_{1,V}$ ($V \uparrow \mathbb{Z}^\nu$) fill the gap between its lower and upper bounds in (2.2). We use the variational arguments developed by Gärtner and Molchanov (1998). To formulate the results, we need some abbreviations and remarks related to the variational problems. To emphasize the dependence of $\lambda_{1,V}$ on the sample $\xi_V$, we denote by $\lambda(\xi_V) := \lambda_{1,V}$ the principal eigenvalue of the operator $\mathcal{H}_V = \kappa\Delta_V + \xi_V$ on $l^2(V)$. As in Section 2.4, let $\mathbb{B}_R(z) \subset V$ be the closed ball of radius $R \geqslant 0$ centered at $z \in V$, and let $\mathbb{B}_R := \mathbb{B}_R(0)$.

Given a ball $\mathbb{B} \subset V$, let $\xi_V^{\mathbb{B}}(x) := \xi_V(x)$ if $x \in \mathbb{B}$, and $\xi_V^{\mathbb{B}}(x) := -\infty$, otherwise. As before, $\mathcal{H}_V^{\mathbb{B}} := \kappa\Delta_V + \xi_V^{\mathbb{B}}$ is interpreted as an operator with zero boundary conditions outside $\mathbb{B}$. We write

$$\xi_V(x) = \xi_{1,V} + h_V(x) \quad (x \in V),$$

where the function $h_V \leqslant 0$ admits the interpretation as the shape of $\xi_V$-values close to the maximum $\xi_{1,V}$. Note that

$$\lambda(\xi_V) = \xi_{1,V} + \lambda(h_V) \quad \text{and} \quad \lambda(\xi_V^{\mathbb{B}}) = \xi_{1,V} + \lambda(h_V^{\mathbb{B}}). \tag{2.26}$$

For a fixed constant $0 < \rho < \infty$, we are interested in the following supremum of $\lambda(h^{\mathbb{B}})$ over $h : \mathbb{B} \to [-\infty; 0]$:

$$\sup\Big\{\lambda(h^{\mathbb{B}}) \colon \sum_{x\in\mathbb{B}} \mathrm{e}^{h(x)/\rho} < 1\Big\}.$$

This variational problem is equivalent to the corresponding variational problem in terms of the functionals

$$S^{\mathbb{B}}(p) := \sum_{x\in\mathbb{B}} \sqrt{p(x)}\Delta\sqrt{p}(x) \quad \text{where} \quad p(y) = 0 \quad \text{for} \quad y \in \mathbb{Z}^\nu \backslash \mathbb{B},$$

and

$$I^{\mathbb{B}}(p) := -\sum_{x\in\mathbb{B}} p(x)\log p(x)$$

for $p(\cdot) \in \mathcal{P}(\mathbb{B})$, the set of probability measures on $\mathbb{B}$. More precisely, for a sequence of balls $\mathbb{B}_R \subset \mathbb{Z}^\nu$, the following formulas hold true according to Rayleigh–Ritz theorem and (Gärtner and Molchanov, 1998, Lemmas 2.17 and 1.10):

$$\begin{aligned} &\sup\Big\{\lambda(h^{\mathbb{B}_R}) \colon \sum_{x\in\mathbb{B}_R} \mathrm{e}^{h(x)/\rho} < 1\Big\} \\ &\quad = \sup\Big\{\lambda(h^{\mathbb{B}_R}) \colon \sum_{x\in\mathbb{B}_R} \mathrm{e}^{h(x)/\rho} = 1\Big\} \\ &\quad = \sup_{p\in\mathcal{P}(\mathbb{B}_R)} \Big(\kappa S^{\mathbb{B}_R}(p) - \rho I^{\mathbb{B}_R}(p)\Big), \end{aligned} \tag{2.27}$$

where the right-hand side of (2.27) converges (as $R \to \infty$) to

$$\sup_{p\in\mathcal{P}(\mathbb{Z}^\nu)} \big(\kappa S(p) - \rho I(p)\big) =: 2\nu\kappa q(\rho/\kappa). \tag{2.28}$$

Here $S(p)$ and $I(p)$ are the corresponding functionals on $\mathcal{P}(\mathbb{Z}^\nu)$= the set of probability measures on lattice $\mathbb{Z}^\nu$. It is easy to check that $q : \mathbb{R}_+ \to (0; 1)$ is convex, strictly decreasing and surjective function; $q(0) = 1$ and $q(\infty) = \lim_{\rho\to\infty} q(\rho) = 0$. Moreover, $q(\rho) = (2\rho\log\rho)^{-1}(1+\mathrm{o}(1))$ as $\rho \to \infty$ (Astrauskas, 2008, Proposition 2.1 and Corollary 4.5). The supremum on both sides of (2.27) and (2.28) is attained. Denote by $h_{\mathrm{opt}}^{\mathbb{B}_R}$ the maximizer for the variational problem on the left-hand side of (2.27). Then $p_{\mathrm{opt}}^{\mathbb{B}_R}$ is the maximizer for the right-hand side of (2.27) if and only if $h_{\mathrm{opt}}^{\mathbb{B}_R} = \rho\log p_{\mathrm{opt}}^{\mathbb{B}_R}$. For this and further properties of the maximizes in (2.27) and (2.28) in the limit case $R = \infty$, see (Gärtner and den Hollander, 1999, Sections 0.3 and 0.4) and (Gärtner et al., 2007, Sections 1.3 and 3).

The following theorem tells us that, under reasonable conditions on $\xi_V$, the principal eigenvalue $\lambda_{1,V}$ of the operator $\mathcal{H}_V = \kappa\Delta_V + \xi_{1,V} + h_V$ in $V$ is approximated (letting first $V \uparrow \mathbb{Z}^\nu$ and then $R \to \infty$) by the local principal eigenvalue of the operator restricted to the regions $\mathbb{B}_{R;\mathrm{opt}} := \mathbb{B}_R(z_V) \subset V$ where $h_V$ is close to $h_{\mathrm{opt}}^{\mathbb{B}_R}$, i.e., relevant regions with optimal potential shape.

**Theorem 2.7.** *Given a constant $0 < \rho < \infty$ and a sequence $R \to \infty$, assume that functions $\xi_V$ satisfy the following conditions:*

$$\lim_{R\to\infty} \limsup_{V} \max_{z\in V} \sum_{y\in\mathbb{B}_R(z)} \exp\left\{\frac{\xi_V(y) - \xi_{1,V}}{\rho}\right\} \leqslant 1 \tag{2.29}$$

*and*

$$\liminf_{R\to\infty} \liminf_{V} \max_{z\in V} \min_{y\in\mathbb{B}_R(z)} \left(\xi_V(y) - \xi_{1,V} - h_{\mathrm{opt}}^{\mathbb{B}_R}(y-z)\right) \geqslant 0. \tag{2.30}$$

*Then*

$$\lim_{V} \left(\lambda_{1,V} - \xi_{1,V}\right) = 2\nu\kappa q(\rho/\kappa). \tag{2.31}$$

*Proof.* Limit (2.31) follows from the results of (Gärtner and Molchanov, 1998, Section 2.4) under the stronger conditions on $\xi_V$ including sparseness of clusters of $\xi_V$-extremes. To prove (2.31) under conditions (2.29) and (2.30), we apply the same arguments as in (Gärtner and Molchanov, 1998, the proof of Theorem 2.16) combined with the following lemma by Biskup and König (2001), which is slightly modified for the operator $\mathcal{H}_V$ with periodic boundary conditions:

**Lemma 2.8** (Biskup and König, 2001, Lemma 4.6)**.** *For each $R \in \mathbb{N}$, $V \supset V_0(R)$ and each $\xi_V$,*

$$\lambda(\xi_V) \leqslant \xi_{1,V} + \max_{z\in V} \lambda\left(h_V^{\mathbb{B}_R(z)}\right) + \mathrm{const}\, R^{-1}$$

*for some (universal)* const $> 0$.

We first obtain the upper bound for $\lambda(\xi_V)$. Condition (2.29) implies that there is a sequence $0 < \varepsilon_R \to 0$ such that

$$\max_{z\in V} \sum_{y\in\mathbb{B}_R(z)} \exp\left\{h_V(y)/\rho\right\} < \exp\left\{\varepsilon_R/\rho\right\} \quad \text{for each} \quad V \supset V_0(R).$$

In view of (2.26), this estimate and Lemma 2.8 yield that

$$\begin{aligned}
&\lambda(\xi_V) - \xi_{1,V} \\
&\leqslant \sup\left\{\lambda(h^{\mathbb{B}_R}):\ h(\cdot) \leqslant 0,\ \sum_{y\in\mathbb{B}_R} \mathrm{e}^{h(y)/\rho} < \mathrm{e}^{\varepsilon_R/\rho}\right\} + \frac{\mathrm{const}}{R} \\
&\leqslant \sup\left\{\lambda(h^{\mathbb{B}_R}):\ h(\cdot) \leqslant 0,\ \sum_{y\in\mathbb{B}_R} \mathrm{e}^{h(y)/\rho} < 1\right\} + \varepsilon_R + \frac{\mathrm{const}}{R}
\end{aligned}$$

for $V$ as above. Taking the limit as first $V \uparrow \mathbb{Z}^\nu$ and then $R \to \infty$, and using (2.27)–(2.28), we arrive at

$$\limsup_{V} \left(\lambda_{1,V} - \xi_{1,V}\right) \leqslant 2\nu\kappa q(\rho/\kappa). \tag{2.32}$$

By combining condition (2.30), the monotonicity property of eigenvalues and assertions (2.27)–(2.28), similarly as in the proof of (2.25) we obtain the lower bound

$$\lambda(\xi_V) - \xi_{1,V} \geqslant \lambda(h_{\text{opt}}^{\mathbb{B}_R}) + \mathrm{o}(1) \to 2\nu\kappa q(\rho/\kappa)$$

letting first $V \uparrow \mathbb{Z}^\nu$ and then $R \to \infty$. This and (2.32) conclude the proof of Theorem 2.7. □

If $\rho = \rho_V \to \infty$ in Theorem 2.7, then the "relevant" regions $\mathbb{B}_{R;\text{opt}}$ shrink to single sites and, therefore, we are in the situation of Theorem 2.3. Meanwhile, if $\rho = \rho_V \to 0$, then we stick to the result of Theorem 2.6.

Let $\xi(\cdot\,)$ be an i.i.d. random field with the distribution function $1-\mathrm{e}^{-Q}$. It will be shown that, if the tails $\mathrm{e}^{-Q}$ are the double exponential (i.e., $t^{-1}\log Q(t) \to 1/\rho$) and satisfy additional RV conditions at $\infty$, then with probability one $\xi_V$ satisfies the assumptions of Theorem 2.7; see Theorem 4.7 and Theorem 3.1(i) for arbitrarily large $R$ and $\theta_R(y) \approx 1-\exp\{h_{\text{opt}}^{\mathbb{B}_R}(y)/\rho\}$ $(y \in \mathbb{B}_R)$. (For continuous $Q$, see Corollaries 2.7, 2.12 and 2.15 in (Gärtner and Molchanov, 1998).) In this case, the second order expansion formula for $\lambda_{1,V}$ holds true (Theorem 6.19).

## 3. Clustering of high-level exceedances of i.i.d. random fields

Let $\xi(x)$, $x \in \mathbb{Z}^\nu$, be an i.i.d. random field with the cumulative hazard function $Q$. The main task of the present section is to investigate the almost sure asymptotic structure of clusters ("islands") of bounded size formed by exceedances of the sample $\xi_V$ as $V \uparrow \mathbb{Z}^\nu$. With the abbreviations in Section 1, we also need additional notation. For $\theta < 1$, put $L_{V,\theta} := f((1-\theta)\log|V|)$ where $f := Q^{\leftarrow}$. (Without loss of generality, we write $L_{V,1} := \sup\{t\colon\ Q(t-) = 0\}$, so that almost surely $\xi(x) \geqslant L_{V,1}$ for each $x$.) Let $\mathbb{B}_R(z) := \{x \in \mathbb{Z}^\nu\colon\ |x-z| \leqslant R\}$, and $\mathbb{B}_R := \mathbb{B}_R(0)$. For fixed $R \in \mathbb{N}\cup\{0\}$ and a function $\theta_R(\cdot)\colon\ \mathbb{B}_R \to (-\infty;1]$, we denote by $\mathbb{V}_R$ the set of balls $\mathbb{B}_R(z) \subset V$, and

$$\mathcal{E}_{V,\theta}^R := \{\mathbb{B}_R(z) \subset V\colon\ \xi(y) \geqslant L_{V,\theta_R(y-z)} \ \text{ for all } \ y \in \mathbb{B}_R(z)\},$$

the subset of clusters of $\xi_V$-exceedances in $\mathbb{V}_R$ over the level function $L_{V,\theta_R(\cdot)}$. We abbreviate

$$r(\mathcal{E}_{V,\theta}^R) := \min\{\mathrm{dist}\,(\mathbb{B},\mathbb{B}')\colon\ \mathbb{B}\in\mathcal{E}_{V,\theta}^R,\ \mathbb{B}'\in\mathcal{E}_{V,\theta}^R,\ \mathbb{B}\neq\mathbb{B}'\} \ \text{ if } \ |\mathcal{E}_{V,\theta}^R| \geqslant 2,$$

and $r(\mathcal{E}_{V,\theta}^R) := |V|^{1/\nu}$ if $|\mathcal{E}_{V,\theta}^R| \leqslant 1$, by convention; here $\mathrm{dist}\,(\mathbb{B},\mathbb{B}')$ stands for the lattice $l^1$-distance between balls $\mathbb{B},\mathbb{B}' \subset V$. If $R = 0$ and $\theta := \theta(0)$, then $\mathcal{E}_{V,\theta} := \mathcal{E}_{V,\theta}^0$ shrinks to the subset of single $\xi_V$-exceedances, so that

$$r(\mathcal{E}_{V,\theta}) = \min\{|x-y|\colon\ x \in \mathcal{E}_{V,\theta},\ y \in \mathcal{E}_{V,\theta},\ x \neq y\}.$$

To formulate the main result of this section, we also need the following abbreviations

$$\mu_R := \sum_{y\in\mathbb{B}_R} (1-\theta_R(y)) > 0 \ \text{ and } \ \theta_{\max,R} := \max_{y\in\mathbb{B}_R} \{\theta_R(y)\colon\ \theta_R(y) < 1\}.$$

**Theorem 3.1** (cf. Theorems 2.2–2.7)**.** *For arbitrarily fixed $R \in \mathbb{N} \cup \{0\}$, the following almost sure limits hold true.*

(i) *If $\mu_R < 1$, then*

$$\liminf_V \frac{\log \left|\mathcal{E}_{V,\theta}^R\right|}{\log |V|} \geqslant 1 - \mu_R.$$

(ii) *If $\mu_R < 1$ and, in addition, $Q$ satisfies the condition*

$$\lim_{t \uparrow t_Q} \frac{Q(t-)}{Q(t)} = 1, \tag{3.1}$$

*then*

$$\lim_V \frac{\log \left|\mathcal{E}_{V,\theta}^R\right|}{\log |V|} = 1 - \mu_R.$$

(iii) *If $\mu_R > 1$ and $Q$ satisfies (3.1), then*

$$\lim_V \left|\mathcal{E}_{V,\theta}^R\right| = 0.$$

(iv) *If $\theta_{\max,R} < \mu_R < 1$ and $Q$ satisfies (3.1), then*

$$\lim_V \frac{\log r\left(\mathcal{E}_{V,\theta}^R\right)}{\log |V|} = \frac{2\mu_R - 1}{\nu}.$$

**Remark 3.2.** (a) Clearly, for arbitrary $Q$ and $\mu_R < 1$,

$$\begin{aligned}
&\mathbb{E}\left|\mathcal{E}_{V,\theta}^R\right| \quad (= \text{the mean number of clusters of exceedances}) \\
&= |\mathbb{V}_R| \prod_{y \in \mathbb{B}_R} \mathbb{P}\left(\xi(y) \geqslant L_{V,\theta_R(y)}\right) \\
&\geqslant \text{const}\, |V| \exp\Big\{ - \sum_{y \in \mathbb{B}_R} Q(L_{V,\theta_R(y)}-)\Big\} \geqslant \text{const}\, |V|^{1-\mu_R} \to \infty,
\end{aligned}$$

according to Lemma A.11(iii) in Appendix.

(b) On the other hand, by Lemma A.11(iii), condition (3.1) implies the asymptotic formula

$$Q(f(s)) = s + \mathrm{o}(s) \quad \text{as} \quad s \to \infty, \tag{3.2}$$

which in turn yields, for $\mu_R \geqslant 0$, the upper bound

$$\log \mathbb{E}|\mathcal{E}_{V,\theta}^R| \leqslant (1 - \mu_R + \mathrm{o}(1)) \log |V|$$

as $|V| \to \infty$.

**Remark 3.3** (see part (iv))**.** If $\theta_{\max,R} < \mu_R < 1$, then $1/2 < \mu_R < 1$.

**Remark 3.4.** In the case $R = 0$ and $0 < \theta < 1/2$, i.e., single rare $\xi_V$-peaks, Theorem 3.1 was proved by Astrauskas (2001). For the Gaussian random field $\xi(\cdot)$ with correlated values, the case $R = 0$ was studied by Astrauskas (2003). Here the results depend slightly on the correlation function of $\xi(\cdot)$. Finally, assertion (i) generalizes Corollary 2.15(b) in (Gärtner and Molchanov, 1998) where the continuity of $Q$ is assumed.

*Proof of Theorem 3.1.* To simplify the proof, we assume throughout that $\theta_R(\cdot) \equiv \theta(\cdot) < 1$ in $\mathbb{B}_R$. The general case is treated similarly.

(i) We denote by $\widetilde{\mathbb{V}}_R \subset \mathbb{V}_R$ the maximal subset of nonintersecting balls $\mathbb{B}_R(z)$ in $V$, so that $|\widetilde{\mathbb{V}}_R| \asymp |V|$. The claimed bound is proved by estimating $\big|\mathcal{E}^R_{V,\theta} \cap \widetilde{\mathbb{V}}_R\big|$ similarly as in the proof of Theorem 1 in Astrauskas (2001), where the exceedances $\{\xi(x) \geqslant L_{V,\theta}\}$ $(x \in V)$ are replaced by mutually independent (multiple) exceedances $\{\xi(\cdot) \geqslant L_{V,\theta(\cdot - z)}$ in $\mathbb{B}_R(z)\}$ $(\mathbb{B}_R(z) \in \widetilde{\mathbb{V}}_R)$. In particular, if $Q(t_Q-) = \infty$, we obtain that, for any $-1 < \delta < 0$ and $V \uparrow \mathbb{Z}^\nu$,

$$\mathbb{P}\left(\big|\mathcal{E}^R_{V,\theta} \cap \widetilde{\mathbb{V}}_R\big| \leqslant (1+\delta)\mathbb{E}\big|\mathcal{E}^R_{V,\theta} \cap \widetilde{\mathbb{V}}_R\big|\right)$$

$$\leqslant \exp\big\{ - \operatorname{const}(\delta) \cdot \mathbb{E}\big|\mathcal{E}^R_{V,\theta} \cap \widetilde{\mathbb{V}}_R\big|(1+\mathrm{o}(1))\big\}$$

for some $\operatorname{const}(\delta) > 0$. Since the right-hand side is summable over $V$ according to the assertion of Remark 3.2(a), we conclude the proof of (i) by using the Borel-Cantelli lemma.

(ii) We only need to estimate $|\mathcal{E}^R_{V,\theta}|$ from above. Fix a function $\theta'(\cdot)\colon\ \mathbb{B}_R \to (-\infty; 1)$ such that $\theta'(\cdot) > \theta(\cdot)$ in $\mathbb{B}_R$, and pick a constant $\delta > 1 - \mu'$ where $\mu' := \sum_{y \in \mathbb{B}_R}(1 - \theta'(y))$. We then apply Chebyshev's inequality and the assertion of Remark 3.2(b) to find that, for any $V \supset V_0$,

$$\mathbb{P}\left(\big|\mathcal{E}^R_{V,\theta'}\big| > |V|^\delta\right) \leqslant \mathbb{E}\big|\mathcal{E}^R_{V,\theta'}\big||V|^{-\delta} \leqslant |V|^{-\mathrm{const}} \tag{3.3}$$

where $\operatorname{const} = \operatorname{const}(\theta'(\cdot), \delta) > 0$. Choose a subsequence $\{V(l)\colon\ l \in \mathbb{N}\} \subset \{V\}$ such that

$$V(l) \ \text{ monotonously increases and } \ |V(l)| = 2^l(1+\mathrm{o}(1)) \ \text{ as } \ l \to \infty. \tag{3.4}$$

Since the right-hand side of (3.3) is summable over the subsequence $\{V(l)\}$, the Borel–Cantelli lemma implies that almost surely $\big|\mathcal{E}^R_{V(l),\theta'}\big| \leqslant |V(l)|^\delta$ for all $l \geqslant l_0(\omega)$. Because of the monotonicity of $\mathcal{E}^R_{V,\theta}$ in $L_{V,\theta(\cdot)}$, we obtain that with probability 1, for any $V$ such that $V(l-1) \subset V \subseteq V(l)$ and any $l \geqslant l_0(\omega; \theta(\cdot), \theta'(\cdot))$, the set $\mathcal{E}^R_{V,\theta}$ is contained in $\mathcal{E}^R_{V(l),\theta'}$, therefore,

$$\big|\mathcal{E}^R_{V,\theta}\big| \leqslant \big|\mathcal{E}^R_{V(l),\theta'}\big| \leqslant |V(l)|^\delta \leqslant \operatorname{const}|V|^\delta.$$

Since $\theta'(\cdot) > \theta(\cdot)$ and $\delta > 1 - \mu'$ are chosen arbitrarily, this estimate yields the upper limit bound for $\log\big|\mathcal{E}^R_{V,\theta}\big|$, as claimed.

As in part (ii), it suffices to prove the assertions of (iii)–(iv) for the subsequence $\{V(l)\}$ (3.4) instead of $\{V\}$.

(iii) We note that $\mathcal{E}^R_{V,\theta} \neq \varnothing$ if and only if there exists $\mathbb{B}_R(z) \subset V$ such that $\xi(\cdot) \geqslant L_{V,\theta(\cdot - z)}$ in $\mathbb{B}_R(z)$. According to the assertion of Remark 3.2(b), the probability of the last event does not exceed $\mathbb{E}|\mathcal{E}^R_{V,\theta}| \leqslant |V|^{-\rho}$ for some $0 < \rho < -1 + \mu$. Therefore, $\mathbb{P}(\mathcal{E}^R_{V,\theta} \neq \varnothing) \leqslant |V|^{-\rho}$. Since the latter is summable over $\{V(l)\}$ (3.4), the Borel-Cantelli lemma yields that almost surely $\mathcal{E}^R_{V(l),\theta} = \varnothing$ for all $l \geqslant l_0(\omega)$, as claimed.

(iv) With $\widetilde{\mathbb{V}}_R \subset \mathbb{V}_R$ defined in part (i), the almost sure upper bound for $r(\mathcal{E}^R_{V,\theta} \cap \widetilde{\mathbb{V}}_R) \geqslant r(\mathcal{E}^R_{V,\theta})$ is derived similarly as in the proof of Theorem 2 of

Astrauskas (2001) where the exceedances $\{\xi(x) \geqslant L_{V,\theta}\}$ $(x \in V)$ are replaced by mutually independent (multiple) exceedances $\{\xi(\cdot) \geqslant L_{V,\theta(\cdot-z)}$ in $\mathbb{B}_R(z)\}$ $(\mathbb{B}_R(z) \in \widetilde{\mathbb{V}}_R)$.

To obtain the lower bound for $r\big(\mathcal{E}^R_{V,\theta}\big)$, we first note that the event $\big\{r\big(\mathcal{E}^R_{V,\theta}\big) = 0, \big|\mathcal{E}^R_{V,\theta}\big| \geqslant 2\big\}$ implies that there exists $\mathbb{B}_R(z) \subset V$ such that $\xi(\cdot) \geqslant L_{V,\theta(\cdot-z)}$ in $\mathbb{B}_R(z)$ and $\xi(y) \geqslant L_{V,\theta_{\max}}$ for some $y \in \big(\mathbb{B}_{3R}(z) \setminus \mathbb{B}_R(z)\big) \cap V$. Therefore, as in the proof of (iii) we obtain that, for fixed $y \in \mathbb{Z}^\nu \setminus \mathbb{B}_R$ and for any $V \supset V_0$,

$$\begin{aligned}&\mathbb{P}\left(r\big(\mathcal{E}^R_{V,\theta}\big) = 0,\ \big|\mathcal{E}^R_{V,\theta}\big| \geqslant 2\right)\\ &\leqslant \text{const}\,|V|\mathbb{P}\left(\xi(\cdot) \geqslant L_{V,\theta(\cdot)} \text{ in } \mathbb{B}_R,\ \xi(\mathrm{y}) \geqslant \mathrm{L}_{\mathrm{V},\theta_{\max}}\right) \leqslant |V|^{-\rho}\end{aligned}$$

for some $0 < \rho < \mu - \theta_{\max}$. Second, similarly as in the proof of part (i), we find that $\mathbb{P}\left(\big|\mathcal{E}^R_{V,\theta}\big| < 2\right) \leqslant |V|^{-\text{const}}$ for any $V \supset V_0$ and some const $> 0$. Summarizing these bounds and picking $0 < \varepsilon < (2\mu - 1)/\nu$ arbitrarily, we get that, for any $V \supset V_0$,

$$\begin{aligned}&\mathbb{P}\left(r\big(\mathcal{E}^R_{V,\theta}\big) < |V|^{\varepsilon}\right)\\ &\leqslant \mathbb{P}\left(1 \leqslant r\big(\mathcal{E}^R_{V,\theta}\big) < |V|^{\varepsilon}, \big|\mathcal{E}^R_{V,\theta}\big| \geqslant 2\right) + |V|^{-\text{const}_1} \leqslant |V|^{-\text{const}_2}\end{aligned} \tag{3.5}$$

for some $\text{const}_i > 0$, where the last probability is estimated similarly as in the proof of Theorem 2 of (Astrauskas, 2001) with mutually independent (multiple) exceedances $\{\xi(\cdot) \geqslant L_{V,\theta(\cdot-z)}$ in $\mathbb{B}_R(z)\}$ instead of $\{\xi(x) \geqslant L_{V,\theta}\}$. Since the right-hand side of (3.5) is again summable over $\{V(l)\}$ (3.4), we conclude from the Borel–Cantelli lemma that almost surely $r\big(\mathcal{E}^R_{V(l),\theta}\big) \geqslant |V(l)|^{\varepsilon}$ for any $l \geqslant l_0(\omega;\varepsilon)$, as claimed. This completes the proof of Theorem 3.1. □

**Remark 3.5.** By the same arguments as in the proof above, the assertions of Theorem 3.1 are extended to the following class of high-level exceedances:

For fixed $R \in \mathbb{N} \cup \{0\}$, we denote by $\mathbb{S}_R$ the set of all subsets $U \subset \mathbb{Z}^\nu$, the diameter of which does not exceed $R$. Let $\mathbb{V}_R := \{U \in \mathbb{S}_R\colon\ U \subset V\}$. For a fixed set of functions $\Theta_R := \big\{\theta_{U,R}(\cdot) \in (-\infty;1)^{|U|}\colon\ U \in \mathbb{S}_R\big\}$, let $\mathcal{E}^R_{V,\Theta} \subset \mathbb{V}_R$ be the subset of elements $U \in \mathbb{V}_R$ such that $\xi(\cdot) \geqslant L_{V,\theta_{U,R}(\cdot)}$ in $U$. I.e., $\mathcal{E}^R_{V,\Theta}$ consists of clusters of exceedances in $\mathbb{V}_R$ over level functions $L_{V,\Theta}$. Denote by $r\left(\mathcal{E}^R_{V,\Theta}\right)$ the minimum distance among elements $U, U' \in \mathcal{E}^R_{V,\Theta}$, $U \neq U'$. Finally, let $\mu_R := \sum_{y\in U}(1 - \theta_{U,R}(y))$ be a positive constant independent of $U \in \mathbb{S}_R$, and write $\theta_{\max,R} := \sup_{U\in\mathbb{S}_R} \max_{y\in U} \theta_{U,R}(y)$. With these notation for $\mathcal{E}^R_{V,\Theta}$ and $r\left(\mathcal{E}^R_{V,\Theta}\right)$, the almost sure assertions (i)–(iv) of Theorem 3.1 hold true.

## 4. Spacings of order statistics of i.i.d. random fields

### *4.1. Spacings of consecutive order statistics*

We first formulate the results for the exponential order statistics $\eta_{K,V}$ and their spacings, which are then transferred to $\xi_{K,V} = f(\eta_{K,V})$ under appropriate conditions for $f$.

Note that the random variables

$$\eta_{1,V} - \eta_{2,V}, \ldots, (|V|-1)(\eta_{|V|-1,V} - \eta_{|V|,V}), |V|\eta_{|V|,V}$$

are mutually independent exponentially distributed with mean 1; see, e.g., (Shorack and Wellner, 1986, pp. 336). This property immediately implies the first assertion of the following lemma.

**Lemma 4.1.** (i) *For fixed $K \in \mathbb{N}$,*

$$\lim_V \mathbb{P}(\eta_{1,V} - \eta_{2,V} > t_1, \ldots, \eta_{K-1,V} - \eta_{K,V} > t_{K-1}, \eta_{K,V} - \log|V| > t) = \left(\prod_{l=1}^{K-1} \mathrm{e}^{-lt_l}\right) \frac{1}{(K-1)!} \int_t^\infty \exp\left\{-Ks - \mathrm{e}^{-s}\right\} \mathrm{d}s$$

*for all $t_l \geqslant 0$ $(1 \leqslant l \leqslant K-1)$ and all $t \in \mathbb{R}$.*

(ii) *For an arbitrary sequence $\{K_V\}$ such that $1 \leqslant K_V \leqslant |V|$,*

$$\limsup_V \sqrt{K_V} \max_{K_V \leqslant k \leqslant |V|} \left| \eta_{k,V} - \log \frac{|V|}{k} \right| < \infty \quad \textit{in probability.}$$

*Proof.* Let us show (ii). Write $\eta_{|V|+1,V} := 0$. By Kolmogorov's inequality (Shorack and Wellner, 1986, pp. 843), we have that

$$\mathbb{P}\left( \max_{K_V \leqslant k \leqslant |V|} \left| \sum_{l=k}^{|V|} \left( \eta_{l,V} - \eta_{l+1,V} - \frac{1}{l} \right) \right| > \left( \frac{C}{K_V} \right)^{1/2} \right) \leqslant \frac{K_V}{C} \sum_{l=K_V}^{|V|} \mathbb{E} \left( \eta_{l,V} - \eta_{l+1,V} - \frac{1}{l} \right)^2 \leqslant \frac{2}{C}$$

for any $C > C_0$ and any $V \supset V_0(C)$. Combining this bound with the following simple estimate

$$\max_{K_V \leqslant k \leqslant |V|} \left( \sum_{l=k}^{|V|} \frac{1}{l} - \log \frac{|V|}{k} \right) \leqslant \frac{1}{K_V} \quad (V \supset V_0),$$

we obtain the claimed assertion of (ii). □

The almost sure asymptotic behavior of the random variables $\eta_{K,V}$ and $\eta_{K,V} - \eta_{K+1,V}$ $(|V| \to \infty)$ is more intricate.

**Lemma 4.2.** *For any fixed constants $K \in \mathbb{N}$ and $m \in \mathbb{N} \setminus \{1\}$, the following almost sure limits hold true.*

(i) $$\liminf_V \frac{\log(\eta_{K,V} - \eta_{K+1,V}) + \sum_{i=2}^{m-1} \log_i |V|}{\log_m |V|} = -1,$$

(ii) $$\limsup_V \frac{\eta_{K,V} - \eta_{K+1,V} - K^{-1} \sum_{i=2}^{m-1} \log_i |V|}{\log_m |V|} = \frac{1}{K}$$

*and*

(iii) $$\limsup_{V} \max_{1\leqslant l\leqslant |V|} \left| \eta_{l,V} - \log \frac{|V|}{l} \right| \frac{1}{\log\log |V|} = 1;$$

*here* $\sum_2^1 \ldots := 0$.

*Proof.* Assertions (i) and (ii) follow from more general results for exponential spacings in (Astrauskas, 2006, Corollary 12). Assertion (iii) follows from the corresponding strong limits for the uniform order statistics $\zeta_{|V|-k+1,V}$ (Shorack and Wellner, 1986, pp. 408 and pp. 420–424) via transformation $\eta_{k,V} = -\log \zeta_{|V|-k+1,V}$ $(1 \leqslant k \leqslant |V|)$. □

We now turn to the case $\xi_{k,V} = f(\eta_{k,V})$. For $p \geqslant 0$, we denote by $A\Pi^p_\infty$ the class of functions $f := Q^{\leftarrow}$ that satisfy

$$\lim_{s\to\infty} f(s)^p \big(f(s+c) - f(s)\big) = \infty \quad \text{for any} \quad c > 0, \tag{4.1}$$

and by $A\Pi^p_0$ the class of functions $f$ that satisfy

$$\lim_{s\to\infty} f(s)^p \big(f(s+c) - f(s)\big) = 0 \quad \text{for any} \quad c > 0, \tag{4.2}$$

and, finally, $\mathrm{O}A\Pi^p$ stands for the class of $f$ satisfying

$$f(s)^p \big(f(s+c) - f(s)\big) \asymp 1 \quad \text{as} \quad s \to \infty, \quad \text{for any} \quad c > 0. \tag{4.3}$$

We see that, if $f$ is in $A\Pi^p_\infty$ or $\mathrm{O}A\Pi^p$, then the right endpoint $t_Q$ is infinity or, equivalently, $f(s) \to \infty$ as $s \to \infty$. Of course, $A\Pi^p_0$ includes the trivial case of finite $t_Q > 0$. The characterization of $A\Pi^p_\infty$, $A\Pi^p_0$ and $\mathrm{O}A\Pi^p$ is given in Lemmas A.6, A.7 and A.8 of Appendix A respectively. In particular, the functions $f := Q^{\leftarrow} \in \mathrm{O}A\Pi^p$ are associated with Weibull type distributions $1 - \mathrm{e}^{-Q}$, where $Q(t) \asymp t^{p+1}$ as $t \to \infty$.

**Theorem 4.3** (Cf. (2.7) and (Astrauskas, 2012; 2013))**.** *For fixed natural $K > l \geqslant 1$ and real $p \geqslant 0$, we have the following limits in probability.*

(i) *If $f \in A\Pi^p_\infty$, then*

$$\lim_V \xi^p_{K,V} (\xi_{l,V} - \xi_{K,V}) = \infty. \tag{4.4}$$

(ii) *If $f \in A\Pi^p_0$, then* $\lim_V \xi^p_{K,V} (\xi_{l,V} - \xi_{K,V}) = 0$.

(iii) *If $f \in \mathrm{O}A\Pi^p$, then* $\xi^p_{K,V} (\xi_{l,V} - \xi_{K,V}) \asymp 1$ *as* $|V| \to \infty$.

*Proof.* Using notation (1.22)–(1.24), rewrite the left-hand side of (4.4) in the form

$$f(\eta_{K,V})^p \left( f(\eta_{K,V} + (\eta_{l,V} - \eta_{K,V})) - f(\eta_{K,V}) \right).$$

The claimed assertions follow by applying Lemma 4.1(i). □

To obtain these limits with probability 1, we need the stronger conditions for $f$. Let us abbreviate

$$d_{m,\gamma}(s) := s\Big(\prod_{i=1}^{m-1} \log_i s\Big)(\log_m s)^{1+\gamma} \quad (s \geqslant s_0).$$

**Theorem 4.4** (Cf. (2.7))**.** *For fixed constants $K \in \mathbb{N}$ and $p \geqslant 0$, the following almost sure limits hold true.*

(i) *If $\lim_{s\to\infty} f(s)^p\big(f(s+1/d_{m,\gamma}(s)) - f(s)\big) = \infty$ for some $m \in \mathbb{N}$ and $\gamma > 0$, then*

$$\lim_V \xi^p_{K+1,V}(\xi_{K,V} - \xi_{K+1,V}) = \infty.$$

(ii) *If $\lim_{s\to\infty} f(s)^p\big(f(s + K^{-1}\log d_{m,\gamma}(s)) - f(s)\big) = 0$ for some $m \in \mathbb{N}$ and $\gamma > 0$, then*

$$\lim_V \xi^p_{K+1,V}(\xi_{K,V} - \xi_{K+1,V}) = 0.$$

(iii) *If $\lim_{s\to\infty} \big(f(s + \log s) - f(s)\big) = 0$, then*

$$\lim_V \big(\xi_{K,V} - f(\log|V|)\big) = 0.$$

*Proof.* Assertions (i)–(ii) follow by the same arguments as in the proof of Theorem 4.3, where one applies Lemma 4.2 instead of Lemma 4.1(i). Assertion (iii) follows from Lemma 4.2 (iii). ☐

### *4.2. Spacings of intermediate order statistics*

We denote by $PI_{<2}$ the class of functions $f := Q^{\leftarrow}$ satisfying the condition

$$\limsup_{s\to\infty} \frac{f((1-\varepsilon)s)}{f(s)} < 1 \quad \text{for some} \quad 0 < \varepsilon < 1/2. \tag{4.5}$$

Class (4.5) is characterized in Lemma A.13 (Appendix A).

**Theorem 4.5** (Cf. (2.8))**.** *Assume that $f \in PI_{<2}$ (4.5). Then for fixed $K \in \mathbb{N}$ and $\theta > \varepsilon$, almost surely*

$$\limsup_V \xi_{[|V|^\theta],V}/\xi_{K,V} < \text{const} < 1. \tag{4.6}$$

*Proof.* By Lemma 4.2(iii), with probability one the random variable $\xi_{K,V} = f(\eta_{K,V})$ is bounded from below by $f(\log|V| - 2\log\log|V|)$ and $\xi_{[|V|^\theta],V}$ is bounded from above by $f((1-\theta)\log|V| + 2\log\log|V|)$ for each $V \supset V_0(\omega)$. Substituting these bounds into the left-hand side of (4.6) and using (4.5), we obtain the claimed assertion. ☐

We denote by $RV_\rho$ the class of nondecreasing functions $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for any $c > 1$, $\lim_{s\to\infty} g(cs)/g(s) = c^\rho$. I.e., $g$ is regularly varying at infinity with index $0 \leqslant \rho \leqslant \infty$. The case $\rho = \infty$ (resp., $\rho = 0$) indicates a rapid variation (resp., slow variation) of the function $g$. See Lemma A.3 in Appendix A for a summary of the well-known properties of the class $RV_\rho$.

**Theorem 4.6** (Cf. (2.14) and (2.24)). *For some* $0 \leqslant \rho \leqslant \infty$, *assume that* $\mathrm{e}^f \in RV_\rho$. *Then, for all constants* $0 \leqslant \varepsilon < \theta < 1$, *almost surely*

$$\lim_V \left(\xi_{[|V|^\varepsilon],V} - \xi_{[|V|^\theta],V}\right) = \rho \log \frac{1-\varepsilon}{1-\theta}.$$

*Proof.* To prove this assertion, use Lemma 4.2(iii) and Lemma A.3(ii) similarly as in the proof of Theorem 4.5. ☐

The following statement is closely related to the result of Theorem 4.6 with $0 < \rho < \infty$. As in Section 2.4, let $\mathbb{B}_R(z)$ denote the closed ball in $V$ with the center $z \in V$ and the radius $R \geqslant 0$.

**Theorem 4.7** (Cf. (2.29) and (Gärtner and Molchanov, 1998)). *For some* $0 < \rho < \infty$, *assume that* $\mathrm{e}^f \in RV_\rho$. *Then, for any fixed* $R \in \mathbb{N}$, *almost surely*

$$\limsup_V \max_{z \in V} \sum_{y \in \mathbb{B}_R(z)} \exp\left\{\left(\xi(y) - \xi_{1,V}\right)/\rho\right\} \leqslant 1.$$

*Proof.* For continuous $Q$, this assertion is a straightforward consequence of Corollary 2.12 in (Gärtner and Molchanov, 1998) and Theorem 4.4(iii) above. (In view of Lemma A.3(ii), the condition of Theorem 4.4(iii) follows from the assumption of Theorem 4.7). If the continuity condition on $Q$ is dropped, one applies slightly modified arguments based on the technique of function inversion, e.g., Lemma A.11 in Appendix A. ☐

Recall that the conditions of Theorems 4.6 and 4.7 are discussed in Lemma A.3. In particular, the assumption of Theorem 4.7 implies that $\log Q(t) = t/\rho + \mathrm{o}(t)$ as $t \to \infty$, i.e., the double exponential tails $\mathrm{e}^{-Q}$.

### *4.3. Minimum of spacings*

We first recall some notation from Section 2.3. For fixed $0 < \theta < 1/2$, we write $\widetilde{\xi}(x) := \xi(x)$ if $\xi(x) < f((1-\theta)\log|V|)$, and $\widetilde{\xi}(x) := 0$, otherwise. For any $z \in V$, let $\widetilde{\lambda}(z)$ denote the principal eigenvalue of the "single peak" Hamiltonian $\kappa\Delta_V + \xi(z)\delta_z + \widetilde{\xi}_V(1-\delta_z)$ in $l^2(V)$. As in (2.13), let $\widetilde{\lambda}_{K,V}$ denote the $K$th extreme order statistics of the random field $\widetilde{\lambda}_V$. For $\kappa = 0$ and $0 < \varepsilon < \theta$, we know from Theorem 3.1(i)($R = 0$) that with probability one $\widetilde{\lambda}_{k,V} \equiv \xi_{k,V}$ for all $1 \leqslant k \leqslant |V|^\varepsilon$ and all large $V$. For any $\kappa \geqslant 0$, we are interested in the asymptotic behavior of the minimum of the gaps $\widetilde{\lambda}_{k,V} - \widetilde{\lambda}_{k+1,V}$ $(1 \leqslant k \leqslant |V|^\varepsilon)$ defined by

$$S_{V,\varepsilon} := \min\left\{\widetilde{\lambda}_{k,V} - \widetilde{\lambda}_{k+1,V} \colon\ 1 \leqslant k \leqslant |V|^\varepsilon\right\}.$$

Given a constant $\mu > 0$, we say that the function $F : \mathbb{R} \to \mathbb{R}$ is *log-Hölder continuous of order* $\mu > 0$ *at infinity*, if $F$ satisfies the following condition:

$$\begin{aligned} &|F(t+s) - F(t-s)|\,|\log s|^\mu = \mathrm{O}(1) \\ &\text{as } \ t \to \infty \text{ and } \ s \downarrow 0 \ \text{ simultaneously.} \end{aligned} \tag{4.7}$$

**Theorem 4.8** (Cf. (2.15)). *Let $t_Q = \infty$, $\kappa \geqslant 0$ and $0 < \varepsilon < \theta < 1/2$, and assume that the distribution tails $\mathrm{e}^{-Q}$ are log-Hölder continuous of order $\mu > 0$ at infinity. For $\kappa > 0$, assume additionally that $\mathrm{e}^f \in RV_\infty$. Then almost surely*

$$\limsup_V \frac{\log\{-\log(S_{V,\varepsilon} \wedge 1)\}}{\log|V|} \leqslant \frac{1+\varepsilon}{\mu}.$$

*Proof.* The assertion follows from Lemmas 3.5 and 4.3 in (Astrauskas, 2008) and Theorem 3.1(ii) above, where $R = 0$ and $0 < \theta < 1/2$. □

In (Astrauskas, 2003), the results of Theorem 4.8 are extended to the Gaussian random fields with correlated values.

We end this section with some generalization of Theorem 4.3(iii) for the functions $f \in \mathrm{O}A\Pi^p$ (4.3) associated with Weibull type distributions $1 - \mathrm{e}^{-Q}$, where $Q(t) \asymp t^{p+1}$ as $t \to \infty$.

**Theorem 4.9** (Cf. Lemma 4.2 in (Astrauskas, 2013)). *For some $p \geqslant 0$, assume that $f \in \mathrm{O}A\Pi^p$ (4.3). Then, for arbitrarily fixed constants $K \in \mathbb{N}$, $0 < \varepsilon < 1$ and any sequence $\{n_V\} \subset \mathbb{N}$ such that $n_V = \mathrm{O}(|V|^\varepsilon)$, we have the following limits in probability:*

(i) $$\xi_{n_V,V} \asymp (\log|V|)^{1/(p+1)} \quad \text{as} \quad |V| \to \infty,$$

*and*

(ii) $$0 < \liminf_V \min_{K+1 \leqslant l \leqslant |V|^\varepsilon} \xi_{l,V}^p (\xi_{K,V} - \xi_{l,V}) \frac{1}{\log l} \leqslant \limsup_V \max_{K+1 \leqslant l \leqslant |V|^\varepsilon} \xi_{l,V}^p (\xi_{K,V} - \xi_{l,V}) \frac{1}{\log l} < \infty.$$

*Proof.* Assertion (i) follows from a combination of the formula $\xi_{k,V} = f(\eta_{k,V})$, Lemma 4.1(ii) and the limit $f(s) \asymp s^{1/(p+1)}$ as $s \to \infty$ (the latter follows from Lemma A.8(iii) with $a(\cdot) \equiv \text{const}$). Assertion (ii) is shown by combining the formula $\xi_{k,V} = f(\eta_{k,V})$ and Lemmas 4.1(ii), A.8(iii) similarly as in the proof of Theorems 4.3 and 4.5 above. □

## 5. Neighboring effects for extremes of i.i.d. random fields

We finally study the asymptotic properties of $\eta_V$-values neighboring to $\eta_V$-peaks. It is then straightforward to extend the results for $\eta(\cdot)$ to $\xi(\cdot) = f(\eta(\cdot))$.

The following lemma tells us that, for fixed $y \neq 0$ and for small $\varepsilon > 0$, asymptotic properties of the random variables $\eta(z_{k,V} + y)$ $(1 \leqslant k \leqslant |V|^\varepsilon)$ and their extremes are the same as in the case of exponential i.i.d. random variables.

**Lemma 5.1.** *For fixed $y \in \mathbb{Z}^\nu \backslash \{0\}$, $0 < \varepsilon < 1/2$ and a sequence of integers $K := K_V = \mathrm{O}(|V|^\varepsilon)$, the following assertions hold true.*

(i) $\lim_V \mathbb{P}\big(\eta(z_{K,V} + y) > t\big) = \mathrm{e}^{-t}$ *for all $t \geqslant 0$.*

(ii) *If, in addition,* $K := K_V \to \infty$, *then*

$$\lim_V \mathbb{P}\big( \max_{1 \leqslant l \leqslant K} \eta(z_{l,V} + y) - \log K \leqslant t \big) = \exp\{-\,\mathrm{e}^{-t}\} \quad \textit{for all} \quad t \in \mathbb{R}.$$

(iii)
$$\lim_{M\to\infty} \limsup_V \Big| \max_{M \leqslant l \leqslant |V|^\varepsilon} \frac{\eta(z_{l,V} + y)}{\log l} - 1 \Big| = 0 \quad \textit{in probability.}$$

*Proof.* Here and in the sequel, we need the following key statement (which is frequently used in (Astrauskas, 2013) as well).

**Lemma 5.2.** *Fix a finite subset* $U \subset \mathbb{Z}^\nu \backslash \{0\}$, $U \neq \varnothing$, *and a sequence of nonrandom real functions* $\{D_l(t_U)\colon\ t_U \in \mathbb{R}^{|U|}\}$ $(l \in \mathbb{N})$. *Abbreviate* $\eta(z;l) := D_l(\{\eta(z+x)\colon x \in U\})$ *for* $z \in \mathbb{Z}^\nu$ *and* $l \in \mathbb{N}$. *Finally, pick a sequence of integers* $K := K_V = \mathrm{O}(|V|^\varepsilon)$ *for some* $0 < \varepsilon < \frac{1}{2}$. *Then, for any* $V$ *and any* $t \in \mathbb{R}$,

$$\left| \mathbb{P}\left( \max_{1 \leqslant l \leqslant K} \eta(z_{l,V}; l) \leqslant t \right) - \prod_{l=1}^{K} \mathbb{P}(\eta(0;l) \leqslant t) \right| \leqslant 3|V|^{-\mathrm{const}},$$

*where* const $> 0$ *does not depend on* $V$ *and* $t$.

Now, part (i) of Lemma 5.1 follows from Lemma 5.2 with $U := \{y\}$, where $D_K(t_U) \equiv t_y$ and $D_l(t_U) \equiv 0$ for $l \neq K$. Part (ii) follows from Lemma 5.2, where $D_l(t_U) \equiv t_y$ $(l \in \mathbb{N})$, combined with Lemma 4.1(i). Finally, by Lemma 5.2 with $D_l(t_U) \equiv t_y / \log l$ $(l \geqslant 2)$, we derive that, for any small $\delta > 0$,

$$\limsup_V \mathbb{P}\Big( \max_{M \leqslant l \leqslant |V|^\varepsilon} \frac{\eta(z_{l,V} + y)}{\log l} > 1 + \delta \Big) \leqslant \sum_{l=M}^{\infty} \mathrm{e}^{-(1+\delta)\log l} \to 0$$

and

$$\limsup_V \mathbb{P}\Big( \max_{M \leqslant l \leqslant |V|^\varepsilon} \frac{\eta(z_{l,V} + y)}{\log l} < 1 - \delta \Big) \leqslant \prod_{l=M}^{\infty} \big(1 - \mathrm{e}^{-(1-\delta)\log l}\big) = 0$$

as $M \to \infty$, i.e., assertion (iii) of Lemma 5.1 is proved. □

*Proof of Lemma 5.2.* Fix a constant $\theta \in (\varepsilon, \frac{1}{2})$, so that $K := K_V \leqslant \frac{1}{2}|V|^\theta$ for each $V \supset V_0$. Write $L_V := (1-\theta)\log|V|$. Denote by $\mathcal{E}_V \subset V$ the subset consisting of sites at which $\eta(\cdot)$ exceeds the level $L_V$, and let $r(\mathcal{E}_V)$ be the minimum distance among sites in $\mathcal{E}_V$; cf. the notation at the beginning of Section 3. We abbreviate by $I$ the intervals $(-\infty, t]$ or $(t, \infty)$, where $t \in \mathbb{R}$. Further, pick $\delta$ to satisfy $0 < \delta < (1-2\theta)/\nu$. Now

$$\begin{aligned}
&\mathbb{P}\Big( \max_{1 \leqslant l \leqslant K} \eta(z_{l,V}; l) \in I \Big) \\
&\leqslant \mathbb{P}\Big( \max_{1 \leqslant l \leqslant K} \eta(z_{l,V}; l) \in I,\ 2^{-1}|V|^\theta \leqslant |\mathcal{E}_V| \leqslant 2|V|^\theta,\ r(\mathcal{E}_V) > |V|^\delta \Big) \\
&\quad + \mathbb{P}\big(|\mathcal{E}_V| < 2^{-1}|V|^\theta\big) + \mathbb{P}\big(|\mathcal{E}_V| > 2|V|^\theta\big) + \mathbb{P}\big(r(\mathcal{E}_V) \leqslant |V|^\delta\big)
\end{aligned}$$

$$=: p(I) + p^{(1)} + p^{(2)} + p^{(3)}.$$

Using the continuity of exponential distribution, similarly as in the proof of Theorem 3.1(ii)–(iv) with $R = 0$ and $0 < \theta < 1/2$, we obtain that $p^{(i)} \leqslant |V|^{-\text{const}}$ for some const $> 0$. Thus, to show the assertion of Lemma 5.2, we need to check that

$$p(I) \leqslant \mathbb{P}\Big(\max_{1\leqslant l\leqslant K} \eta(\widetilde{x}_l; l) \in I\Big) + |V|^{-\text{const}_1} \tag{5.1}$$

for a fixed (nonrandom) subset $\widetilde{V} := \{\widetilde{x}_l : 1 \leqslant l \leqslant K\} \subset \mathbb{Z}^\nu$ such that $r(\widetilde{V}) > |V|^\delta$.

Let $\sum_{V'}$ be the sum over all subsets $V' \subset V$ with the properties: $\frac{1}{2}|V|^\theta \leqslant |V'| \leqslant 2|V|^\theta$ and $r(V') > |V|^\delta$. We denote by $\sum_{\{x_l\}}$ the sum over all permutations $x_1, \ldots, x_{|V'|}$ of the subset $V'$. Write $p_V :\, = |V|^{\theta-1}$. Then, for $V \supset V_0$,

$$\begin{aligned} p(I) &\leqslant \sum_{V'} \sum_{\{x_l\}} \mathbb{P}\Big(\max_{1\leqslant l\leqslant K} \eta(x_l; l) \in I, \\ &\qquad \eta(x_1) \geqslant \eta(x_2) \geqslant \ldots \geqslant \eta(x_{|V'|}) \geqslant L_V, \max_{x\in V\setminus V'} \eta(x) < L_V\Big) \\ &\leqslant \sum_{V'} \sum_{\{x_l\}} \mathbb{P}\Big(\max_{1\leqslant l\leqslant K} \eta(x_l; l) \in I, \eta(x_1) \geqslant \eta(x_2) \geqslant \ldots \geqslant \eta(x_{|V'|}) \geqslant L_V, \\ &\qquad \max\{\eta(x) : x \in V\setminus((V' + U) \cup V')\} < L_V\Big), \end{aligned} \tag{5.2}$$

where $V' + U$ denotes the algebraic sum of the subsets $V'$ and $U$. Since all the random variables are mutually independent, the double sum on the right-hand side of (5.2) is equal to

$$\begin{aligned} &\mathbb{P}\Big(\max_{1\leqslant l\leqslant K} \eta(\widetilde{x}_l; l) \in I\Big) \sum_{V'} p_V^{|V'|} (1 - p_V)^{|V| - (|U|+1)|V'|} \\ &\qquad \leqslant \mathbb{P}\Big(\max_{1\leqslant l\leqslant K} \eta(\widetilde{x}_l; l) \in I\Big) + \text{const}\, |V|^{2\theta-1} \quad (V \supset V_0), \end{aligned}$$

since $|V'| p_V \asymp |V|^{2\theta-1}$ via the notation. This completes the proof of (5.1). Lemma 5.2 is proved. □

We now turn to the case $\xi(\cdot) = f(\eta(\cdot))$.

**Theorem 5.3** (Cf. (2.10))**.** *Fix $y \in \mathbb{Z}^\nu \setminus \{0\}$ and $K \in \mathbb{N}$. Then*

$$\limsup_V \big|\xi(z_{K,V} + y)\big| < \infty \quad \textit{in probability}\,.$$

*Proof.* The assertion follows from Lemma 5.1(i). □

To the end of this section, let us fix constants $0 < \varepsilon < \theta < 1/2$. With $L_{V,\theta}$ as in Section 3, we write $\widetilde{\xi}(x) := \xi(x)$ if $\xi(x) < L_{V,\theta}$, and $\widetilde{\xi}(x) := 0$, otherwise. For natural $K \geqslant 1$ and $l > K$, we put

$$\chi_{K,V}(l) := \xi_{K,V}^2 \Big( \xi_{K,V} + 2\nu\kappa^2 \xi_{K,V}^{-1} - \xi_{l,V} \\ - \kappa^2 \sum_{|x|=1} \big( \xi_{K,V} - \widetilde{\xi}(z_{l,V} + x) \big)^{-1} \Big) \mathbb{1}_{\{\xi_{K,V} > L_{V,\theta}\}}; \tag{5.3}$$

here $\mathbb{1}_{\Omega'} := \mathbb{1}_{\Omega'}(\omega)$ denotes the indicator of $\Omega' \subset \Omega$. To study the asymptotic behavior of variables (5.3), we introduce the class $SA\Pi_\infty^2$ of functions $f := Q^{\leftarrow}$ such that

$$\lim_{s\to\infty} \inf_{a\in(c,\theta s)} \Big( f(s)^2 \big( f(s+a) - f(s) \big) - \frac{f(2a)}{c} \Big) = \infty \\ \text{for any } 0 < c < 1 \quad \text{and some } 0 < \theta < 1/2. \tag{5.4}$$

The class $SA\Pi_\infty^2$ is a strict subset of $A\Pi_\infty^2$ (4.1). The following theorem provides some generalization of limit (4.4) for $p = 2$.

**Theorem 5.4** (Cf. (2.11))**.** *Fix $K \in \mathbb{N}$. If $f$ belongs to the classes $SA\Pi_\infty^2$ (5.4) and $PI_{<2}$ (4.5) with $\varepsilon$ and $\theta$ as above, then*

$$\lim_V \min_{K+1\leqslant l\leqslant 2|V|^\theta} \chi_{K,V}(l) = \infty \quad \textit{in probability.}$$

*Proof.* We begin with estimating $\chi(l) := \chi_{K,V}(l)$ for $K+1 \leqslant l \leqslant M$; $M \geqslant M_0$. Let $\Omega_{V,M}^{(1)} \in \mathcal{F}$ denote the subset of configurations $\xi_V^{(\omega)}$ satisfying the following three inequalities:

$$\max_{|x|=1} \max_{K\leqslant l\leqslant M} \xi(z_{l,V} + x) \leqslant f(2\log M), \tag{5.5}$$

$$\xi_{K+1,V} > 0 \quad \text{and} \quad \frac{\widetilde{\xi}(\cdot)}{\xi_{K,V}} \leqslant \frac{L_{V,\theta}}{\xi_{K,V}} \leqslant \text{const}' < 1 \tag{5.6}$$

for const$'$ > const$(\varepsilon)$ specified in Theorem 4.5. According to Lemma 5.1(ii) and Theorem 4.5, we obtain that $\limsup_V \mathbb{P}(\Omega \backslash \Omega_{V,M}^{(1)}) \to 0$ as $M \to \infty$. On the other hand, expanding the sum $\sum_{|x|=1}$ in (5.3) over powers of $\widetilde{\xi}(z_{l,V} + x)/\xi_{K,V}$ with $K+1 \leqslant l \leqslant M$, we get that, for any $M \geqslant M_0$ and any $V \supset V_0(M)$, the inequalities (5.5) and (5.6) imply the following estimate

$$\min_{K+1\leqslant l\leqslant M} \chi(l) \geqslant \big(\xi_{K,V}\big)^2 \big(\xi_{K,V} - \xi_{K+1,V}\big) - \text{const}\, f(2\log M),$$

where const $> 0$ does not depend on $V$ and $M$. From this implication and Theorem 4.3(i) with $p = 2$, we obtain that, for any $C > 0$,

$$\limsup_V \mathbb{P}\big( \min_{K+1\leqslant l\leqslant M} \chi(l) \leqslant C \big) \leqslant \limsup_V \mathbb{P}\big(\Omega \backslash \Omega_{V,M}^{(1)}\big) \to 0 \tag{5.7}$$

as $M \to \infty$.

It only remains to estimate $\chi(l) := \chi_{K,V}(l)$ for $M \leqslant l \leqslant 2|V|^\theta$. Using formulas (1.23) and (1.24), we represent $\xi_{l,V}$ and $\xi(z_{l,V}+x)$ in the form:

$$\xi_{l,V} = f\left(\log\frac{|V|}{l} + \rho_V(l)\right) \quad \text{and} \quad \xi(z_{l,V}+x) = f(\eta(z_{l,V}+x)),$$

where $\rho_V(l) := \eta_{l,V} - \log(|V|/l)$. Denote by $\Omega^{(2)}_{V,M} \in \mathcal{F}$ the subset of configurations $\xi^{(\omega)}(\cdot) = f(\eta^{(\omega)}(\cdot))$ satisfying (5.6) and the following three inequalities:

$$|\rho_V(K)| \leqslant \frac{1}{3}\log M, \qquad \max_{M\leqslant l\leqslant 2|V|^\theta} |\rho_V(l)| < 1 \tag{5.8}$$

and

$$\max_{|x|=1}\max_{M\leqslant l\leqslant 2|V|^\theta} \frac{\eta(z_{l,V}+x)}{\log l} < \frac{3}{2}. \tag{5.9}$$

We then apply Lemma 4.1(ii) to $\rho_V(l)$ and Lemma 5.1(iii) to the left-hand side of (5.9) to obtain that $\limsup_V \mathbb{P}(\Omega\backslash\Omega^{(2)}_{V,M}) \to 0$ as $M \to \infty$. We now write $\xi_+ := \xi \vee 0$ and note that, for any $M \geqslant M_0$ and any $V \supset V_0(M)$, inequalities (5.6), (5.8) and (5.9) imply the following estimate:

$$\begin{aligned}
&\min_{M\leqslant l\leqslant 2|V|^\theta} \chi(l) \\
&\geqslant \min_{M\leqslant l\leqslant 2|V|^\theta}\left[\xi^2_{K,V}(\xi_{K,V} - \xi_{l,V}) - \text{const}\sum_{|x|=1}\xi_+(z_{l,V}+x)\right] \\
&\geqslant \min_{M\leqslant l\leqslant 2|V|^\theta}\left[f\left(\log\frac{|V|}{\sqrt{M}}\right)^2\left(f\left(\log\frac{|V|}{\sqrt{M}}\right) - f\left(\log\frac{|V|}{l}+1\right)\right)\right. \\
&\qquad\qquad \left. - \text{const}' f\left(\frac{3}{2}\log l\right)\right].
\end{aligned} \tag{5.10}$$

Using this implication combined with the fact that, by condition (5.4), the right-hand side of (5.10) tends to infinity as $|V| \to \infty$, we obtain that, for any $C > 0$,

$$\limsup_V \mathbb{P}\left(\min_{M\leqslant l\leqslant 2|V|^\theta}\chi(l) \leqslant C\right) \leqslant \limsup_V \mathbb{P}\big(\Omega\backslash\Omega^{(2)}_{V,M}\big) \to 0$$

as $M \to \infty$. This limit and (5.7) yield the assertion of Theorem 5.4. ☐

## 6. Poisson limit theorems for the largest eigenvalues

This section is to provide an overview of current results on the extreme value theory for the spectrum of the Anderson Hamiltonian $\mathcal{H}_V = \kappa\Delta_V + \xi_V$, $V \uparrow \mathbb{Z}^\nu$, with an i.i.d. potential $\xi(\cdot)$. The results under consideration are taken from (Astrauskas and Molchanov, 1992), (Astrauskas, 2007; 2008; 2012; 2013), (Bishop and Wehr, 2012), (Germinet and Klopp, 2013), (Gärtner and Molchanov, 1998) and (Biskup and König, 2016).

In Section 6.1, we give Poisson limit theorems for the largest eigenvalues and the corresponding localization centers, provided the distribution tails $\mathrm{e}^{-Q}$ of $\xi(0)$ are heavier than the double exponential function (Theorems 6.2 and 6.9). These limit theorems are then complemented and illustrated by the distributions with polynomially decaying tails, Weibull distributions and those with fractional double exponential tails (resp., Examples 6.11, 6.12 and 6.13).

In Section 6.2, we first give the second order expansion formulas for the largest eigenvalues, provided the tails $\mathrm{e}^{-Q}$ are lighter than the double exponential function (Theorem 6.14). For bounded $\xi(0)$, further extensions of this result are discussed.

Section 6.3 provides the second order expansion formulas for the principal eigenvalue in the case of double exponential tails (Theorem 6.19), which are further extended up to Poisson limit theorems for eigenvalues.

In Section 6.4, we comment and compare the proofs of Poisson limit theorems stated in Sections 6.1 and 6.3. We mention, en passant, that Theorems 6.2, 6.9, 6.14 and 6.19 simply follow from the corresponding results of Sections 2–5.

### *6.1. Distribution tails heavier than the double exponential function*

The extreme value theory for i.i.d. random variables $\xi(x)$ deals with the asymptotic behavior of the $K$th largest values $\xi_{K,V}$ of the sample $\xi_V$ as $V \uparrow \mathbb{Z}^\nu$. It is well known that for suitable normalizing constants $a_V > 0$ and $b_V$, the nontrivial limiting (max-stable) distributions $G(\cdot)$ for $\mathbb{P}\big((\xi_{1,V} - b_V)a_V \leqslant \cdot\big)$ are either Weibull law $D_\beta(t) := \exp\{-(-t)^\beta\}$ $(t < 0)$ or Fréchet law $G_\beta(t) = \exp\{-t^{-\beta}\}$ $(t \geqslant 0)$ for some $\beta > 0$, or Gumbel law $G_{\exp}(t) = \exp\{-\mathrm{e}^{-t}\}$ $(-\infty < t < \infty)$; see, e.g., (Resnick, 1987). Note that the weak convergence of maxima to Gumbel law is equivalent to the limit

$$\lim_V |V|\mathbb{P}\big(\xi(0) > b_V + t/a_V\big) = \mathrm{e}^{-t} \quad \text{for all} \quad t \in \mathbb{R}. \tag{6.1}$$

On the other hand, limit (6.1) implies that the point process $\mathcal{N}_V^\xi$ on $[-1/2; 1/2]^\nu \times \mathbb{R}$, defined by

$$\mathcal{N}_V^\xi := \sum_{z \in V} \delta_{\Xi_V(z)} \quad \text{where} \quad \Xi_V(z) := \big(z|V|^{-1/\nu}, (\xi(z) - b_V)a_V\big), \tag{6.2}$$

converges weakly (as $V \uparrow \mathbb{Z}^\nu$) to the Poisson process on $[-1/2; 1/2]^\nu \times \mathbb{R}$ with the intensity measure $\mathrm{d}x \times \mathrm{e}^{-t}\mathrm{d}t$, i.e., the product of Lebesgue measure on $[-1/2; 1/2]^\nu$ and that defined by the increasing function $\log G_{\exp}(\cdot)$ on $\mathbb{R}$; see (Leadbetter et al., 1983).

The necessary and sufficient conditions for (6.1) to hold are generally formulated in terms of $\Gamma$-variation of the function $\mathrm{e}^Q$ at the right endpoint $t_Q$ or, equivalently, in terms of $\Pi$-variation of its inverse $f{\circ}\log$ (Resnick, 1987). We say that $f := Q^{\leftarrow}$ is in the class $A\Pi$, if there exists a function $a : (-\infty; t_Q) \to \mathbb{R}_+$ such that

$$\lim_{s\to\infty} \frac{f(s+c)-f(s)}{a(f(s))} = c \quad \text{for any} \quad c \in \mathbb{R}_+. \tag{6.3}$$

Here $a(\cdot)$ is called an auxiliary function. The class $A\Pi$ (6.3) is an argument-additive version of the original class of $\Pi$-varying functions $f \circ \log$ considered, e.g., in (Resnick, 1987, Section 0.4.3). In Lemma A.1 of Appendix A, we recall the well-known characterization of the class $A\Pi$ in terms of $Q$.

**Lemma 6.1** (Resnick, 1987, Sections 0.4.3 and 1.1)**.** *Limit (6.1) holds true if and only if $f \in A\Pi$ for some auxiliary function $a(\cdot)$. In this case, the normalizing constants can be chosen $b_V = f(\log|V|)$ and $a_V = 1/a(b_V)$.*

We now formulate Poisson limit theorems for the largest eigenvalues $\lambda_{K,V}$ of the random Schrödinger operator $\mathcal{H}_V = \kappa\Delta_V + \xi_V$ introduced in Section 2.1. Throughout this subsection, we assume that $\mathrm{e}^f \in RV_\infty$, so that $\log Q(t) = \mathrm{o}(t)$ as $t \to \infty$ by Lemma A.3 with $\rho = \infty$. This class of distributions includes Weibull distributions (1.5) for arbitrary $\alpha > 0$ and those with fractional double exponential tails (1.8) with $\gamma < 1$. Using the notation from Section 4.3, for fixed small $0 < \theta < 1/2$, we write $\widetilde{\xi}(x) := \xi(x)$ if $\xi(x) < L_{V,\theta} := f((1-\theta)\log|V|)$, and $\widetilde{\xi}(x) := 0$, otherwise. For any $z \in V$, denote by $\widetilde{\lambda}(z)$ the principal eigenvalue of the "single peak" Hamiltonian $\kappa\Delta_V + \xi(z)\delta_z + \widetilde{\xi}_V(1-\delta_z)$. Let $\widetilde{\lambda}_{K,V}$ be the $K$th order statistics of the stationary random field $\widetilde{\lambda}(\cdot)$ in $V$, and let $z_{\tau(K),V} \in V$ stand for its location defined by $\widetilde{\lambda}(z_{\tau(K),V}) := \widetilde{\lambda}_{K,V}$. (Recall that the sites $z_{l,V} \in V$ $(1 \leqslant l \leqslant |V|)$ are associated with the variational series (1.1) based on $\xi_V$.) Note that, for $Z := z_{\tau(K),V}$ and $K \in \mathbb{N}$ fixed, the eigenvalues $\widetilde{\lambda}(Z)$ are expanded into a certain power series in the variables $\xi(Z)$ and $\widetilde{\xi}(x)$ $(x \in V)$; cf. (2.20)–(2.22).

**Theorem 6.2** (see Theorem 4 in (Astrauskas, 2007) and Theorem 5.2 in (Astrauskas, 2008))**.** *Let $t_Q = \infty$ and $\mathrm{e}^f \in RV_\infty$, and assume that $\mathrm{e}^{-Q}$ is log-Hölder continuous of order $\mu > (1+\theta)\nu/(1-2\theta)$ at infinity for some small $\theta > 0$ as above, i.e., (4.7) holds true. Then the following assertions* (I)–(II) *hold true:*

(I) (Poisson limit theorem) *Assume, additionally, that there exist the normalizing constants $A_V > 0$ and $B_V$ such that*

$$\lim_V |V|\mathbb{P}(\widetilde{\lambda}(0) > B_V + A_V^{-1}t) = \mathrm{e}^{-t} \quad \textit{for any} \quad t \in \mathbb{R}, \tag{6.4}$$

*and define the point process $\mathcal{N}_V^\lambda$ on $[-1/2; 1/2]^\nu \times \mathbb{R}$ by*

$$\mathcal{N}_V^\lambda := \sum_{k=1}^{|V|} \delta_{\Lambda_V(k)} \quad \textit{where} \quad \Lambda_V(k) := \left( \frac{z_{\tau(k),V}}{|V|^{1/\nu}}, (\lambda_{k,V} - B_V)A_V \right); \tag{6.5}$$

*then $\mathcal{N}_V^\lambda$ converges weakly to the Poisson process $\mathcal{N}$ on $[-1/2; 1/2]^\nu \times \mathbb{R}$ with the intensity measure $\mathrm{d}\,x \times \mathrm{e}^{-t}\mathrm{d}\,t$.*

(II) (Exponential localization) *Fix a constant $\varepsilon$ such that $0 < \varepsilon < \theta$ and write $M_V := \log(L_{V,\varepsilon} - L_{V,\theta})$ (so that $M_V \to \infty$); then with probability one*

$$\limsup_{V} \max_{1 \leqslant K \leqslant |V|^{\varepsilon}} \max_{x \neq z_{\tau(K),V}} \frac{\log |\psi(x; \lambda_{K,V})|}{M_V |x - z_{\tau(K),V}|} \leqslant -1. \tag{6.6}$$

*Sketch of the proof of Theorem 6.2(I).* Using Theorem 4.6 with $\rho = \infty$, Theorems 3.1 and 4.8 with $R = 0$ and $0 < \varepsilon < \theta < 1/2$, we obtain that almost surely $\xi_V$ satisfies the assumptions of Theorem 2.3, where $K \in \mathbb{N}$ is fixed and $N := [|V|^{\theta}]$. Theorem 2.3 implies that almost surely $\lambda_{K,V} = \widetilde{\lambda}_{K,V} + \mathrm{O}\big(\exp\{-|V|^{(1+\theta)/\mu}\}\big)$ as $|V| \to \infty$, for fixed $K \in \mathbb{N}$. This asymptotic formula in turn yields that the point process $\mathcal{N}_V^{\lambda}$ is approximated by the corresponding point process $\widetilde{\mathcal{N}}_V^{\lambda}$ where $\lambda_{k,V}$ are replaced by $\widetilde{\lambda}_{k,V}$ $(1 \leqslant k \leqslant |V|)$; see the proof of Theorem 4 in (Astrauskas, 2007). The weak convergence of $\widetilde{\mathcal{N}}_V^{\lambda}$ to $\mathcal{N}$ is shown by checking Leadbetter's mixing conditions for the random field $\widetilde{\lambda}(\cdot)$ (Astrauskas, 2007, Lemma 6). This concludes the proof of the theorem. □

In Corollaries 6.3–6.7 below, we give the alternative conditions on $Q$ (where $\log Q(t) = \mathrm{o}(t)$) for the Poisson convergence of the largest eigenvalues to hold.

**Corollary 6.3** (*Specification of the normalizing constants $A_V > 0$ and $B_V$ in (6.4) for some examples of potential distributions)***.** Let $Q(t) = t^{\alpha}$ for $t \geqslant 0$ where $\alpha > 0$ (Weibull distribution), or $Q(t) = \mathrm{e}^{t^{\gamma}}$ for $t \geqslant t_0$; $0 < \gamma < 1$ (fractional double exponential distribution). Consequently, $Q$ satisfies the regularity and continuity conditions of Theorem 6.2. Moreover, the equations for the normalizing constants $A_V > 0$ and $B_V$ in (6.4) are derived by applying a certain iteration scheme for $\widetilde{\lambda}_{1,V}$ as in (2.20)–(2.22) combined with Laplace's method for the corresponding integrals (Astrauskas, 2008, Section 6), (Astrauskas, 2016); see also (Astrauskas, 2013, Section 3). From these equations one derives the explicit expansion formulas for $B_V$ and hence those for the eigenvalues $\lambda_{K,V}$ up to the random max-stable fluctuations of order $\mathrm{O}(A_V^{-1})$; cf. Examples 6.12 and 6.13 below.

**Corollary 6.4.** (*Specification of the normalizing constants $A_V > 0$ and $B_V$ in (6.4) under general RV conditions on potential distributions*). Let $t_Q = \infty$ and, for some large $t_0$, assume that $Q : [t_0; \infty) \to \mathbb{R}_+$ is (locally) absolutely continuous with the positive density $Q' : [t_0; \infty) \to \mathbb{R}_+$ obeying the following conditions:

$$\lim_{t \to \infty} \frac{Q'(t + C)}{Q'(t)} = 1 \quad \text{for any} \quad C > 0, \tag{6.7}$$

and

$$\liminf_{t \to \infty} Q'(t) > 0. \tag{6.8}$$

Consequently, the function $Q$ satisfies the regularity and continuity conditions of Theorem 6.2. Then the centralizing constants $B_V$ in (6.4) are defined by the equation:

$$\mathbb{P}(\widetilde{\lambda}(0) > B_{V}) = |V|^{-1} \quad (V \supset V_0), \tag{6.9}$$

and the normalizing constants $A_V > 0$ in (6.4) and $a_V > 0, b_V$ in (6.1) are specified as follows:

$$A_V = a_V := Q'(b_V) \quad \text{where} \quad b_V := f(\log|V|) \quad (V \supset V_0). \tag{6.10}$$

Therefore, for any $t \in \mathbb{R}$ and $|V| \to \infty$,

$$|V|\mathbb{P}\Big(\xi(0) > b_V + \frac{t}{a_V}\Big) \to \mathrm{e}^{-t} \quad \text{and} \quad |V|\mathbb{P}\Big(\widetilde{\lambda}(0) > B_{V} + \frac{t}{a_V}\Big) \to \mathrm{e}^{-t}, \tag{6.11}$$

where

$$B_V = b_V + \mathrm{o}(1). \tag{6.12}$$

Corollary 6.4 ensures that both the distribution functions $\mathbb{P}(\xi(0) \leqslant t) = 1 - \mathrm{e}^{-Q(t)}$ and $\mathbb{P}(\widetilde{\lambda}(0) \leqslant t)$ $(t \in \mathbb{R})$ are in the domain of attraction of the max-stable Gumbel law $G_{\exp}(\cdot)$; cf. Lemma 6.1 and the assertions before this lemma. Note also that the additional condition (6.8) is to exclude the heavy-tailed ("subexponential") distributions $1 - \mathrm{e}^{-Q}$ which are considered in Theorem 6.9(C0) and Example 6.11 below.

*Proof of Corollary 6.4.* We observe from Lemmas A.4(I) and A.3 with $\rho = \infty$ that conditions (6.7) and (6.8) imply the regularity and continuity conditions of Theorem 6.2. Further, from Lemma A.4(III) and Lemma A.1 we derive that $f \in A\Pi$ (6.3) with the auxiliary function $a(\cdot) \equiv 1/Q'(\cdot)$ in $[t_0; \infty)$; therefore, by Lemma 6.1 we obtain the first limit in (6.11).

To prove the second limit in (6.11), we first notice that, for each $V \supset V_0$, there is a solution $B_V$ of equation (6.9) because of the continuity of the distribution function of $\widetilde{\lambda}(0)$. Let us show (6.12). Since $\widetilde{\lambda}(0) \geqslant \xi(0)$, we get from (6.9) that

$$|V|^{-1} = \mathbb{P}(\widetilde{\lambda}(0) > B_{V}) \geqslant \mathbb{P}(\xi(0) > B_{V}) = \mathrm{e}^{-Q(B_V)},$$

therefore, $B_V \geqslant b_V = f(\log|V|)$ for $V \supset V_0$. If $\xi(0) \geqslant L_{V,\varepsilon} := f((1-\varepsilon)\log|V|)$ for some $0 < \varepsilon < \theta$, then we get from (2.20)–(2.22) that almost surely $\widetilde{\lambda}(0) \leqslant \xi(0) + \beta_V$ for some (nonrandom) $0 < \beta_V \downarrow 0$ as $|V| \to \infty$. Thus, for $V \supset V_0$,

$$\begin{aligned} |V|^{-1} &= \mathbb{P}(\widetilde{\lambda}(0) > B_{V}) = \mathbb{P}(\widetilde{\lambda}(0) > B_{V}, \xi(0) \geqslant L_{V,\varepsilon}) \\ &\leqslant \mathbb{P}(\xi(0) > B_{V} - \beta_V) = \mathrm{e}^{-Q(B_V - \beta_V)}, \end{aligned}$$

therefore, $B_V \leqslant b_V + \beta_V = b_V + \mathrm{o}(1)$. These estimates imply (6.12), as claimed. To prove the second limit in (6.11), we also need the following observations. First, since $\liminf_V a_V > 0$, it follows that $\limsup_V |V|\mathbb{P}(\xi(0) \geqslant b_V + M) \to 0$

as $M \to \infty$. Second, for any $M \geqslant M_0$ and any $V \supset V_0(M)$, if $\xi(0) \leqslant b_V - M$, then $\widetilde{\lambda}(0) < b_V - M/2$. These two assertions imply that, for any $t \in \mathbb{R}$,

$$|V|\mathbb{P}\big(\widetilde{\lambda}(0) > B_V + ta_V^{-1}\big) = |V|\mathbb{P}\big(\widetilde{\lambda}(0) > B_V + ta_V^{-1}, |\xi(0) - b_V| < M\big) + \mathrm{o}_{V,M}(1), \quad (6.13)$$

where $\mathrm{o}_{V,M}(1) \to 0$ letting first $V \uparrow \mathbb{Z}^\nu$ and then $M \to \infty$. Thus, it suffices to check that, for any $t \in \mathbb{R}$, any $M \geqslant M_0(t)$ and $V \uparrow \mathbb{Z}^\nu$,

$$\mathbb{P}\big(\widetilde{\lambda}(0) > B_V + ta_V^{-1}, |\xi(0) - b_V| < M\big) = \mathrm{e}^{-t}\mathbb{P}\big(\widetilde{\lambda}(0) > B_V, |\xi(0) - b_V| < M\big)(1 + \mathrm{o}(1)). \quad (6.14)$$

To prove (6.14), we follow the arguments of the paper (Biskup and König, 2016, Section 7.1) which are now simplified and adapted to our case $\log Q(t) = \mathrm{o}(t)$. (Recall that this paper considers the case of double exponential tails of potential, i.e., $\log Q(t) \approx t/\rho$.) The main idea here is the observation that the shift of the eigenvalue $\widetilde{\lambda}(0)$ by $ta_V^{-1}$ is achieved by the corresponding shift of the single $\xi_V$-peak $\xi(0)$ on the left-hand side of (6.14). Indeed, write $\xi^{(t)} := \xi(0) - ta_V^{-1}$, and denote by $\lambda^{(t)}$ the principal eigenvalue of the Hamiltonian $\kappa\Delta_V + \xi^{(t)}\delta_0 + \widetilde{\xi}_V(1-\delta_0)$ in $l^2(V)$. Note that $\xi^{(0)} = \xi(0)$ and $\lambda^{(0)} = \widetilde{\lambda}(0)$. Fix $t \geqslant 0$. Comparing expansion formulas (2.20)–(2.22) for $\lambda^{(t)}$ with those for $\widetilde{\lambda}(0)$, we find that, for any (small) $\varepsilon > 0$, any $M \geqslant M_0(t, \varepsilon)$ and any $V \supset V_0(M, t, \varepsilon)$,

$$\text{if} \quad |\xi(0) - b_V| < M, \quad \text{then} \quad \lambda^{(t+\varepsilon)} \leqslant \widetilde{\lambda}(0) - ta_V^{-1} \leqslant \lambda^{(t)};$$

therefore, we obtain the following bounds for the left-hand side of (6.14):

$$\begin{aligned} &\mathbb{P}\big(\lambda^{(t+\varepsilon)} > B_V, |\xi^{(t+\varepsilon)} - b_V| < M/2\big) \\ &\quad\leqslant \mathbb{P}\big(\widetilde{\lambda}(0) > B_V + ta_V^{-1}, |\xi(0) - b_V| < M\big) \\ &\quad\leqslant \mathbb{P}\big(\lambda^{(t)} > B_V, |\xi^{(t)} - b_V| < 2M\big). \end{aligned} \quad (6.15)$$

Since $\varepsilon > 0$ is arbitrarily small, it suffices to prove limit (6.14) for the upper and lower bounds in (6.15). We write $\lambda^{(t)}(\xi_V) = \lambda^{(t)}$ to emphasize the dependence of $\lambda^{(t)}$ on the sample $\xi_V = \{\xi(x)\}_{x \in V}$. Let us consider the functions $\lambda^{(t)}(s_V)$ of $s_V = \{s(x)\}_{x \in V} \in \mathbb{R}^{|V|}$ and the corresponding integrals on the right hand-side of (6.15) with respect to the probability measure $\prod_{x \in V} \mathrm{d}F(s(x))$; here $F := 1 - \mathrm{e}^{-Q}$ stands for the distribution function of $\xi(0)$ with the density $p(\cdot) := F'(\cdot)$ in $[t_0; \infty)$. By the change of variables $u(0) := s(0) - ta_V^{-1}$ and $u(x) := s(x)$ for all $x \in V \setminus \{0\}$, we get that $\lambda^{(t)}(s_V) = \lambda^{(0)}(u_V)$; therefore, for fixed $M > 0$ and $V \uparrow \mathbb{Z}^\nu$,

$$\mathbb{P}\big(\lambda^{(t)} > B_V, |\xi^{(t)} - b_V| < M\big) = \int_{\mathbb{R}^{|V|}} \mathbb{1}_{\left\{\lambda^{(t)}(s_V) > B_V, |s(0) - ta_V^{-1} - b_V| < M\right\}} \prod_{x \in V} \mathrm{d}F(s(x))$$

$$= \int_{\mathbb{R}^{|V|}} \mathbb{1}_{\left\{\lambda^{(0)}(u_V) > B_V, |u(0) - b_V| < M\right\}} \frac{p(u(0) + t a_V^{-1})}{p(u(0))} \prod_{x \in V} \mathrm{d}F(u(x))$$

$$= \mathbb{E}\left(\mathbb{1}_{\left\{\lambda^{(0)}(\xi_V) > B_V, |\xi(0) - b_V| < M\right\}} \frac{p(\xi(0) + t a_V^{-1})}{p(\xi(0))}\right) \tag{6.16}$$

$$= \mathbb{P}\big(\lambda^{(0)}(\xi_V) > B_V, |\xi(0) - b_V| < M\big)\big(\mathrm{e}^{-t} + \mathrm{o}(1)\big)$$

by applying Lemma A.4(IV) to the ratio of the densities in the last expectation (6.16) where $a_V = Q'(b_V)$. Formula (6.16) combined with (6.15) implies (6.14) for $t \geqslant 0$, as claimed. Since the case $t \leqslant 0$ is treated similarly, this concludes the proof of assertions of Corollary 6.4. □

**Corollary 6.5** (*Suppression of the* log*-Hölder continuity of* $\mathrm{e}^{-Q}$ *in Theorem 6.2)***.** Let $t_Q = \infty$. Assume that $\mathrm{e}^f \in RV_\infty$, and let assumption (6.4) be fulfilled with

$$A_V = \mathrm{O}\big((L_{V,\varepsilon} - L_{V,\varepsilon'})^{|V|^{(1-2\delta)/\nu}}\big) \quad \text{for some constants} \quad 0 < \varepsilon < \varepsilon' < \delta < \frac{1}{2}. \tag{6.17}$$

Then the point process $\mathcal{N}_V^{\lambda}$ (6.5) converges weakly to the Poisson process $\mathcal{N}$ as in Theorem 6.2(I) and, moreover, the $K$th eigenfunction obeys exponential localization (6.6) in probability.

To *prove Corollary 6.5*, we again apply Theorem 2.3. So we need to show that the samples $\xi_V$ satisfy limits (2.9), (2.14) and (2.15) in probability with the same abbreviation as in the proof of Theorem 6.2. First, since the condition $\mathrm{e}^f \in RV_\infty$ implies (3.1) (see Lemma A.3 with $\rho = \infty$), we may apply Theorem 3.1(iv) with $R = 0$ to obtain limit (2.9) with $N = [|V|^\theta]$ where $\varepsilon' < \theta < \delta$. Second, by Theorem 4.6 with $\rho = \infty$, the condition $\mathrm{e}^f \in RV_\infty$ yields (2.14) with fixed $K \in \mathbb{N}$ and $N$ as above. It remains to prove limit (2.15) in probability with those $K$ and $N$. As mentioned in the proof of Theorem 6.2, assumption (6.4) implies that the point process $\widetilde{\mathcal{N}}_V^{\lambda}$, based on the sample $\widetilde{\lambda}_V$, converges weakly to the corresponding Poisson process. This convergence in turn yields that with probability $1 + \mathrm{o}(1)$ the normalized spacings $A_V\big(\widetilde{\lambda}_{k,V} - \widetilde{\lambda}_{k+1,V}\big)$ are bounded away from zero as $V \uparrow \mathbb{Z}^\nu$, for any fixed $k \in \mathbb{N}$ (Astrauskas, 2007, Corollary 1(jj)). Combining this with the upper bound (6.17) for $A_V$ and observing from Lemma 4.2(iii) and Theorem 3.1(iv) ($R = 0$) that almost surely $\xi_{K,V} - \xi_{N,V} \geqslant L_{V,\varepsilon} - L_{V,\varepsilon'}$ and $r_{N,V} \geqslant |V|^{(1-2\delta)/\nu}$ for any $V \supset V_0$, we arrive at limit (2.15) in probability with $N = [|V|^\theta]$, as claimed. The assertions of Corollary 6.5 are proved. □

**Corollary 6.6.** (*The second order expansion formula for the top eigenvalues*). Let again $t_Q = \infty$, and $\mathrm{e}^f \in RV_\infty$. Assume, in addition, that $f(s + \log s) - f(s) \to 0$ as $s \to \infty$. Then, for fixed $K \geqslant 1$, with probability one

$$\lim_V \big(\lambda_{K,V} - f(\log|V|)\big) = \lim_V \big(\xi_{K,V} - f(\log|V|)\big) = 0.$$

Notice that the additional condition of Corollary 6.6 is to exclude the heavy-tailed ("subexponential") distributions $1-\mathrm{e}^{-Q}$, or in other words, the class of i.i.d. potentials whose extremes possess sharp random fluctuations; see Lemma 4.4(iii).

*Proof of Corollary 6.6.* We first obtain from Theorem 3.1(iv) with $R=0$ and Theorem 4.6 with $\rho=\infty$ that the samples $\xi_V$ $(V\uparrow\mathbb{Z}^\nu)$ satisfy almost surely the conditions of Theorem 2.1(ii) and Remark 2.4 with $N:=[|V|^\theta]$ for some $0<\theta<1/2$. Thus, using the lower bound for $\lambda_{K,V}$ (Theorem 2.1(ii)) and the almost sure limit (2.23) for $\lambda_{1,V}$ (Remark 2.4) combined with Theorem 4.4 (iii), we arrive at the assertion of Corollary 6.6. □

**Corollary 6.7.** (*Localization centers*). Assume that $Q$ satisfies the conditions of Theorem 6.2 on the regular increase (i.e. $\mathrm{e}^f\in RV_\infty$) and the log-Hölder continuity at infinity. Then almost surely the eigenfunction $\psi(\cdot\,;\lambda_{K,V})$ is asymptotically delta-function at the site $z_{\tau(K),V}\in V$ for each $K=\mathrm{o}(|V|^\varepsilon)$; see Theorem 6.2(II). Consequently, any site $z_{\tau(k),V}$ in (6.5) can alternatively be defined as a localization center of the eigenfunction $\psi(\cdot\,;\lambda_{k,V})$, viz.

$$\psi(z_{\tau(k),V};\lambda_{k,V}):=\max_{1\leqslant l\leqslant|V|}\psi(z_{l,V};\lambda_{k,V})\quad\text{for some}\quad\tau(k)=\tau_V(k),\tag{6.18}$$

for all $1\leqslant k\leqslant|V|$. □

The latter definition of the sites $z_{\tau(k),V}$ in (6.5) is more natural in the context of the localization theory for the Anderson Hamiltonians.

The asymptotic behavior of the localization indices $\tau(K)=\tau_V(K)$ is studied by Astrauskas (2013).

**Lemma 6.8** (Astrauskas, 2013, Theorem 2.1)**.** *Assume that the condition of Theorem 6.2 on the log-Hölder continuity of* $\mathrm{e}^{-Q}$ *at infinity holds true. Fix* $K\in\mathbb{N}$.

(i) *If* $f\in OA\Pi^2$ (4.3), *then*

$$\limsup_V\,\tau_V(K)<\infty\qquad\textit{in probability.}$$

(ii) *If* $f\in A\Pi_0^2$ (4.2) *and* $\mathrm{e}^f\in RV_\infty$, *then*

$$\lim_V\tau_V(K)=\infty\quad\textit{and}\quad\lim_V\frac{\log\tau_V(K)}{\log|V|}=0\qquad\textit{in probability.}$$

Recall that, in Theorem 6.2 and Lemma 6.8, the condition $\mathrm{e}^f\in RV_\infty$ implies $\log Q(t)=\mathrm{o}(t)$ (see Lemma A.3 with $\rho=\infty$ ); the condition $f\in OA\Pi^2$ yields $Q(t)\asymp t^3$ (see Lemma A.8(ii) with $p=2$); finally, the condition $f\in A\Pi_0^2$ implies $t^{-3}Q(t)\to\infty$ as $t\to\infty$ (see Lemma A.7 with $p=2$).

In the case $Q(t)=\mathrm{o}(t^3)$ as $t\to\infty$ (for example, Weibull distribution (1.5) with $\alpha<3$), the eigenvalue $\lambda_{K,V}$ approaches the $K$th extreme value of $\xi_V$ as $V\uparrow\mathbb{Z}^\nu$, for fixed $K\geqslant 1$. For such distributions, we obtain a simplified version of Poisson limit theorems for the largest eigenvalues:

**Theorem 6.9** (see Theorem 5 in (Astrauskas, 2007) and Theorem 2.5 in (Astrauskas, 2012))**.** *Let $t_Q = \infty$, and $f \in A\Pi$ (6.3) for some auxiliary function $a(\cdot)$, and assume that either of the following conditions* (C0)–(C2) *holds true:*

(C0) $\lim_{s\to\infty} a(s) = \infty$,
(C1) $\lim_{s\to\infty} sa(s) = \infty$ *and* $f \in PI_{<2}$ *(4.5)*

*or*

(C2) $f \in SA\Pi^2_\infty$ *(5.4) and* $f \in PI_{<2}$. *(4.5)*

*Write now* $b_V := f(\log|V|)$, $A_V = a_V := 1/a(b_V)$ *and*

$$B_V := \begin{cases} b_V, & \textit{under conditions (C0) or (C1)},\\ b_V + 2\nu\kappa^2 b_V^{-1}, & \textit{under condition (C2)}. \end{cases}$$

*Define the point process* $\mathcal{N}^\lambda_V$ *on* $[-1/2;1/2]^\nu \times \mathbb{R}$ *by*

$$\mathcal{N}^\lambda_V := \sum_{k=1}^{|V|} \delta_{\Lambda_V(k)} \quad \textit{where} \quad \Lambda_V(k) := \left(\frac{z_{k,V}}{|V|^{1/\nu}}, (\lambda_{k,V} - B_V)A_V\right). \tag{6.19}$$

*Then* $\mathcal{N}^\lambda_V$ *converges weakly to the Poisson process* $\mathcal{N}$ *on* $[-1/2;1/2]^\nu \times \mathbb{R}$ *with the intensity measure* $\mathrm{d}\,x \times \mathrm{e}^{-t}\mathrm{d}\,t$.

*Sketch of the proof.* Conditions (6.3) and (C0) imply that $f \in A\Pi^0_\infty$ (4.1). Therefore, by Theorem 4.3(i) with $p = 0$, the samples $\xi_V$ satisfy the condition of Theorem 2.2(i), consequently, $\lambda_{K,V} = \xi_{K,V} + \mathrm{o}(1)$ in probability as $|V| \to \infty$, for fixed $K \in \mathbb{N}$. Similarly, (6.3) and (C1) imply that $f$ is in the classes $A\Pi^1_\infty$ (4.1) and $PI_{<2}$ (4.5). Therefore, by combining Theorem 4.3(i) for $p = 1$, Theorems 4.5 and 3.1(iv) with $R = 0$ and $0 < \varepsilon < \theta < 1/2$, we obtain that the samples $\xi_V$ satisfy the conditions of Theorem 2.2(ii) with $N = [|V|^\theta]$. Consequently, $\lambda_{K,V} = \xi_{K,V} + \mathrm{O}(\xi^{-1}_{K,V})$ in probability, for fixed $K \in \mathbb{N}$. Using these asymptotic expansion formulas for $\lambda_{K,V}$, we obtain that in the cases (C0) and (C1) the point process $\mathcal{N}^\lambda_V$ (6.19) is approximated by the corresponding point process $\mathcal{N}^\xi_V$ with $\xi$ instead of $\lambda$; see the proof of Theorem 2.5(ii) in (Astrauskas, 2012). Since $\mathcal{N}^\xi_V$ converges weakly to $\mathcal{N}$ (Leadbetter et al. 1983), this concludes the proof of the theorem for (C0) and (C1).

In the case of (C2), we combine Theorem 4.3(i) for $p = 2$ and Theorems 4.5, 3.1(iv), 5.3 and 5.4 for $R = 0$ and $0 < \varepsilon < \theta < 1/2$, to find that the samples $\xi_V$ satisfy the conditions of Theorem 2.2(iii) in probability, with $N$ and $K$ as above. Consequently, $\lambda_{K,V} = \xi^0_{K,V} + \mathrm{O}(\xi^{-2}_{K,V})$ in probability, where $\xi^0_{K,V}$ is the $K$th extreme value of the i.i.d. field $\xi^0(\cdot) := \xi(\cdot) + 2\nu\kappa^2/(\xi(\cdot) \vee 1)$ in $V$. Using this limit and applying the same arguments as above with $\xi$ replaced by $\xi^0$, we obtain the assertion of the theorem in the case (C2). □

From Lemmas A.6 and A.13 we know that condition (C0) (resp., (C1) or (C2)) implies that $Q(t) = \mathrm{o}(t)$ (resp., $Q(t) = \mathrm{o}(t^2)$ or $Q(t) = \mathrm{o}(t^3)$) as $t$ tends to infinity.

The following corollary provides some limiting distributions for the top eigenvalues and the corresponding localization centers, which immediately follow from the Poisson convergence results of Theorems 6.2 and 6.9; see (Astrauskas, 2007). We also refer the reader to (Leadbetter et al., 1983, Chapter 5) for a detailed survey on Poisson limit theorems and their applications concerning the extremal properties of random fields.

**Corollary 6.10.** (*Eigenvalue statistics*). Assume that the conditions of Theorem 6.2 or Theorem 6.9 are fulfilled, with the normalizing constants $A_V > 0$ and $B_V$ specified herein. Then, for fixed $K \geqslant 1$ and $V \uparrow \mathbb{Z}^\nu$, we have the following assertions:

(i) The normalized spectral gaps

$$(\lambda_{1,V} - \lambda_{2,V})A_V, \ldots, (\lambda_{K-1,V} - \lambda_{K,V})A_V, (\lambda_{K,V} - B_V)A_V$$

are asymptotically mutually independent and have limiting joint distributions with the density

$$\exp\left\{-t_1 - \cdots - (K-1)t_{K-1} - Kt_K - \mathrm{e}^{-t_K}\right\}$$

for all $t_k \geqslant 0$ $(1 \leqslant k \leqslant K-1)$ and all $t_K \in \mathbb{R}$.

(ii) The normalized localization centers

$$\frac{z_{\tau(1),V}}{|V|^{1/\nu}}, \frac{z_{\tau(2),V}}{|V|^{1/\nu}}, \ldots, \frac{z_{\tau(K),V}}{|V|^{1/\nu}}$$

are asymptotically mutually independent, and each of them is asymptotically uniformly distributed on $[-1/2; 1/2]^\nu$.

(iii) As a consequence of (ii), the distance between the localization centers is of order $|V|^{1/\nu}$, i.e., for all $1 \leqslant l < k \leqslant K$,

$$\left|z_{\tau(k),V} - z_{\tau(l),V}\right| \asymp |V|^{1/\nu} \quad \text{in probability.}$$

□

We now give three examples of distributions $1 - \mathrm{e}^{-Q}$, where $\log Q(t) = \mathrm{o}(t)$.

**Example 6.11** (Astrauskas, 2012)**.** *Polynomially decaying distributions.* For some $\beta > 0$, assume that $f \circ \log \in RV_{1/\beta}$ or, equivalently, $\mathrm{e}^Q \in RV_\beta$. The latter is the sufficient and necessary condition for the distribution $1 - \mathrm{e}^{-Q}$ to be in the domain of attraction of the max-stable Fréchet law $G_\beta(\cdot)$, or equivalently, the following limit holds true:

$$\lim_V |V|\mathbb{P}\big(\xi(0) > tf(\log|V|)\big) = t^{-\beta} \quad \text{for all} \quad t \in \mathbb{R}_+;$$

see, e.g., (de Haan and Ferreira, 2006, Chapter 1). Since $f \in A\Pi^0_\infty$ (4.1), from Theorem 4.3(i) with $p = 0$ and Theorem 2.2(i) we see that $\lambda_{K,V} = \xi_{K,V} + \mathrm{o}(1)$ in probability, for fixed $K \in \mathbb{N}$. Using this limit and denoting $B_V \equiv 0$ and $a_V = 1/f(\log|V|)$, we obtain similarly as in the proof of Theorem 6.9(C0) that the point process $\mathcal{N}_V^\lambda$ (6.19) converges weakly to the Poisson process on $[-1/2; 1/2]^\nu \times \mathbb{R}_+$ with the intensity measure $\beta \mathrm{d}\,x \times t^{-\beta-1}\mathrm{d}\,t$.

**Example 6.12** (Grenkova et al., 1990; Astrauskas and Molchanov, 1992; Astrauskas, 2008)**.** *Weibull distributions.* Let $Q(t) = t^\alpha$ for $t \geqslant 0$, where $\alpha > 0$. For $\alpha \geqslant 1$, the function $Q$ satisfies conditions (6.7) and (6.8) of Corollary 6.4. For $\alpha < 3$, the inverse function $f(s) := Q^{\leftarrow}(s) = s^{1/\alpha}$ ($s \geqslant 0$) satisfies conditions (6.3) and (C2) of Theorem 6.9. Therefore, by Theorems 6.2, 6.9 and Corollary 6.4,

$$\lim_V |V|\mathbb{P}\Big(\xi(0) > b_V + \frac{t}{a_V}\Big) = \lim_V |V|\mathbb{P}\Big(\widetilde{\lambda}(0) > B_V + \frac{t}{a_V}\Big) = \mathrm{e}^{-t} \quad (t \in \mathbb{R}).$$

Consequently, the point processes $\mathcal{N}_V^\xi$ (6.2) and $\mathcal{N}_V^\lambda$ (6.5) converge weakly to the same Poisson process $\mathcal{N}$ as in Theorem 6.2, where the normalizing constants can be chosen as follows: $b_V = (\log |V|)^{1/\alpha}$, $A_V = a_V = Q'(b_V) = \alpha b_V^{\alpha-1}$ and

(a) $B_V = b_V$ if $\alpha < 2$,
(b) $B_V = b_V + 2\nu\kappa^2 b_V^{-1}$ if $2 \leqslant \alpha < 3$

and, as $|V| \to \infty$,

(c) $B_V = b_V + 2\nu\kappa^2 b_V^{-1} + \mathrm{O}(b_V^{-\frac{\alpha+1}{\alpha-1}})$ if $\alpha \geqslant 3$.

For $\alpha \geqslant 3$, asymptotic equations for $B_V$ are given in (Astrauskas, 2008, Section 6).

In the case $\alpha < 1$, we obtain the following almost sure asymptotic bounds for the eigenvalues and their spacings for any fixed $K \in \mathbb{N}$ and $m \in \mathbb{N}\backslash\{1\}$:

$$\limsup_V \frac{|\lambda_{K,V} - b_V| a_V}{\log_2 |V|} = \frac{1}{K},$$

$$\liminf_V \left( \log\Big((\lambda_{K,V} - \lambda_{K+1,V})a_V\Big) + \sum_{i=2}^{m-1} \log_i |V| \right) \Big/ \log_m |V| = -1$$

and

$$\limsup_V \left( (\lambda_{K,V} - \lambda_{K+1,V})a_V - \frac{1}{K}\sum_{i=2}^{m-1} \log_i |V| \right) \Big/ \log_m |V| = \frac{1}{K},$$

with $a_V$ and $b_V$ as above; here $\log_m := \log_{m-1}(\log)$ for $m \geqslant 2$. For any $\alpha > 0$, these strong limits for $\xi$ instead of $\lambda$ are proved in (Astrauskas, 2006, Section 3). Therefore, the case of $\lambda$ and $\alpha < 1$ is derived by the same arguments as in the proof of Theorem 6.9(C1) above, where one explores Theorem 4.4(i) ($p = 1$) instead of Theorem 4.3(i).

**Example 6.13** (Astrauskas, 2013; 2016)**.** *Distributions with fractional double exponential tails.* Let $Q(t) = \mathrm{e}^{t^\gamma}$ for $t \geqslant t_0$, where $0 < \gamma < 1$. Obviously, $Q$ satisfies conditions (6.7) and (6.8) of Corollary 6.4, therefore,

$$\lim_V |V|\mathbb{P}\Big(\xi(0) > b_V + \frac{t}{a_V}\Big) = \lim_V |V|\mathbb{P}\Big(\widetilde{\lambda}(0) > B_V + \frac{t}{a_V}\Big) = \mathrm{e}^{-t} \quad (t \in \mathbb{R}).$$

Consequently, the point processes $\mathcal{N}_V^\xi$ (6.2) and $\mathcal{N}_V^\lambda$ (6.5) converge weakly to the same Poisson process $\mathcal{N}$ as in Theorem 6.2; here

$$b_V = (\log\log|V|)^{1/\gamma}, \quad A_V = a_V = Q'(b_V) = \gamma b_V^{\gamma-1}\log|V|$$

and

$$B_V = b_V + c_1\frac{b_V^{\gamma-1}}{\log b_V} + c_2\frac{b_V^{\gamma-1}\log\log b_V}{(\log b_V)^2} + c_3\frac{b_V^{\gamma-1}}{(\log b_V)^2}\big(1+\mathrm{o}(1)\big)$$

as $|V| \to \infty$, where $c_1 := \nu\kappa^2\gamma(1-\gamma)^{-1}$, $c_2 := c_1(\gamma-1)^{-1}$ and $c_3 := c_2\log\frac{2(1-\gamma)\sqrt{\mathrm{e}}}{\kappa\gamma}$. The last formula and the asymptotic equations for $B_V$ are derived in (Astrauskas, 2016); see also (Astrauskas, 2013, Section 3).

### *6.2. Distribution tails lighter than the double exponential function*

Throughout this subsection, we assume that the upper tails $\mathrm{e}^{-Q}$ are lighter than the double exponential function. This class of distributions includes fractional double exponential tails (1.8) with $\gamma > 1$ and bounded tails ($t_Q < \infty$).

We start with the second order asymptotic expansion formula for the largest eigenvalues $\lambda_{K,V}$ of $\mathcal{H}_V = \kappa\Delta_V + \xi_V$.

**Theorem 6.14.** *Let $\mathrm{e}^f \in RV_0$, and fix $K \in \mathbb{N}$. Then with probability 1*

$$\lim_V(\lambda_{K,V} - f(\log|V|)) = 2\nu\kappa.$$

*Proof.* We apply part (i) of Theorem 3.1, where $R \in \mathbb{N}$ is fixed, $\theta_R(\cdot) \equiv \theta$ is a constant, and $m := |\mathbb{B}_R|$. Thus, with probability one there is a ball $\mathbb{B}_R(z_V)$ such that $\xi(\cdot) \geqslant L_{V,\theta}$ in $\mathbb{B}_R(z_V)$ for some $\theta \in \big(\frac{m-1}{m}; \frac{m}{m+1}\big)$ and each $V \supset V_0(\omega; R)$. Therefore, by Theorem 4.4(iii) and Lemma A.3(ii) with $\rho = 0$, almost surely

$$\xi(\cdot) - \xi_{1,V} \geqslant L_{V,\theta} - L_{V,0} + \mathrm{o}(1) = \mathrm{o}(1) \text{ uniformly in } \mathbb{B}_R(z_V),$$

as $|V| \to \infty$, for any $R \in \mathbb{N}$. The latter means that with probability one the samples $\xi_V$ satisfy the condition of Theorem 2.6, therefore, $\lambda_{K,V} = L_{V,0} + 2\nu\kappa + \mathrm{o}(1)$, as claimed. Theorem 6.14 is proved. □

From the proof of Theorems 2.6 and 6.14 we see that the top eigenvalue $\lambda_{K,V}$ of $\mathcal{H}_V$ is approximated by the corresponding eigenvalue of the Hamiltonian restricted to the (random) relevant regions $\mathbb{B}_{\mathrm{opt}} := \mathbb{B}_{R_V}(z_V) \subset V$ where $\xi(\cdot)$ is close to $\xi_{1,V}$ and the diameter of which tends to infinity as $|V| \to \infty$.

Bishop and Wehr (2012) have obtained more accurate asymptotic bounds for the principal eigenvalue $\lambda_{1,V}$ of the one-dimensional Schrödinger operators in $V \subset \mathbb{Z}$, with the Bernoulli i.i.d. potential. In particular, their results imply the following

**Theorem 6.15** (Bishop and Wehr, 2012)**.** *Let $\nu = 1$, and suppose that the random sequence $\xi(\cdot)$ has a common Bernoulli distribution: $a = \mathbb{P}(\xi(0) = 1)$ and $1 - a = \mathbb{P}(\xi(0) = 0)$, so that $t_Q = 1$. Then with probability one*

$$\lambda_{1,V} = t_Q + 2\kappa + (\log|V|)^{-2}(-D + \mathrm{o}(1)) \quad \textit{as} \quad V \uparrow \mathbb{Z},$$

*where $D = D(\kappa, a) > 0$ is the universal constant depending on $\kappa$ and $a$.* □

In the proof of this theorem, the authors have established that almost surely the relevant region $\mathbb{B}_{\mathrm{opt}} \subset V$ is the longest consecutive sequence of sites in $V$ with $\xi(\cdot)$ equal to 1, so that the size of $\mathbb{B}_{\mathrm{opt}}$ is of order $\log|V|$. See, e.g., the review paper by Binswanger and Embrechts (1994) for the strong and weak limit theorems for the length $|\mathbb{B}_{\mathrm{opt}}|$ as $V \uparrow \mathbb{Z}$.

Recently, Germinet and Klopp (2013) have proved the Poisson limit theorem for the top eigenvalues under nonlinear renormalization, i.e., for the so-called unfolded eigenvalues. Write, as above, $N(\lambda)$ $(\lambda \in \mathbb{R})$ for the integrated density of states, i.e., the nonrandom distribution function of eigenvalues defined as the almost sure limit of the empirical distribution function $N_V(\lambda) := \#\{k\colon\ \lambda_{k,V} \leqslant \lambda\}/|V|$ as $|V| \to \infty$ (Kirsch, 2008).

**Theorem 6.16** (Germinet and Klopp, 2013, Theorem 2.3)**.** *Assume that $\mathcal{H}_V$ is the one-dimensional Anderson Hamiltonian ($\nu = 1$), where the potential is bounded ($|\xi(0)| \leqslant \mathrm{const}$ ) with the distribution density $p(\cdot) := \mathrm{e}^{-Q(\cdot)}Q'(\cdot)$. Assume, in addition, that the density $p(\cdot)$ is bounded and does not decay too fast at $t_Q$ (say, $p(t) = \mathrm{e}^{-(t_Q - t)^{\mathrm{o}(1)}}$ or $p(t) = \mathrm{e}^{-(t_Q - t)^{-\vartheta}}$ as $t \uparrow t_Q$, for $0 < \vartheta < 1/2$). Define the point process $\mathcal{M}_V^\lambda$ on the positive half-axis $\mathbb{R}_+$ by*

$$\mathcal{M}_V^\lambda := \sum_{k=1}^{|V|} \delta_{|V|(1 - N(\lambda_{k,V}))}.$$

*Then $\mathcal{M}_V^\lambda$ converges weakly to the Poisson process on $\mathbb{R}_+$ with the intensity measure $\mathrm{d}t$, the Lebesgue measure.* □

For $\nu \geqslant 2$, this Poisson limit theorem was shown to hold if the Laplacian $\Delta$ is replaced by some translation invariant operator $\mathcal{T}\psi(x) = \sum_{y \in \mathbb{Z}^\nu} T(y)\psi(x - y)$, where $T(\cdot\,)$ is a real nonrandom function decaying exponentially at infinity.

The proof of Theorem 6.16 relies on the improved versions of Wegner and Minami estimates that control the structure of eigenvalues $\lambda_{k,V}$ in a small neighborhood $I$ of the upper spectral edge, so the amount $N(I)$ is allowed to be exponentially small in $|I|^{-1}$; see Section 1.3 for more explanations. See also (Minami, 2007) for a detailed background of Poisson convergence results for unfolded spectral values. It is important for applications that this convergence result is given in terms of the integrated density of states, the main quantity in the theory of random Schrödinger operators. However, in the proof of Theorem 6.16, neither extreme value theory, nor links to the asymptotic geometric properties of random potential are explored.

**Corollary 6.17.** Assume that $\mathcal{H}_V$ is the one-dimensional Anderson Hamiltonian ($\nu = 1$), where the potential has the distribution density $p(\cdot)$ satisfying the conditions of Theorem 6.16. Then, for fixed $K \geqslant 1$ and $V \uparrow \mathbb{Z}$, we have the following asymptotic formulas in probability:

(i) If $p(t) = \mathrm{e}^{-(t_Q - t)^{\mathrm{o}(1)}}$ as $t \uparrow t_Q$, then

$$\lambda_{K,V} = t_Q + 2\kappa - (\log |V|)^{-2+\mathrm{o}(1)}.$$

(ii) If there is $0 < \vartheta < 1/2$ such that $p(t) = \mathrm{e}^{-(t_Q - t)^{-\vartheta}}$ as $t \uparrow t_Q$, then

$$\lambda_{K,V} = t_Q + 2\kappa - (\log |V|)^{-2/(1+2\vartheta)+\mathrm{o}(1)}.$$

□

Corollary 6.17 can be proved by using the limit theorems for the unfolded eigenvalues and asymptotic expansion formulas for the tails $1-N(\cdot)$ at the upper spectral edge derived, e.g., in (Klopp, 1998; Biskup and König, 2001). See also (Klopp, 2000) and Section 3.5 of (Kirsch and Metzger, 2007) for a detailed discussion on the edge asymptotics of the integrated density of states. Thus, the bifurcations in the asymptotic behavior of the top eigenvalues are caught by those in the tail behavior of the integrated density of states, i.e. “Lifshits tails”. Notice that the asymptotics in Corollary 6.17(i) agree with the results in the Bernoulli case (Theorem 6.15). Meanwhile, if the tails $\mathrm{e}^{-Q}$ decays faster at $t_Q$, then the fluctuations of $\lambda_{K,V}$ are much sharper (Corollary 6.17(ii)) than those in the Bernoulli case (Theorem 6.15).

For the tails lighter than the double exponential function including the case $t_Q < \infty$, some heuristics on the asymptotic formulas for $\lambda_{1,V}$ ($\nu \geqslant 1$) and their relations to the long-time asymptotic formulas for the parabolic Anderson model have earlier been discussed by Biskup and König (2001), and van der Hofstad et al. (2006). Their assumptions are given in terms of scaling and regularity properties of the cumulant generating function $\log \mathbb{E}\,\mathrm{e}^{t\xi(0)}$ as $t \to \infty$.

Recently, Biskup et al. (2014) have proved the “homogenized” versions of limit theorems for the largest eigenvalues of the (scaled) finite-volume discrete Schrödinger operators $\mathcal{H}^{(\varepsilon)}$ with a bounded random potential. Their results assert that, as the scale parameter $\varepsilon$ tends to zero, then 1) the largest eigenvalues of $\mathcal{H}^{(\varepsilon)}$ converge in probability to the corresponding eigenvalues of the limiting (nonrandom) finite-volume continuous Schrödinger operator, and 2) the fluctuations of the largest eigenvalues centered by their means are Gaussian in limit.

We notice that the conditions of the above statements imply that $t^{-1} \log Q(t) \to \infty$ as $t \uparrow t_Q > 0$ (see Lemma A.3 with $\rho = 0$ below).

We end this subsection with a discussion on the following important model of spatially continuous random Schrödinger operators:

**Example 6.18.** (*Schrödinger operators in $\mathbb{R}^\nu$ with a bounded Poisson potential of obstacles*). Let $\Delta^{\mathrm{cont}}$ be the $\nu$-dimensional continuum Laplacian. Define the random potential $\xi(\cdot)$ by

$$\xi(x) = -\sum_i W(x + x_i) \quad (x \in \mathbb{R}^\nu); \tag{6.20}$$

here $\{x_i\}$ is a Poisson point process in $\mathbb{R}^\nu$ with the constant intensity $\mu > 0$; $W(\cdot)$ is a fixed nonnegative compactly supported, bounded measurable function, $W(\cdot)$ is non-identically zero Lebesgue-a.e. The potential $\xi(\cdot)$ is known as a *Poisson field of "soft" obstacles*. Denote by $V := [-s; s]^\nu \subset \mathbb{R}^\nu$ the cubes of the volume $|V|$ such that $V \uparrow \mathbb{R}^\nu$. Let us consider the principal Dirichlet eigenvalue $\lambda_{1,V}^{\text{cont}} < 0$ of the operator $\Delta^{\text{cont}} + \xi(\cdot)$ in $V$. The eigenvalue $\lambda_{1,V}^{\text{cont}}$ satisfies the following asymptotic formula (e.g., Sznitman, 1998): As $V \uparrow \mathbb{R}^\nu$, almost surely

$$\lambda_{1,V}^{\text{cont}} = (\log |V|)^{-2/\nu}(-C(\nu, \mu) + \mathrm{o}(1)), \tag{6.21}$$

where $C(\nu, \mu) > 0$ is the universal constant depending on $\nu$ and $\mu$; see below.

Let us *sketch a derivation of the lower bound* for $\lambda_{1,V}^{\text{cont}}$. By the monotonicity property of eigenvalues, we have the bound $\lambda_{1,V}^{\text{cont}} \geqslant \lambda_{1,\mathbb{A}}^0$, where $\lambda_{1,\mathbb{A}}^0$ is the local principal Dirichlet eigenvalue of the operator $\Delta^{\text{cont}} + \xi(\cdot)$ restricted to the obstacle-free connected open region $\mathbb{A} \subset V$. We now maximize the eigenvalue $\lambda_{1,\mathbb{A}}^0$ over such $\mathbb{A}$. First, since the probability of region $\mathbb{A}$ to have no points $\{x_i\}$ is equal to $\mathrm{e}^{-\mu|\mathbb{A}|}$ and the number of disjoint shifts of $\mathbb{A}$ is of order $|V|$, we obtain from the Borel-Cantelli lemma that the volume $|\mathbb{A}|$ should be approximately equal to $\mu^{-1} \log |V|$; cf. the proof of Theorem 3.1 above. On the other hand, we have from the Faber-Krahn inequality that the principal Dirichlet eigenvalues $\lambda_{1,\mathbb{A}}^0$ of the Laplacian $\Delta^{\text{cont}}$ in regions $\mathbb{A}$ of the constant volume achieve their maximum at the ball. Thus, $\lambda_{1,V}^{\text{cont}} \geqslant \lambda_{1,\mathbb{B}_{\text{opt}}}^0(1 + \mathrm{o}(1))$, where $\lambda_{1,\mathbb{B}_{\text{opt}}}^0$ is the principal Dirichlet eigenvalue of the operator $\Delta^{\text{cont}}$ in the ball $\mathbb{B}_{\text{opt}} := \mathbb{B}_{R_V}(z_V)$ centered at some random $z_V \in V$ with the radius $R_V := \left(|\mathbb{B}_1|^{-1}\mu^{-1} \log |V|\right)^{1/\nu}$, and $|\mathbb{B}_1|$ is the volume of the unit ball $\mathbb{B}_1 \subset \mathbb{R}^\nu$. Consequently, as $V \uparrow \mathbb{R}^\nu$, almost surely

$$\lambda_{1,V}^{\text{cont}} \geqslant \lambda_{1,\mathbb{B}_{\text{opt}}}^0(1+\mathrm{o}(1)) = R_V^{-2}\left(\lambda_{1,\mathbb{B}_1}^0 + \mathrm{o}(1)\right) = (\log |V|)^{-2/\nu}(-C(\nu, \mu)+\mathrm{o}(1)),$$

where $C(\nu, \mu) := -\lambda_{1,\mathbb{B}_1}^0(|\mathbb{B}_1|\mu)^{2/\nu} > 0$. This lower bound can be shown to be equal to the upper bound for $\lambda_{1,V}^{\text{cont}}$, concluding the proof of (6.21). Notice also that the rough upper bound for $\lambda_{1,V}^{\text{cont}}$ can be derived by using the spatially continuous version of Lemma 2.8 above.

Summarizing, we conclude that the principal Dirichlet eigenvalue $\lambda_{1,V}^{\text{cont}}$ is approximated, as $V \uparrow \mathbb{R}^\nu$, by the local principal Dirichlet eigenvalue $\lambda_{1;\mathbb{B}_{\text{opt}}}^0$ of the operator restricted to the relevant region $\mathbb{B}_{\text{opt}} := \mathbb{B}_{R_V}(z_V) \subset V$, so that $\lambda_{1,V}^{\text{cont}} \leftrightarrow \mathbb{B}_{\text{opt}}$. These observations and formulas agree with the corresponding formulas for the discrete Anderson models in $\mathbb{Z}$ with the Bernoulli i.i.d. potential (Theorem 6.15).

As already mentioned, formula (6.21) and the more explicit asymptotic bounds for the principal Dirichlet eigenvalue $\lambda_{1,V}^{\text{cont}}$ ($V \uparrow \mathbb{R}^\nu$) were proved by Sznitman (1998) exploring his original method of enlargement of obstacles. By this method, the geometry of the spatial regions where $\xi(\cdot) > 0$ and $\xi(\cdot) \equiv 0$ is reduced to the simpler geometry of regions associated with the modified potential in the spectral problems, without changing the eigenvalues very much.

Finally, notice that the asymptotic bounds for the principal eigenvalues are crucial for study of the intermittent behavior of a Brownian motion in a Poisson field of obstacles; cf. Section 7 below. □

### *6.3. The double exponential tails*

In the double exponential case (1.14), Gärtner and Molchanov (1998, Theorem 2.16) have obtained the second order expansion formula for the principal eigenvalue $\lambda_{1,V}$ of $\mathcal{H}_V = \kappa\Delta_V + \xi_V$ by claiming a continuity of $Q$. We now provide their result with the continuity condition removed.

**Theorem 6.19.** *If* $\mathrm{e}^f \in RV_\rho$ *for some* $0 < \rho < \infty$*, then with probability 1*

$$\lim_V (\lambda_{1,V} - f(\log|V|)) = 2\nu\kappa q(\rho/\kappa),$$

*where the nonrandom function $q$ is defined in Section 2.5.*

*Proof.* We check the conditions of Theorem 2.7. First, by Theorem 4.7, almost surely $\xi_V$ satisfies condition (2.29). To prove (2.30), we fix constants $R \in \mathbb{N}$, $\delta > 0$, and write

$$\theta(y) := \theta_R(y) := 1 - \exp\left\{ \left( h_{\mathrm{opt}}^{\mathbb{B}_R}(y) - \delta \right) / \rho \right\} \quad (y \in \mathbb{B}_R),$$

where the nonrandom function $h_{\mathrm{opt}}^{\mathbb{B}_R}(\cdot)$ is defined in Section 2.5. Consequently, $\theta(\cdot)$ satisfies the assumptions of Theorem 3.1(i). Combining the statements of Theorem 3.1(i), Lemma A.3(ii) and Theorem 4.4(iii), we obtain the following assertion with probability one: for any $V \supset V_0(\omega; \delta, R)$ there is $z_V \in V$ such that

$$\xi(y) \geqslant L_{V,\theta(y-z_V)} \geqslant \xi_{1,V} + h_{\mathrm{opt}}^{\mathbb{B}_R}(y - z_V) - 2\delta \quad \text{for all} \quad y \in \mathbb{B}_R(z_V).$$

Since $R \in \mathbb{N}$ and $\delta > 0$ are arbitrary constants, this estimate concludes the proof of the almost sure limit (2.30). Now, Theorems 2.7 and 4.4(iii) imply the assertion of Theorem 6.19. □

From the proof of Theorems 2.7 and 6.19 we see that almost surely the eigenvalue $\lambda_{1,V}$ approaches (as $|V| \to \infty$) the local principal eigenvalue in the random region, where $\xi(\cdot) \approx \xi_{1,V} + h_{\mathrm{opt}}^{\mathbb{B}_R}(\cdot)$ for $R$ arbitrarily large, so that $\lambda_{1,V}$ is associated with the (random) "relevant island" of high $\xi_V$-values of optimal shape, the diameter of which is asymptotically bounded. From Theorem 3.1(iv), Remark 3.5 and the last assertion of Lemma A.3, it follows that the "islands" of $\xi_V$-extremes are located asymptotically far away from each other. Moreover, if the constant $\rho/\kappa$ is large enough, these "islands" are located in the neighborhood of single extremely high $\xi_V$-peaks; see (Astrauskas, 2008, Theorem 4.4 and Corollary 4.5) and (Astrauskas, 2013, Theorem 2.1(iii)).

For arbitrary $0 < \rho < \infty$, Poisson limit theorems for the largest eigenvalues and the corresponding localization centers are proved by Biskup and König

(2016); see also the survey by König (2016) on these limit theorems and related topics. To formulate their result, we again define the sites $z_{\tau(k),V} \in V$ by (6.18), i.e., the localization centers of the $k$th eigenfunctions $\psi(\cdot\,;\lambda_{k,V})$ $(1 \leqslant k \leqslant |V|)$ of the Hamiltonian $\mathcal{H}_V = \kappa\Delta_V + \xi_V$.

**Theorem 6.20** (Biskup and König, 2016, Theorem 1.2)**.** *Let $t_Q = \infty$, and assume that $Q$ is a continuously differentiable function such that*

$$\lim_{t\to\infty} \frac{Q'(t)}{Q(t)} = \frac{1}{\rho} \quad \textit{for some} \;\; 0 < \rho < \infty.$$

*Then the following assertions (I) and (II) hold true:*

(I) (Poisson limit theorem) *There are constants $B_V = f(\log|V|) + 2\nu\kappa q(\rho/\kappa) + \mathrm{o}(1)$ and $A_V = \rho^{-1}\log|V|$ such that the point process $\mathcal{N}_V^{\lambda}$ (6.5) converges weakly to the Poisson process $\mathcal{N}$ on $[-1/2;1/2]^\nu \times \mathbb{R}$ with the intensity measure $\mathrm{d}\,x \times \mathrm{e}^{-t}\mathrm{d}\,t$.*

(II) (Exponential localization) *As $V \uparrow \mathbb{Z}^\nu$ and $K \geqslant 1$ fixed, we have with probability $1 + \mathrm{o}(1)$ that there exist non-random constants $C > 0$, $M > 0$, $C' > 0$ and $M' > 0$ such that*

$$|\psi(x;\lambda_{K,V})| \leqslant C \exp\{-M|x - z_{\tau(K),V}|\} \quad \textit{for all} \quad x \in V,$$

*and*

$$|\psi(x;\lambda_{K,V})| \leqslant C' \exp\left\{-M'(\log\log|V|)\big|x - z_{\tau(K),V}\big|\right\} \quad \textit{for} \quad |x - z_{\tau(K),V}| \geqslant \log|V|,$$

*i.e., the $K$th eigenfunction is highly concentrated in the neighborhood of its localization center.*

For sufficiently large $\rho$, i.e. $\rho > \rho_0$, Poisson limit theorems and localization theorems for the top eigenvalues were earlier proved by Astrauskas (2007; 2008; 2013). For $\rho > \rho_0$, the corresponding localization properties present an interesting intermediate case between the single site concentration property, i.e. $\lambda_{K,V} \leftrightarrow z_{\tau(K),V}$, in the case $\rho = \infty$ (Theorem 6.2(II)) and the non-single site concentration property, i.e. $\lambda_{K,V} \leftrightarrow \mathbb{B}^K_{\mathrm{opt}}$, for $0 \leqslant \rho < \rho_0$ (Section 6.2 and Theorem 6.20).

From Remark A.5(i) and Lemma A.3, we notice that the conditions of Theorem 6.20 imply $\mathrm{e}^f \in RV_\rho$, i.e., the assumption of Theorem 6.19. Moreover, from Remark A.5(ii), Lemma A.1 and Lemma 6.1 with $a(\cdot) \equiv 1/Q'(\cdot)$ we also see that the conditions of Theorem 6.20 yield the limit (6.1) with $b_V = f(\log|V|)$ and $a_V = A_V = \rho^{-1}\log|V|$. Consequently, the distribution $1 - \mathrm{e}^{-Q}$ is in the domain of attraction of the max-stable Gumbel law $G_{\exp}(\cdot)$. We finally notice that the conditions of Theorem 6.19 or Theorem 6.20 imply the limit $f(s) = (\rho + \mathrm{o}(1))\log s$ as $s \to \infty$, which in turn is equivalent to $\log Q(t) = \rho^{-1}t + \mathrm{o}(t)$ as $t \to \infty$; see Lemma A.3.

### *6.4. Some comments on the proofs*

In this section, we briefly comment and compare the proof of Theorems 2.2, 2.3, 6.2, and 6.9 by Astrauskas and Molchanov (1992) and Astrauskas (2007; 2008; 2012; 2013) ("relevant single peak" approximation) and the proof of Theorem 6.20 by Biskup and König (2016) ("relevant island" approximation).

**(RSP)** *"Relevant single peak" approximation.* As already mentioned in Section 1.2, the proof of Theorems 2.2, 2.3, 6.2, and 6.9 is based on the finite-rank perturbation arguments and the analysis of Green functions involving the cluster expansion over paths. To be more precise, fix $Z := z_{\tau(K),V} \in V$, the localization center of the $K$th eigenfunction, and denote by $\lambda(Z)$ the extreme eigenvalue of $\mathcal{H}_V = \kappa\Delta_V + \xi_V$ associated with the site $Z$. Let $\mathcal{G}_V^{(Z)}(\lambda; \cdot, \cdot)$ be the Green function of the Hamiltonian $\kappa\Delta_V + (1 - \delta_Z)\xi_V$ on $l^2(V)$. Under the conditions of Theorem 2.2 or 2.3 (i.e. sparseness and difference in height of $\xi_V$-peaks as $V \uparrow \mathbb{Z}^\nu$), the eigenvalue $\lambda(Z)$ is a solution to the dispersion equation

$$\mathcal{G}_V^{(Z)}(\lambda; Z, Z) = \frac{1}{\xi(Z)} \tag{6.22}$$

and the corresponding eigenfunction is $\mathcal{G}_V^{(Z)}(\lambda(Z); \cdot, Z)$. By expanding the Green function $\mathcal{G}_V^{(Z)}(\lambda; \cdot, \cdot)$ over paths, one proves that equation (6.22) is approximated by the corresponding equation $\widetilde{\mathcal{G}}_V(\lambda; Z, Z) = 1/\xi(Z)$ for the principal eigenvalue of the "single peak" Hamiltonian $\kappa\Delta_V + \widetilde{\xi}_V + \xi(Z)\delta_Z$; here $\widetilde{\mathcal{G}}_V(\lambda; \cdot, \cdot)$ stands for the Green function of the operator $\kappa\Delta_V + \widetilde{\xi}_V$. Again expanding $\widetilde{\mathcal{G}}_V(\lambda; \cdot, \cdot)$ over paths, one finds that the eigenvalue $\lambda(Z)$ of $\mathcal{H}_V$ is approximated by a certain (nonlinear) function on $\widetilde{\xi}_V$ and $\xi(Z)$; cf. (2.20)–(2.22). Moreover, because of the sparseness of $\xi_V$-peaks, the extreme eigenvalues $\lambda(Z)$ become asymptotically independent, so that they obey asymptotic Poisson behavior as $V \uparrow \mathbb{Z}^\nu$ (see Theorems 6.2 and 6.9).

We notice that the analysis of the Green functions combined with the finite-rank perturbation theory is essential to study the largest eigenvalues of the finite-volume operators $\mathcal{H}_V$ in the "relevant single peak" approximation. Recall that these techniques also play a crucial role in the proof of the Anderson localization for the infinite-volume Hamiltonian $\mathcal{H}$ (Kirsch, 2008; Stolz, 2011); see also Section 1.3 above.

**(RI)** *"Relevant island" approximation.* Recently, Biskup and König (2016) have developed novel arguments to prove Poisson limit theorems for the largest eigenvalues in the case of double exponential tails (see Theorem 6.20 above). As in the single-peak approximation, the analysis of the extreme eigenvalues is here based on controlling the dependence of an eigenvalue on the geometric properties of $\xi_V$-peaks and the associated regions in $V$. This enables to identify the "relevant" regions $\mathbb{B}_{\mathrm{opt}}^K := \mathbb{B}_{R_V}(z_V^K) \subset V$ (where $\xi(\cdot)$ is high and of the optimal shape) such that the $K$th largest eigenvalue $\lambda_{K,V}$ of $\kappa\Delta_V + \xi_V$ is approximated by the local principal eigenvalue $\lambda_{1,\mathbb{B}^K_{\mathrm{opt}}}$ of the Hamiltonian restricted to $l^2(\mathbb{B}^K_{\mathrm{opt}})$ (cf. also Theorems 2.7 and 6.19 and their proofs in the present survey). In other words, the eigenvalues associated with a block of "relevant islands" of high $\xi_V$-values can be determined by the local principal eigenvalues associated with separate "relevant islands". It is worth noticing that the conditions of Theorem 6.20 imply that the islands of high $\xi_V$-values are located extremely far from each other as $V \uparrow \mathbb{Z}^\nu$. (See also Theorem 3.1 for the related limits under the continuity assumption (3.1)). The proof of Theorem 6.20 involves the following procedures on a simplification of potential configurations: 1) those regions, where the potential possesses the lower values, are deleted from $V$ (*domain truncation and component trimming*); 2) for the radius $R_V$ tending to infinity slowly, the analysis of the local principal eigenvalues in all balls $\mathbb{B}_{R_V}(z) \subset V$ is reduced to the consideration of independent identically distributed local principal eigenvalues in disjoint balls in $V$ (*coupling to i.i.d. variables*); 3) the local principal eigenvalue in the region $\mathbb{B}^K_{\mathrm{opt}}$ is separated from other local eigenvalues in $\mathbb{B}^K_{\mathrm{opt}}$ (*reduction to one eigenvalue per component*); 4) extremal type limit theorems for the local principal eigenvalues in $\mathbb{B}_{R_V}(z)$ are comparable to each other for the different increase rate of $R_V \to \infty$, with the same normalizing constants $A_V$ and $B_V$ (*stability with respect to partition side*), and so on.

Summarizing, the main idea of the proof of Theorem 6.20 explores the straightforward geometric arguments controlling the dependence of eigenvalues on potential configurations, rather than the techniques of resolvents or Green functions. This is in contrast to the relevant single peak approximation in (RSP), where the Green functions are the main object of analysis. On the other hand, although most of the proof of Theorem 6.20 is based on deterministic arguments, we are not able to reformulate this assertion in terms of $\xi_V$-extremes (like in Theorems 2.2–2.7 above), except for the case of sufficiently large $\rho$ considered in (Astrauskas, 2008, Theorem B.3).

## 7. Applications to the parabolic Anderson model

### *7.1. The parabolic Anderson model*

The parabolic Anderson model (PAM) is the Cauchy problem for the following heat equation with random potential:

$$
\begin{aligned}
&\frac{\partial u(s,x)}{\partial s} = \kappa \sum_{|y|=1} (u(s,x+y) - u(s,x)) + \xi(x) u(s,x), \quad s \geqslant 0,\ x \in \mathbb{Z}^\nu, \\
&u(0,x) = \delta_0(x), \quad x \in \mathbb{Z}^\nu;
\end{aligned}
\tag{7.1}
$$

here, as above, $\xi(\cdot)$ is an i.i.d. random field (potential) with distribution $\mathbb{P}(\xi(0) > t) = \mathrm{e}^{-Q(t)}$; $\delta_0$ is the Kronecker delta function at the origin (i.e., the localized initial datum of the problem); the variable $s \geqslant 0$ is referred to as a time. The equation has almost surely a unique nonnegative solution, provided $\xi(0) \vee 0$ has a finite moment of order $> \nu$ (Gärtner and Molchanov, 1990).

The PAM appears in the context of population dynamics, chemical kinetics, magnetism and turbulence, etc. (e.g., Gärtner and Molchanov, 1990; Molchanov, 1994). The following interpretation of the solution $u$ is well-known in the mathematical literature (e.g., Molchanov, 1994): Let $(X(s)\colon\ s \geqslant 0)$ be a continuous-time random walk in $\mathbb{Z}^\nu$ with a generator $\kappa\Delta_{\mathrm{dif}}$, where $\Delta_{\mathrm{dif}}\psi(x) := \sum_{|y-x|=1}(\psi(y) - \psi(x))$. Let $\xi^+(\cdot) \geqslant 0$ and $\xi^-(\cdot) \geqslant 0$ be independent random i.i.d. fields on $\mathbb{Z}^\nu$, and write $\xi(\cdot) := \xi^+(\cdot) - \xi^-(\cdot)$. For a fixed realization $(\xi^+(\cdot); \xi^-(\cdot))$, consider a system of particles which obey the following diffusion and branching mechanism:

1) at time $s = 0$, there is a single particle at the origin;
2) particles move independently of each other according to the random walk $X(\cdot)$;
3) at the site $x$ a particle disappears with intensity $\xi^-(x)$ and splits into two new particles with intensity $\xi^+(x)$, which further move according to $X(\cdot)$.

Then, for a fixed realization $(\xi^+(\cdot); \xi^-(\cdot))$, the solution $u(s, x)$ to (7.1) is the expected number of particles at the site $x$ at time $s$, where the expectation is taken over a branching mechanism and diffusion (but not over random medium $\xi^+(\cdot), \xi^-(\cdot)$). Thus, the sum $U(s) := \sum_{x\in\mathbb{Z}^\nu} u(s, x)$ is the expected total mass of particles at time $s$. We see from the Feynman-Kac formula (7.2) below that $U(s)$ is equal to $U(s, 0)$, where $U(s, \cdot)$ is the solution to equation (7.1) with the homogeneous initial datum $U(0, \cdot) \equiv 1$ instead of the localized one.

For i.i.d. random potentials, the PAM exhibits an *intermittency effect*: As $s \to \infty$, the overwhelming contribution to the total mass $U(s) = \sum_x u(s, x)$ of the solution $u$ to (7.1) comes from a small number of spatially separated and relatively small islands of large $u(s, \cdot)$-values, i.e., *intermittent islands*. This is in contrast to the case of constant potential $\xi(\cdot) \equiv \mathrm{const}$, for which the solution $u(s, \cdot)$ is spread over the spatial ball of radius $\mathrm{O}(\sqrt{s})$ as $s \to \infty$, i.e., *diffusion effect*.

Various aspects of long-time intermittent behavior of the PAM (asymptotic expansion formulas for the total mass $U(s)$ and its statistical moments, concentration properties for the solutions $u(s, \cdot)$, etc.) have been intensively studied, during the last two decades, by mathematicians Molchanov, Gärtner, Sznitman, König, Biskup, Mörters, den Hollander, Sidorova, van der Hofstad, and their colleagues. See (König, 2016) for a recent survey on the subject and references therein. The main technical tools of intermittency theory are a spectral representation and the Feynman-Kac formula for the solutions $u$, $U$. Recall the latter formula:

$$
\begin{aligned}
u(s, x) &= \mathbf{E}_x\left[\exp\left\{\int_0^s \xi(X(a))\,\mathrm{d}a\right\} u(0, X(s))\right] \\
&= \mathbf{E}_x\left[\exp\left\{\int_0^s \xi(X(a))\,\mathrm{d}a\right\} \delta_0\left(X(s)\right)\right], \quad s \geqslant 0,\ x \in \mathbb{Z}^\nu; \qquad (7.2)
\end{aligned}
$$

here the expectation $\mathbf{E}_x$ is taken with respect to the random walk $X(\cdot)$ in $\mathbb{Z}^\nu$ as above, conditioned by $X(0) = x$. Looking at (7.2), we see that the intermittent behavior is determined by competition between two factors: (i) extremely large exponential factor in (7.2) associated with a portion of the trajectories $X[0;s] := (X(a)\colon\ 0 \leqslant a \leqslant s)$ spending much time at spatial regions where $\xi(\cdot)$ is high, and (ii) very small probabilities of such trajectories in (7.2). In the model, the potential necessitates concentration properties of $u$; meanwhile, the Laplacian forces these properties to be less expressed. It turns out that, as the upper tails of potential distribution get heavier, the $\xi_V$-extremes get more pronounced as $V = V(s) \uparrow \mathbb{Z}^\nu$ (cf. Section 1.2); therefore, the long-time intermittent properties (in particular, mass concentration properties) of the PAM become stronger. In (Gärtner and Molchanov, 1998; Gärtner and den Hollander (1999); Gärtner et al., 2007), the emphasize has been made on the double exponential tails (1.14). Such tails indicate the critical situation between formation of widely-spaced single peaks of $u(s,\cdot\,)$ in the case of tails heavier than in (1.14) (e.g., Gärtner et al., 2007; König et al., 2009; Sidorova and Twarowski, 2014; Fiodorov and Muirhead, 2014), and formation of widely-spaced extremely large "islands" of higher values in the behavior of $u(s,\cdot\,)$ for the tails lighter than in (1.14) (Biskup and König, 2001; van der Hofstad et al., 2006). The results of these papers suggest that the optimal strategy of particles is to move quickly to the spatial region where the potential values are high and of preferred shape, and to stay here for the remaining time.

In this section, we will focus on the representation of the solutions $u$ and $U$ in the spectral terms of the Anderson Hamiltonian $\mathcal{H}_V = \kappa\Delta_V + \xi_V$ where $V = V(s) \uparrow \mathbb{Z}^\nu$. In view of this representation, we will discuss some techniques of the extreme value theory for eigenvalues, which can be applied to study intermittent properties of the PAM; cf. Theorems 7.1–7.2 below.

### *7.2. Asymptotic expansion formulas for the total mass*

The first result in this direction was obtained by Gärtner and Molchanov (1998), who particularly derived the second-order asymptotic formula for the logarithm of the total mass $U(s)$ $(s \to \infty)$ with probability one (and, as a by-product, the corresponding result for the principal eigenvalue of the operator $\mathcal{H}_V$), provided the potential distribution satisfies mild RV conditions and has all positive exponential moments finite; cf. the assumptions of Theorem 7.1 below. Let us *sketch the proof of the almost sure asymptotic formula* for the PAM following the arguments of their paper. The first key observation is that the solutions $u$ and $U$ are approximated, as $s \to \infty$, by their finite-volume analogues $u_{V(s)}$ and $U_{V(s)}$, respectively. I.e., $u_{V(s)}$ and $U_{V(s)}$ are solutions to the corresponding equations in $V(s)$ with the Dirichlet boundary condition; here $V(s) \subset \mathbb{Z}^\nu$ denote cubes centered at the origin, whose size length is of order $s(\log s)^c$ for some constant $c > 1$. In particular, almost surely

$$U(s) = U_{V(s)}(s) + \mathrm{o}(1) \quad \text{and} \quad u(s,x) = u_{V(s)}(s,x) + \mathrm{o}(1) \quad \text{as} \quad s \to \infty \qquad (7.3)$$

uniformly in $x$, by using the standard cut-off procedure for the solutions $u$ and $U$, based on the following facts: 1) the overwhelming asymptotic contribution to the Feynman-Kac representation of $u$ and $U$ is given by trajectories $X(\cdot)$ which stay inside the box $V(s)$ during the whole time interval $[0,s]$ i.e. $X[0;s] \subset V(s)$; and 2) the contribution from trajectories $X(\cdot)$ visiting the complement of $V(s)$ during the time interval $[0,s]$ is much smaller. On the other hand, the solution $u_V$ with $V := V(s)$ admits the spectral representation

$$u_V(s,x) = \sum_{k=1}^{|V|} \mathrm{e}^{\lambda_{k,V} s - 2\nu\kappa s} \psi(0;\lambda_{k,V})\psi(x;\lambda_{k,V}) \quad \text{for each } s \geqslant 0 \text{ and } x \in V, \tag{7.4}$$

and, therefore, the total mass $U_V(s) = \sum_{x\in V} u_V(s,x)$ has the following representation

$$U_V(s) = \sum_{k=1}^{|V|} \mathrm{e}^{\lambda_{k,V} s - 2\nu\kappa s} \big(\psi(\cdot\,;\lambda_{k,V}),1\big)_V \psi(0;\lambda_{k,V}) \quad \text{for each } s \geqslant 0; \tag{7.5}$$

here $\big(\psi,\varphi\big)_V$ denotes the inner product of the functions $\psi$ and $\varphi$ in $l^2(V)$, and 1 stands for the function taking everywhere value 1. Recall that $\lambda_{k,V}$ and $\psi(\cdot\,;\lambda_{k,V})$ are the $k$th eigenvalue and the corresponding eigenfunction of the Hamiltonian $\mathcal{H}_V = \kappa\Delta_V + \xi_V$. Moreover, the eigenfunctions are chosen to form an orthonormal basis of $l^2(V)$, and the principal eigenfunction to be strictly positive in $V$. We may also assume, without loss of generality, that $\psi(0;\lambda_{k,V}) \geqslant 0$ for all $k$.

Let us prove that the main asymptotic contribution to the logarithm of $U_V(s)$ in (7.5) comes from the first term associated with the principal eigenvalue, and the contribution from the other terms is asymptotically negligible. Looking at (7.5) and applying the Cauchy-Schwarz inequality and Parseval's identity, we easily obtain the upper bound

$$U_V(s) \leqslant \mathrm{e}^{\lambda_{1,V} s - 2\nu\kappa s}\sqrt{|V|} \tag{7.6}$$

for each $s \geqslant 0$ and each $V$. To derive the lower bound for $U(s)$, one needs more sophisticated arguments: Let $\underline{V} = \underline{V}(s)$ denote centered cubes of side length of order $s(\log s)^{-c}$ for some $c > 1$, and the site $z_1 \in \underline{V}$ as a localization center of the principal eigenfunction of the operator $\mathcal{H}_{\underline{V}} = \kappa\Delta_{\underline{V}} + \xi_{\underline{V}}$. Recall the Feynman-Kac representation for $U(s)$:

$$U(s) = \mathbf{E}_0\left[\exp\left\{\int_0^s \xi(X(a))\,\mathrm{d}a\right\}\right], \quad s \geqslant 0,$$

so that $U(s) \geqslant \underline{U}(s)$, where $\underline{U}(s)$ is the same expectation $\mathbf{E}_0[\cdot]$ when restricted to the particle trajectories $X[0;s] \subset \underline{V}(s)$, which initially move from the origin to the site $z_1$ until time 1, then stay in $\underline{V}(s)$ time interval of length $s-1$, at the end of which the particles return to $z_1$ and the remaining time move freely. Assuming that $\xi(\cdot)$ is bounded from below (so percolation effects of very low values of $\xi(\cdot)$ are neglected), we obtain from the strong Markov property that

$\underline{U}(s) = v(s-1,z_1)\,\mathrm{e}^{\mathrm{o(s)}}$ almost surely, where $v(s,x)$ is the expectation over trajectories in $\underline{V}(s)$ starting from $x \in \underline{V}(s)$ and ending at $z_1$ during the time interval $[0;s]$, viz.

$$v(s,x) = \mathbf{E}_x\left[\exp\left\{\int_0^s \xi(X(a))\,\mathrm{d}a\right\}\mathbb{1}_{\{X[0;s]\subset\underline{V}(s)\}}\delta_{z_1}(X(s))\right].$$

Thus, $v(s,\cdot)$ is the solution to equation (7.1) in $\underline{V}(s)$ with the initial datum $\delta_{z_1}$ instead of $\delta_0$. Using the spectral representation for $v(s-1,z_1)$ (where all terms are nonnegative!) combined with the previous estimates, we finally obtain the lower bound for $U(s)$: As $s\to\infty$, almost surely

$$\begin{aligned} U(s) &\geqslant v(s-1,z_1)\,\mathrm{e}^{\mathrm{o(s)}} \geqslant \mathrm{e}^{\lambda_{1,\underline{V}}s-2\nu\kappa s+\mathrm{o(s)}}\psi(z_1;\lambda_{1,\underline{V}})^2 \\ &\geqslant \mathrm{e}^{\lambda_{1,\underline{V}}s-2\nu\kappa s+\mathrm{o(s)}}|\underline{V}(s)|^{-1} = \mathrm{e}^{\lambda_{1,\underline{V}}s-2\nu\kappa s+\mathrm{o(s)}}. \end{aligned}$$

From this formula and (7.6), we get that almost surely

$$\lambda_{1,\underline{V}(s)} - 2\nu\kappa + \mathrm{o}(1) \leqslant s^{-1}\log U(s) \leqslant \lambda_{1,V(s)} - 2\nu\kappa + \mathrm{o}(1) \quad \text{as} \quad s\to\infty.$$

Applying the almost sure asymptotic formulas for the principal eigenvalue $\lambda_{1,V}$ in Corollary 6.6 ($\rho=\infty$), Theorem 6.19 ($0<\rho<\infty$) and Theorem 6.14 ($\rho=0$) of the present paper, we can now derive the corresponding asymptotics for the total mass $U(s)$:

**Theorem 7.1** (Gärtner and Molchanov, 1998; Section 2.1)**.** *Let essinf $\xi(0) > -\infty$. Assume that $f := Q^{\leftarrow}$ satisfies the following RV conditions: there is a constant $0\leqslant\rho\leqslant\infty$ such that $\mathrm{e}^f \in RV_\rho$ and, additionally, $f(a+\log a)-f(a)\to 0$ as $a\to\infty$. Then with probability 1*

$$\frac{\log U(s)}{s} = f(\nu\log s) - 2\nu\kappa(1-q(\rho/\kappa)) + \mathrm{o}(1) \quad \textit{as} \quad s\to\infty. \tag{7.7}$$

*Here the nonrandom constants $q(\rho)$ are specified in Section 2.5; in particular, $q(\rho)$ are strictly decreasing in $\rho$; $q(0)=1$ and $q(\infty)=0$.*

Recall that the condition $\mathrm{e}^f\in RV_\rho$ with $0<\rho<\infty$ (resp., $\rho=\infty$ and $\rho=0$) ensures the double exponential upper tails (1.14) (resp., heavier and lighter upper tails than the double exponential) of the potential distribution $1-\mathrm{e}^{-Q}$; see Lemma A.3 of Appendix A. The additional RV condition of the theorem is to exclude heavy-tailed distributions of potential. Therefore, the first term on the right-hand side of (7.7) is equal (with the accuracy $\mathrm{o}(1)$) to the largest values of the potential in $V(s)$; see Theorem 4.4(iii). The second term describes the shape of the potential in the neighborhood of its maxima and is specified by deterministic variational principles; see Section 2.5. From Sections 1.2, 2 and 6, we know that the principal eigenvalue $\lambda_{1,V(s)}$ of the operator $\mathcal{H}_{V(s)} = \kappa\Delta_{V(s)}+\xi_{V(s)}$ is approximated (as $s\to\infty$) by the local principal eigenvalue in the connected region $\mathbb{A}_{\mathrm{opt}}(s)\subset V(s)$ where the potential $\xi_{V(s)}$ possesses high values of a particular preferred shape. The logarithmic asymptotics of $U(s)$ is therefore fully specified by these high values of the potential.

Gärtner and Molchanov (1998) derived also the second-order expansion formulas for statistical moments of the total mass $U(s)$ as $s \to \infty$. For the upper distributional tails of $\xi(0)$ lighter than the double exponential, Biskup and König (2001), van der Hofstad et al. (2006) obtained more accurate expansion formulas for statistical moments and almost sure behavior of $U(s)$. The spatial correlation structure for the PAM was investigated by Gärtner and den Hollander (1999). See also (Molchanov and Zhang, 2012) for the Anderson parabolic model with $\Delta_{\text{dif}}$ replaced by the fractional Laplacian $-(-\Delta_{\text{dif}})^{\theta}$, $0 < \theta < 1$, where the potential has Weibull type tails. In these papers, refined variational arguments were involved to obtain additional information on intermittent islands of solutions $u(s, \cdot)$ and the asymptotic structure of related $\xi_{V(s)}$-extremes (their size, optimal shape, etc., as $s \to \infty$). See also (König, 2016) for a recent survey on the subject.

Van der Hofstad et al. (2008) considered the case of i.i.d. potentials with heavy upper tails, i.e., polynomially decaying (Pareto) distributions and Weibull distributions (1.5) with $\alpha < 1$. Thus, all positive exponential moments of the potential are infinite, in contrary to the assumptions of Theorem 7.1. For such classes of potentials, they proved extremal type limit theorems and almost sure asymptotic bounds for the logarithm of the total mass $U(s)$.

### *7.3. Asymptotic concentration formulas*

However, the rough asymptotic expansion formulas for the PAM (like in Theorem 7.1) provide only appropriate information on the geometric structure of intermittent islands of the solutions $u(s, \cdot)$ to (7.1). Recall that the intermittent islands are formed by those highly concentrated $u(s, \cdot)$-values which give the main contribution to the total mass $U(s) = \sum_x u(s, x)$, and the contribution from the complement of these islands is negligible as $s \to \infty$. Recently, there has been a considerable attention to the following mathematical problems regarding a geometric characterization of intermittency effect:

1) description of the shape and location of intermittent islands;
2) description of the shape of potential values which generate intermittent islands;
3) specification of the minimal number of these islands, etc.

Thus, taking into account (7.3), one needs to prove the following concentration formula for the total mass $U(s)$: As $s \to \infty$,

$$U(s) \sim U_{V(s)}(s) \sim \sum_{k=1}^{n(s)} \sum_{x \in \mathbb{A}^k_{\text{opt}}(s)} u(s, x) \tag{7.8}$$

in the sense of asymptotic equivalence almost surely or in probability, where $\mathbb{A}^k_{\text{opt}}(s)$ are believed to present random connected regions (i.e. intermittent islands) in $V(s)$ at a large distance from each other, such that the diameter of each

$\mathbb{A}^k_{\mathrm{opt}}(s)$ is much smaller than this distance; $n(s)$ are relatively small numbers, and $V(s) \uparrow \mathbb{Z}^\nu$ are centered cubes as above.

Let us give a heuristic explanation of formula (7.8) in the spectral terms of the operator $\mathcal{H}_V = \kappa\Delta_V + \xi_V$ in $V = V(s)$, provided the conditions of Theorem 7.1 are fulfilled. To this end, we need more careful inspection of spectral representation formulas (7.4)–(7.5), by applying extreme value theory for eigenvalues including Poisson limit theorems and localization properties discussed in Sections 1.2, 2, and 6. First, notice that the exponents of the top eigenvalues are essentially larger than those associated with lower eigenvalues. Therefore, it suffices to consider the sum of a few first terms of (7.4)–(7.5) associated with the largest eigenvalues $\lambda_{k,V}$, $1 \leqslant k \leqslant n := n(s)$; the other terms in (7.4)–(7.5) associated with the lower eigenvalues are asymptotically negligible. Thus, the random field $u(s,\cdot)$ $(s \to \infty)$ may be interpreted as a superposition of a few wave functions $\psi(\cdot\,;\lambda_{k,V})$ for $1 \leqslant k \leqslant n$. The general theory of Anderson localization suggests that the $k$th eigenfunction $\psi(\cdot\,;\lambda_{k,V})$ is exponentially localized at the $s$-dependent center $z_{\tau(k),V} \in V$, and it is highly concentrated in a $s$-dependent neighborhood $\mathbb{A}^k_{\mathrm{opt}} \subset V$ of the site $z_{\tau(k),V}$. The regions $\mathbb{A}^k_{\mathrm{opt}} \subset V$ possess a relatively small size and are asymptotically far from each other according to Poissonian behavior of the localization centers $z_{\tau(k),V}|V|^{-1/\nu}$; $k \geqslant 1$. Moreover, $\xi(\cdot)$ possesses in $\mathbb{A}^k_{\mathrm{opt}}$ the deterministic optimal shape specified by the variational principles; cf. Section 2.5. From these observations when applied to (7.4) as $s \to \infty$, we see that the function

$$\mathrm{e}^{\lambda_{k,V}s - 2\nu\kappa s}\psi(0;\lambda_{k,V})\psi(\cdot;\lambda_{k,V})$$

is a very good approximation for $u_V(s,\cdot)$ in the region $\mathbb{A}^k_{\mathrm{opt}}$ for each $1 \leqslant k \leqslant n$. This in turn suggests that the mass concentration formula (7.8) holds true, where $V = V(s) \uparrow \mathbb{Z}^\nu$ as above, $n = n(s)$ are relatively small numbers, and $\mathbb{A}^k_{\mathrm{opt}} \subset V$ are the relevant regions defined above. Thus, the intermittent islands $\mathbb{A}^k_{\mathrm{opt}}(s)$ have relatively small size and are far away from each other as $s \to \infty$. This concludes the heuristic explanation of formula (7.8).

Gärtner et al. (2007) proved the concentration formula (7.8) almost surely, for potential distributions satisfying the conditions of Theorem 7.1 with $0 < \rho \leqslant \infty$, i.e., the double exponential upper tails and heavier than the double exponential. They showed that almost surely $n(s) = s^{\mathrm{o}(1)}$, and the connected regions $\mathbb{A}^k_{\mathrm{opt}}(s) \subset V(s)$ are asymptotically bounded only when defined properly (see above), and the distance between them is of order $s^{1-\mathrm{o}(1)}$. For $\rho = \infty$, the regions $\mathbb{A}^k_{\mathrm{opt}}(s)$ shrink to singletons. Moreover, the shape of the solutions $u(s,\cdot)$ and potential values in $\mathbb{A}^k_{\mathrm{opt}}(s)$ are specified (via the variational formulas) by the local principal eigenfunction and the principal eigenvalue in $\mathbb{A}^k_{\mathrm{opt}}(s)$. This agrees with the heuristics given above; cf. also Sections 1.2, 2, and 6 treating the extreme value theory for eigenvalues of the Hamiltonian $\mathcal{H}_V$. However, the proof of the asymptotic concentration becomes complicated by applying straightforwardly the asymptotic results for the top spectrum of $\mathcal{H}_V$, because of the possibly different signs of the $k$th eigenfunctions in (7.4)–(7.5) where the factor $\psi(0;\lambda_{k,V})$ should be also taken into account. Instead, the authors explore the Feynman-Kac formulas for $u(s,\cdot)$, $U(s)$ as well as for the principal eigenfunctions with a slightly modified potential. To prove the exponential decay of the principal eigenfunctions, they apply a decomposition technique for the trajectories of the random walk in the corresponding Feynman-Kac representations.

Sznitman (1998) earlier proved similar mass concentration results for a Brownian motion in $\mathbb{R}^\nu$ among Poisson obstacles; here the potential $\xi(\cdot)$ is given by formula (6.20). In particular, the spatial regions $\mathbb{A}^k_{\rm opt}(s) \subset \mathbb{R}^\nu$ in (7.8) were shown to have no obstacles and unboundedly increase almost surely as $s \to \infty$. The optimal strategy of the Brownian particle during the time period $[0;s]$ is to move quickly to one of the obstacle-free regions $\mathbb{A}^k_{\rm opt}(s)$ of the optimal shape, i.e. the ball of radius const $(\log s)^{1/\nu}$, and to stay here for the remaining time. Notice that the intermittent behavior of this model is rather similar to that of the spatially discrete PAM with the potential bounded from above. The related asymptotic results for the principal Dirichlet eigenvalues are discussed in Example 6.18 of the present survey.

However, in (Gärtner et al., 2007) and (Sznitman, 1998), the problem of the minimal number of intermittent islands was not considered. This problem in precise setting was solved by several mathematicians for Pareto distributions

$$\mathbb{P}(\xi(0) > t) = \mathrm{e}^{-Q(t)} = t^{-\beta} \quad (t \geqslant 1) \tag{7.9}$$

with $\beta > \nu$, and Weibull distributions (1.5) with arbitrary $\alpha > 0$. Recall that the choice of $\beta > \nu$ in (7.9) is to guarantee the existence and uniqueness of the solution $u(s,\cdot)$ to equation (7.1). Write, as above, $U(s) := \sum_x u(s,x)$ for the total mass.

**Theorem 7.2** (e.g., König et al., 2009; Sidorova and Twarowski, 2014; Fiodorov and Muirhead, 2014)**.** *Assume that the potential has either Pareto distribution with $\beta > \nu$, or Weibull distribution (1.5) with arbitrary $\alpha > 0$ . Then there exists a random process $Z_{\rm opt}(s)$ $(s \geqslant 0)$ with values in $\mathbb{Z}^\nu$ such that*

$$\lim_{s\to\infty} \frac{u\left(s, Z_{\rm opt}(s)\right)}{U(s)} = 1 \quad \textit{in probability.} \tag{7.10}$$

This theorem states the complete localization property for $u(s,\cdot)$ as $s \to \infty$, which is the strongest case of mass concentration formula (7.8) in probability with $n(s) \equiv 1$ and $\mathbb{A}^1_{\rm opt}(s) \equiv \{Z_{\rm opt}(s)\}$, a singleton.

For the Weibull distribution with $0 < \alpha < 2$, asymptotic formula (7.10) was proved by Sidorova and Twarowski (2014). This result was extended by Fiodorov and Muirhead (2014) to an arbitrary $\alpha > 0$. See also (Muirhead and Pymar, 2014) for the proof of the single-site concentration property for a random walk in a random environment (instead of the standard random walk like in the PAM) for the Weibull-distributed i.i.d. potential.

Earlier, König et al. (2009) proved (7.10) for the Pareto-distributed potential. They also established a two-site concentration property for the PAM: Almost surely $U(s) \sim u(s, Z^{(1)}(s)) + u(s, Z^{(2)}(s))$ as $s \to \infty$, where $Z^{(k)}(s)$ are distinct random processes with values in $\mathbb{Z}^\nu$. I.e., one obtains (7.8) with probability one, where $n(s) \equiv 2$, $\mathbb{A}^1_{\rm opt}(s)$ and $\mathbb{A}^2_{\rm opt}(s)$ are two singletons. This is the strongest almost sure version of the localization property, since for i.i.d. random potentials, the random field $u(s,\cdot)$ is asymptotically concentrated on at least two distinct sites in $\mathbb{Z}^\nu$. The reason of this fact lays on the observation that the localization sites are changing infinitely often as $s \to \infty$; thus, at very rare time moments when particles move from the previous localization site to a new one, the mass of particles should be concentrated on at least two different sites.

We recall from Sections 3–5 of the present survey that, for potential distributions as in Theorem 7.2, the $\xi_V$-peaks are strongly pronounced as $V \uparrow \mathbb{Z}^\nu$; therefore, the single-site concentration formula, no surprise, holds true according to the general picture of intermittency based on the Feynman-Kac formulas. On the other hand, for such potential distributions, there is a very precise extreme value theory for eigenvalues of the operators $\mathcal{H}_V = \kappa\Delta_V + \xi_V$, which can (and does) provide powerful techniques for the investigation of single-site concentration properties; cf. Sections 1.2, 2.2–2.3 and 6.1. See also (König, 2016) for a recent survey on the subject.

Let us *sketch the proof of Theorem 7.2 for the Weibull-distributed potentials* by applying the extreme value theory for eigenvalues. We follow the terminology and ideas of (Fiodorov and Muirhead, 2014). First, one obtains finite-volume approximation formulas (7.3) with "good" accuracy, where $V = V(s)$ are centered cubes in $\mathbb{Z}^\nu$ with the size length of order $s(\log s)^{1/\alpha}$. Then, let us look at the spectral representation formulas (7.4) and (7.5) for $u_V(s,\cdot)$ and $U_V(s)$. Because of the factor $\big(\psi(\cdot\,;\lambda_{k,V}), u(0,\cdot)\big)_V = \psi(0;\lambda_{k,V}) \geqslant 0$ $(1 \leqslant k \leqslant |V|)$ in (7.4), we need to study *the penalised spectrum*

$$\Psi(s;k) := \lambda_{k,V} + s^{-1}\log\psi(0;\lambda_{k,V}) - 2\nu\kappa \quad (1 \leqslant k \leqslant |V|)$$

instead of the usual $\operatorname{Spect}(\mathcal{H}_V) = \big\{\lambda_{k,V}:\ 1 \leqslant k \leqslant |V|\big\}$. Let us rewrite $u_V$ (7.4) in the terms of the penalised spectrum $\Psi(s) := \{\Psi(s;k):\ 1 \leqslant k \leqslant |V|\}$:

$$u_V(s,x) = \sum_{k=1}^{|V|} \mathrm{e}^{s\Psi(s;k)}\psi(x;\lambda_{k,V}) \quad (x \in V). \tag{7.11}$$

For $1 \leqslant l \leqslant |V|$, denote by $\Psi_{l,V}(s)$ the $l$th largest value in the sample $\Psi(s)$. It will turn out that the gap $\Psi_{1,V}(s) - \Psi_{2,V}(s)$ between the the first largest $\Psi_{1,V}(s)$ and the second largest $\Psi_{2,V}(s)$ in $\Psi(s)$ is sufficiently large; and moreover, the eigenfunctions $\psi(\cdot;\lambda_{k,V})$ decay exponentially (so that $\psi(0;\lambda_{k,V}) > 0$), for each $k \leqslant |V|^\varepsilon$ with $\varepsilon > 0$ small enough; cf. also Theorem 6.2, Corollary 6.10 and Example 6.12 of the present paper. We shall prove that the right-hand side of (7.4)–(7.5) is dominated by just one term associated with $\Psi_{1,V}(s)$; therefore, the localization center of the corresponding eigenfunction should be the concentration site for the random field $u_V(s,\cdot)$ as $s \to \infty$. To be more precise, let $\lambda_{\rm opt} := \lambda_{k^*,V}$ and $\psi_{\rm opt}(\cdot)$ denote the eigenvalue and eigenfunction of the operator $\mathcal{H}_V$ associated with the first largest value $\Psi_{1,V}(s)$ among the penalised spectrum $\Psi(s)$, i.e.

$$\Psi_{1,V}(s) = \lambda_{k^*,V} + s^{-1}\log\psi_{\rm opt}(0) - 2\nu\kappa;$$

here $k^* < |V|^\varepsilon$ with $\varepsilon > 0$ small enough. Let $Z_{\mathrm{opt}}(s) \in V$ stand for the localization center of the eigenfunction $\psi_{\mathrm{opt}}(\cdot)$. Also, for random processes $Y(s)$ and $W(s)$, we write $Y(s) \approx W(s)$ as $s \to \infty$, if the difference $Y(s) - W(s)$ tends to zero sufficiently fast in probability. Using this abbreviation and applying Cauchy-Schwarz inequality for eigenfunctions in (7.11), we obtain that

$$\max_x \left| \frac{u_V(s,x)}{\mathrm{e}^{s\Psi_{1,V}(s)}} - \psi_{\mathrm{opt}}(x) \right| \leqslant |V| \exp\left\{-s\left(\Psi_{1,V}(s) - \Psi_{2,V}(s)\right)\right\} \approx 0,$$

therefore, for the total mass $U_V(s)$ we have that

$$\left| \frac{U_V(s)}{\mathrm{e}^{s\Psi_{1,V}(s)}} - (\psi_{\mathrm{opt}}, 1)_V \right| \leqslant |V|^2 \exp\left\{-s\left(\Psi_{1,V}(s) - \Psi_{2,V}(s)\right)\right\} \approx 0,$$

as $s \to \infty$. Consequently,

$$\frac{u_V(s, Z_{\mathrm{opt}}(s))}{U_V(s)} \approx \frac{\psi_{\mathrm{opt}}(Z_{\mathrm{opt}}(s))}{(\psi_{\mathrm{opt}}, 1)_V} \approx 1$$

as $s \to \infty$, because of the sharp exponential decay of $\psi_{\mathrm{opt}}(\cdot)$. Since $u(s,\cdot) \approx u_V(s,\cdot)$ in $V$ and $U(s) \approx U_V(s)$, the last formula implies (7.10), as claimed.

However, the penalised spectral values are too complicated to handle. In order to study the spacings of the largest values in $\Psi(s)$ as well as further properties of the concentration site $Z_{\mathrm{opt}}(s)$, one needs a good approximation for $\Psi_{k,V}(s)$ by a simpler function on potential configurations. To this end, we introduce the following auxiliary quantities: Write $J := [(\alpha-1)/2]$ (= the integer part) for $\alpha \geqslant 1$, and $J = 0$ otherwise. Given $z \in V$, let $\lambda^{(J)}(z)$ denote the principal eigenvalue of the single-peak Hamiltonian

$$\kappa\Delta_V + \sum_{y:\ 1 \leqslant |y-z| \leqslant J} \widetilde{\xi}(y)\delta_y + \xi(z)\delta_z \quad \text{on} \quad l^2(V).$$

I.e., $\lambda^{(J)}(z)$ is the local principal eigenvalue on the lattice ball of radius $J$ with a single $\xi_V$-peak at $z$ surrounded by the island of lower $\xi_V$-values. Let $\lambda^{(J)}_{k,V} := \lambda^{(J)}(z_{\tau(k),V})$ denote the $k$th largest value of the sample $\{\lambda^{(J)}(z):\ z \in V\}$. We now observe from Section 6.1 of the present survey that the eigenvalue $\lambda_{k,V}$ is approximately equal to the local eigenvalue $\lambda^{(J)}_{k,V}$, i.e., $\lambda_{k,V} \approx \lambda^{(J)}_{k,V}$ as $V = V(s) \uparrow \mathbb{Z}^\nu$ and $k < |V|^\varepsilon$ with $\varepsilon > 0$ small enough. This observation and more careful inspection of the exponential decay of the $k$th eigenfunction $\psi(\cdot;\lambda_{k,V})$ at the origin (cf. Astrauskas, 2008; Section 6) suggest that, with high probability, the $K$th largest value $\Psi_{K,V}(s)$ of the penalised spectrum $\Psi(s)$ is approximately equal to the $K$th largest value $\Upsilon^{(J)}_{K,V}(s)$ of the *penalisation functional*

$$\Upsilon^{(J)}(s,z) := \lambda^{(J)}(z) - \frac{|z|}{s} \cdot \frac{\log\log s}{\alpha} - 2\nu\kappa \quad (z \in V). \tag{7.12}$$

Comparing the penalised spectral values to (7.12), we observe, in addition, that the quantity $\alpha^{-1}\log\log s = \log b_{V(s)} + \mathrm{O}(1)$ is the nonrandom rate of the exponential decay of the $k$th eigenfunction $\psi(\cdot;\lambda_{k,V})$ described in Proposition 1.3; moreover, $\log\psi(0;\lambda_{k,V})$ asymptotically behaves like $-|z_{\tau(k),V}|\log b_V$ as $V\uparrow\mathbb{Z}^\nu$. Recall also that, for each $z\in V$, the eigenvalue $\lambda^{(J)}(z)$ is a certain (nonlinear) function of the sample $\{\xi(z+x)\colon\ 0\leqslant|x|\leqslant J\}$; cf. (2.20)–(2.22). Now the concentration site $Z_{\mathrm{opt}}(s)$ can be defined as the maximizer of the random field $\Upsilon^{(J)}(s,\cdot)$ in $V$; here $\Upsilon^{(J)}(s,z)$ is a certain function of both the sample $\{\xi(z+x)\colon\ 0\leqslant|x|\leqslant J\}$ and the site $z$. Similarly as in Theorem 6.2 above, one obtains Poisson limit theorems for the (normalized) penalisation functionals $\Upsilon^{(J)}(s,\cdot)$ and their locations. This limit theorem implies the limiting distributions for the normalized concentration site $Z_{\mathrm{opt}}(s)$ as well as for the spacings $\Upsilon^{(J)}_{K,V}(s)-\Upsilon^{(J)}_{K+1,V}(s)$ of the random field $\Upsilon^{(J)}(s,\cdot)$ in $V$. The latter in turn implies the existence of a sufficiently large gap between the largest values in the penalised spectrum $\Psi(s)$, as claimed. This assertion concludes the heuristic proof of the single-site concentration property (7.10) of the solution $u(s,\cdot)$. □

As stated in (Fiodorov and Muirhead, 2014), the above considerations are a starting point in obtaining more information on the asymptotic behavior of the concentration site $Z_{\mathrm{opt}}(s)$ and the shape of potential in its neighborhood as $s\to\infty$, provided the random potential has Weibull distributions. In particular, Poisson limit theorems for the penalisation functional (7.12) imply that the site $Z_{\mathrm{opt}}(s)\in V(s)$ is of order $s$ up to logarithmic corrections and has the limiting distribution as a product of univariate Laplace distributions. Recall also that the random field $\Upsilon^{(J)}(s,\cdot)$ has the finite range $(=J)$ of dependency; see (Fiodorov and Muirhead, 2014) where $J$ is called as the *radius of influence*. For sufficiently heavy tails (Weibull distributions with $\alpha<2$), the eigenvalue $\lambda^{(J)}(z)$ in (7.12) may be replaced by $\xi(z)$, so that $\Upsilon^{(J)}(s,z)$ $(z\in V)$ are independent non-identically distributed random variables. (Recall that the case $\alpha<2$ was studied in (Sidorova and Twarowski, 2014) by exploring the Feynman-Kac representations). These observations are crucial for describing the shape of $\xi(\cdot)$ in the neighborhood of the concentration site $Z_{\mathrm{opt}}(s)$: As $s\to\infty$, with probability $1+\mathrm{o}(1)$ the single peak $\xi\left(Z_{\mathrm{opt}}(s)\right)$ is extremely high:

$$\xi\left(Z_{\mathrm{opt}}(s)\right)=b_{V(s)}(1+\mathrm{o}(1)),$$

where $b_V:=(\log|V|)^{1/\alpha}$; cf. Example 6.12 above. Meanwhile, the neighboring values $\xi(x)$ $(1\leqslant|x-Z_{\mathrm{opt}}(s)|\leqslant J)$ are essentially lower: there exists a strictly decreasing nonrandom function $d(\cdot)\colon[1;J]\mapsto[0;1)$ such that

$$\xi\,(x)\asymp b_{V(s)}^{d(|x-Z_{\mathrm{opt}}(s)|)}\quad\text{for } 1\leqslant|x-Z_{\mathrm{opt}}(s)|\leqslant J.$$

This characterization of the concentration site $Z_{\mathrm{opt}}(s)$ agrees with the asymptotic results for the largest eigenvalues $\lambda_{K,V}$ and eigenfunctions $\psi(\cdot\,;\lambda_{K,V})$ of the Hamiltonian $\mathcal{H}_V=\kappa\Delta_V+\xi_V$ as $V\uparrow\mathbb{Z}^\nu$ and $K\geqslant1$ fixed, provided $\xi(0)$ has Weibull distribution; cf. Sections 1.2, 2.2–2.3, and 6.1 of the present paper. In particular, let us look at the asymptotic expansion formulas for $\lambda_{K,V}$ (Example 6.12) to observe that the leading term $b_V$ comes from an isolated high peak $\xi(z_{\tau(K),V})$; meanwhile, the further terms of order o(1) come from the neighboring values $\xi(x)$ $(1 \leqslant |x - z_{\tau(K),V}| \leqslant J)$ with the same influence radius $J$ as in the PAM above; cf. also (Astrauskas, 2008; Section 6). Moreover, the eigenfunction $\psi(\cdot\,;\lambda_{K,V})$ is highly concentrated at the site $z_{\tau(K),V}$, the localization center (cf. Theorem 6.2(II)); and the neighboring values $\xi(x)$ $(1 \leqslant |x - z_{\tau(K),V}| \leqslant J)$ have significant influence on the asymptotic behavior of the localization index $\tau(K) = \tau_V(K)$ and on the concentration degree of $\psi(\cdot\,;\lambda_{K,V})$ in the neighborhood of $z_{\tau(K),V}$; see (Astrauskas, 2013). These observations suggest the following conclusion: the lighter are the tails of potential, the larger is the influence radius in both models; thus, the weaker are the localization properties of both models, PAM and (time-independent) Anderson Hamiltonian.

It is worth mentioning, at the heuristic level of rigor, that Section 2 (resp., Section 6) of the present paper exhibits all classes of "typical" configurations of the potential (resp., RV classes of potential distributions) which are thought to guarantee certain concentration properties for the solutions $u$ to equation (7.1). For instance, the results of Sections 2.3 and 6.1 (i.e., "relevant single-peak" approximation) should be preliminaries to establish a single-site concentration property in probability for solutions $u$, provided the upper tails of potential are heavier than the double exponential. In particular, the results of Section 2.2 and Theorem 6.9 ("sharp single-peak" approximation) are related to the single-site concentration property for $u$ with zero influence radius $J = 0$, provided the distributional tails are heavier than Weibull's tails with $\alpha = 3$. In view of the results of Sections 2.5 and 6.3 ("relevant island" approximation in the double exponential case), it can be conjectured that, with high probability, the solution $u(s,\cdot)$ exhibits entire concentration on a single island, the diameter of which is asymptotically bounded or increases very slowly as $s \to \infty$. See (König, 2016) for a heuristic explanation of this conjecture.

If the upper tails of potential are lighter than those of double exponential including bounded tails ($t_Q < \infty$), no results on concentration properties for the PAM are known, with the exception of the very special (but important) spatially continuous model of Brownian motion in a Poisson field of obstacles studied by Sznitman (1998).

## Appendix A: Regular variation

In this section, we study the classes of functions $f := Q^{\leftarrow}$ (the left-continuous inverse of the cumulative hazard function) introduced in Sections 4–6. These classes are characterized in terms of $Q$. The tail behavior of $Q(t)$ as $t \uparrow t_Q$ is also treated. In Section A.1, we recall the classical results on the domain of attraction of Gumbel max-stable law and regular variation $RV_\rho$. The classes $A\Pi^p_\infty$ (4.1), $A\Pi^p_0$ (4.2) and $\mathrm{O}A\Pi^p$ (4.3) are studied in Section A.2, and $PI_{<2}$ (4.5) in Section A.3. Finally, examples and counterexamples are given in Section A.4.

### A.1. The domain of attraction of Gumbel max-stable distribution and regular variation

We now give the well-known characterization statements for the distribution function to be in the domain of attraction of the max-stable Gumbel law $G_{\exp}(\cdot)$.

**Lemma A.1** ((Resnick, 1987; de Haan and Ferreira, 2006))**.** *The following assertions* (i)–(iii) *are equivalent:*

(i) $f \in A\Pi$ *(6.3) with an auxiliary function* $a(\cdot) > 0$*;*
(ii) *there exists another auxiliary function* $a_1 : (-\infty; t_Q) \to \mathbb{R}_+$ *such that*

$$Q(t + ca_1(t)) - Q(t) \to c \quad as \quad t \uparrow t_Q, \quad for\ any \quad c \in \mathbb{R};$$

(iii) *there exist functions* $b : (-\infty; t_Q) \to \mathbb{R}$ *and* $a_2 : (-\infty; t_Q) \to \mathbb{R}_+$ *such that*

$$Q(t) = b(t) + \int_{t_0}^{t} 1/a_2(s)\,\mathrm{d}s \qquad (t < t_Q),$$

*here* $b(t) \to \overline{b} \in \mathbb{R}$ $(t \uparrow t_Q)$, *the function* $a_2$ *is locally absolutely continuous with the density* $a_2'(t) \to 0$ $(t \uparrow t_Q)$ *and, for* $t_Q < \infty$, $a_2(t) \to 0$ $(t \uparrow t_Q)$.

*In this case,* $a \circ f(s) = a_1 \circ f(s)(1 + \mathrm{o}(1)) = a_2 \circ f(s)(1 + \mathrm{o}(1))$ *as* $s \to \infty$. *Moreover, the limit in (ii) with an auxiliary function* $a_1(\cdot) > 0$ *implies that* $f \in A\Pi$ *(6.3) with the same auxiliary function* $a(\cdot) \equiv a_1(\cdot)$.

**Example A.2.** For $t_Q = \infty$, $p \geqslant 0$ and $B > 0$, consider the subclass $A\Pi_B^p \subset A\Pi$ associated with the auxiliary function $a_2(s) := B(p+1)^{-1}s^{-p}$ in Lemma A.1(iii). In this case, $Q(t) = B^{-1}t^{p+1} + \mathrm{const} + \mathrm{o}(1)$, i.e., $1 - \mathrm{e}^{-Q}$ are Weibull type distributions. In the next section, we extend the subclass $A\Pi_B^p$ to the boundary cases $B = \infty$, $B = 0$ and O-type asymptotics.

Let us discuss the class $RV_\rho$ of (nondecreasing) functions, which are regularly varying at infinity with index $\rho$. Recall that, for $0 < \beta < \infty$, the condition $f \circ \log \in RV_{1/\beta}$ is sufficient and necessary for the distribution $1 - \mathrm{e}^{-Q}$ to be in the domain of attraction of max-stable Fréchet law $G_\beta(t) := \exp\left\{-t^{-\beta}\right\}$ $(t > 0)$; cf. Example 6.11. We now explore the class $RV_\rho$ to characterize the double exponential type distributions.

**Lemma A.3.** *For* $t_Q = \infty$ *and* $0 \leqslant \rho \leqslant \infty$, *the following assertions are equivalent:*

(i) $\mathrm{e}^f \in RV_\rho$*;*
(ii) $f(s) - f(\delta s) \to -\rho \log \delta$ *as* $s \to \infty$, *for any* $0 < \delta < 1$*;*
(iii) $Q(t + C)/Q(t) \to \mathrm{e}^{C/\rho}$ *as* $t \to \infty$, *for any* $C > 0$.

*For any* $0 \leqslant \rho \leqslant \infty$, *either of* (i)–(iii) *implies that* $\lim_{t\to\infty} t^{-1} \log Q(t) = \rho^{-1}$. *Finally, for any* $0 < \rho \leqslant \infty$, *either of* (i)–(iii) *yields that* $Q(t-)/Q(t) \to 1$ *as* $t \to \infty$, *i.e., the continuity condition (3.1).*

*Proof.* The equivalence of (i)–(iii) follows from Theorems 1.5.12, 2.4.7 and Propositions 2.4.4(iv) and 1.3.6(i) in (Bingham et al., 1987) combined with the observation in (Resnick, 1987, Sect. 0.2) that $Q(t-\varepsilon) \leqslant f^{\leftarrow}(t) \leqslant Q(t)$ for all $t \in \mathbb{R}$ and all $\varepsilon > 0$. For $\rho = \infty$ and $\rho = 0$, the equivalence of (i)–(iii) is also proved, respectively, in Lemma A.6 ($p = 0$) and Lemma A.7 ($p = 0$) of the present paper adapted for the argument-additive functions; see also (de Haan and Ferreira, 2006, Chapter 1 and Appendix B.1) for the case $0 < \rho < \infty$. □

The following lemma is compounded of Lemma A.1 and Lemma A.3 with $\rho = \infty$, provided that there exists the density of the distribution $1 - \mathrm{e}^{-Q}$.

**Lemma A.4** (Cf. Corollary 6.4)**.** *Let $t_Q = \infty$. For some large $t_0$, assume that $Q : [t_0; \infty) \to \mathbb{R}_+$ is (locally) absolutely continuous with the positive density $Q' : [t_0; \infty) \to \mathbb{R}_+$ obeying the following conditions:*

$$\lim_{t\to\infty} \frac{Q'(t+C)}{Q'(t)} = 1 \quad \text{for any} \quad C > 0, \tag{A.1}$$

*and*

$$\liminf_{t\to\infty} Q'(t) > 0. \tag{A.2}$$

*Then the following limits* (I)–(IV) *hold true:*

(I) $\lim_{t\to\infty} Q(t+u)/Q(t) = \lim_{t\to\infty} Q'(t+u)/Q'(t) = 1$ *uniformly in compact sets of $u \in \mathbb{R}$;*
(II) $\liminf_{t\to\infty} Q(t)/t > 0$*;*
(III) $\lim_{t\to\infty} \big(Q(t + va_1(t)) - Q(t)\big) = v$ *uniformly in compact sets of $v \in \mathbb{R}$, with $a_1(\cdot) \equiv 1/Q'(\cdot)$ in $[t_0; \infty)$;*
(IV) *with $p(t) := \mathrm{e}^{-Q(t)}Q'(t)$ $(t \geqslant t_0)$ as the distribution density and $a_1(\cdot)$ as in part* (III),
$$\lim_{t\to\infty} \frac{p(t+u+va_1(t))}{p(t+u)} = \mathrm{e}^{-v}$$
*uniformly in compact sets of $v, u \in \mathbb{R}$.*

*Proof.* (I) By L'Hôpital's rule, we obtain the first limit for any $u \in \mathbb{R}$. The uniform convergence follows from Theorem 1.2.1 in (Bingham et al., 1987) adapted for the argument-additive functions.

(II) The assertion follows from (A.2).

(III) Writing

$$Q(t + va_1(t)) - Q(t) - v = v \int_0^1 \Big(a_1(t)Q'(t + \theta va_1(t)) - 1\Big)\,\mathrm{d}\theta \quad \text{for} \quad t \geqslant t_0$$

and applying assertion (I) and condition (A.2), we easily obtain the claimed limit.

(IV) Let us rewrite the ratio under the limit in the form:

$$\frac{p(t+u+va_1(t))}{p(t+u)} = \exp\Big\{ - \big(Q(t+u+va_1(t)) - Q(t+u)\big)\Big\} \times$$

$$\frac{Q'(t+u+va_1(t))}{Q'(t)}\left(\frac{Q'(t+u)}{Q'(t)}\right)^{-1}; \tag{A.3}$$

$t \geqslant t_0$. Since $a_1(\cdot)$ is a bounded function, assertion (I) implies that the last two ratios on the right-hand side of (A.3) converge to 1 locally uniformly in $u, v \in \mathbb{R}$. It remains to prove the uniform convergence of the exponent on the right-hand side of (A.3). By the theorem of continuous convergence (see, e.g., p. 2 in (Resnick, 1987)), it suffices to check that, for arbitrary functions $u(t) \to u$ and $v(t) \to v$, the following limit holds true:

$$Q\big(t+u(t)+v(t)a_1(t)\big) - Q\big(t+u(t)\big) - v(t) \to 0 \quad \text{as} \quad t \uparrow \infty.$$

This is shown similarly as in part (III), so we omit the details. Lemma A.4 is proved. □

**Remark A.5** (Cf. Theorem 6.20)**.** Let $t_Q = \infty$. For some $t_0$, assume that $Q : [t_0; \infty) \to \mathbb{R}_+$ is (locally) absolutely continuous with the positive density $Q'$ satisfying the following condition: $(\log Q)'(t) \to \rho^{-1}$ as $t \to \infty$, for some $0 < \rho < \infty$. Then the following limits hold true:

(i) $\lim_{t\to\infty} Q(t+C)/Q(t) = \lim_{t\to\infty} Q'(t+C)/Q'(t) = \mathrm{e}^{C/\rho}$ uniformly in compact sets of $C \in \mathbb{R}$;

(ii) with $a(\cdot) \equiv 1/Q'(\cdot)$ in $[t_0; \infty)$,

$$\lim_{t\to\infty} \big(Q(t+Ca(t)) - Q(t)\big) = C \quad \text{for any} \quad C \in \mathbb{R}.$$

*Proof of the assertions of Remark A.5.* (i) Write $\log Q$ in the form:

$$\log Q(t) = \text{const} + \frac{t}{\rho} + \int_{t_0}^{t} \varepsilon(s)\,\mathrm{d}s \quad (t \geqslant t_0),$$

where $\varepsilon(t) := (\log Q)'(t) - \rho^{-1} \to 0$ as $t \to \infty$. Using this representation and the conditions of Remark A.5, we obtain the claimed limits for any $C \in \mathbb{R}$. The uniform convergence follows from Theorem 1.5.2 in (Bingham et al., 1987) adapted for the argument-additive functions.

(ii) Since $a(t) = \mathrm{o}(1)$, the claimed limit is derived similarly as in the proof of Lemma A.4(III). □

### *A.2. Classes* $\mathrm{A\Pi}^p_\infty$, $\mathrm{A\Pi}^p_0$ *and* $\mathrm{OA\Pi}^p$

Recall that, for $p \geqslant 0$, the classes $A\Pi^p_\infty$, $A\Pi^p_0$ and $\mathrm{O}A\Pi^p$ consist of functions $f := Q^{\leftarrow}$ satisfying, respectively, $f(s)^p\big(f(s+c) - f(s)\big) \to \infty$, $\to 0$ and $\asymp 1$ as $s \to \infty$, for any $c > 0$; cf. (4.1)–(4.3). We first formulate the results of this section. To avoid trivialities, we restrict ourselves to the case $t_Q = \infty$.

**Lemma A.6.** *For any* $p \geqslant 0$, $f \in A\Pi^p_\infty$ *if and only if*

$$\lim_{t\to\infty} \big(Q(t+ct^{-p}) - Q(t)\big) = 0 \quad \textit{for any} \quad c > 0. \tag{A.4}$$

*In this case,*

$$Q(t) = \mathrm{o}(t^{p+1}) \quad as \quad t \to \infty. \tag{A.5}$$

**Lemma A.7.** *For any $p \geqslant 0$, $f \in A\Pi_0^p$ if and only if*

$$\lim_{t\to\infty} \big(Q(t + ct^{-p}) - Q(t)\big) = \infty \quad for\ any \quad c > 0. \tag{A.6}$$

*In this case,*

$$Q(t)t^{-p-1} \to \infty \quad as \quad t \to \infty. \tag{A.7}$$

**Lemma A.8.** *For any $p \geqslant 0$ and $f \in \mathrm{O}A\Pi^p$, the following assertions hold true:*

(i) *there is a constant $c > 0$ such that $Q(t + ct^{-p}) - Q(t) \asymp 1$ as $t \to \infty$;*
(ii) *$Q(t) \asymp t^{p+1}$ as $t \to \infty$;*
(iii) *if a function $a : \mathbb{R}_+ \to \mathbb{R}_+$ is chosen to satisfy $\liminf_{s\to\infty} a(s) \geqslant c_1 > 0$ and $\liminf_{s\to\infty}(s - a(s)) \geqslant c_2 > 0$, then*

$$\begin{aligned} &\mathrm{const}\,(s - a(s))s^{-p/(p+1)} \leqslant f(s) - f(a(s)) \\ &\leqslant \mathrm{const}'\Big(s^{1/(p+1)} - a(s)^{1/(p+1)} + a(s)^{-p/(p+1)}\Big) \end{aligned}$$

*for any $s \geqslant s_0$ and for some $\mathrm{const}' \geqslant \mathrm{const} > 0$.*

Before proving Lemmas A.6–A.8, we provide an example of $Q$ satisfying assertion (i) of Lemma A.8 such that $f := Q^{\leftarrow}$ does not belong to $\mathrm{O}A\Pi^p$ for $p \geqslant 0$, and further on, two technical lemmas for later use.

**Example A.9.** For $p \geqslant 0$, write $Q(t) := t^{p+1} + [t]$, $t \geqslant 0$. Note that, for each $c > 0$,

$$Q(t + ct^{-p}) - Q(t) = c(p+1) + g(t) + \mathrm{o}(1) \quad \text{as} \quad t \to \infty,$$

where $0 \leqslant g(t) := [t + ct^{-p}] - [t] = \mathrm{O}(1)$. I.e., $Q$ satisfies the assertion of Lemma A.8(i) for any $c > 0$. However, for each $t := n \in \mathbb{N}$, we get $Q(n) - Q(n-) = 1$, therefore, $f := Q^{\leftarrow} \notin \mathrm{O}A\Pi^p$ according to Lemma A.10 below.

**Lemma A.10.** *If $\liminf_{s\to\infty} f(s)^p\big(f(s+c) - f(s)\big) > 0$ for each $c > 0$ and for some $p \geqslant 0$, then*

$$\lim_{t\to\infty} \big(Q(t) - Q(t-)\big) = 0,$$

*i.e., $Q$ is continuous at infinity.*

*Proof.* Assume for a moment that there exists a sequence $t_n \to \infty$ such that $Q(t_n) - Q(t_n-) \to c^0 > 0$. This limit implies that $s_n := Q(t_n) \to \infty$ and, in addition, that $f(s_n - c) = f(s_n)$ for any $0 < c < c^0$ and any $n \geqslant n_0(c)$, contradicting the assumption of the lemma. This completes the proof of the claimed assertion. □

**Lemma A.11** (Resnick, 1987, pp. 4)**.** *For all $s \in \mathbb{R}_+$ and $t \in (-\infty; t_Q)$, the following assertions hold true:*

(i) $f(s) \leqslant t$ *if and only if* $s \leqslant Q(t)$;
(ii) $f(s) > t$ *if and only if* $s > Q(t)$;
(iii) $Q(f(s)-) \leqslant s \leqslant Q(f(s))$.

We now are in a position to prove Lemmas A.6–A.8. To simplify the proceedings, we need the following abbreviations:

$$f_p(s;c) := f(s)^p\big(f(s+c) - f(s)\big) \quad \text{and} \quad Q_p(t;c) := Q(t+ct^{-p}) - Q(t).$$

*Proof of Lemma A.6.* Assume first that (A.4) holds true. I.e., for each $\varepsilon > 0$ there is $s_0 = s_0(\varepsilon) > 0$ such that

$$Q_p(f(s);\varepsilon^{-1}) < \varepsilon \quad \text{for all} \quad s \geqslant s_0. \tag{A.8}$$

Since (A.4) implies a continuity of $Q$ at infinity, from Lemma A.11(iii) we have that $Q(f(s)) \leqslant s + \varepsilon$ for each $s \geqslant s_0$. This and (A.8) yield that $Q\big(f(s) + \varepsilon^{-1}f(s)^{-p}\big) < s + 2\varepsilon$ for each $s \geqslant s_0$. Inverting $Q$ (see Lemma A.11(ii)), we get that $f_p(s;2\varepsilon) > 1/\varepsilon$ for each $\varepsilon > 0$ and each $s \geqslant s_0(\varepsilon)$. I.e., $f \in A\Pi^p_\infty$.

To prove the inverse implication, assume for a moment that there are a sequence of reals $t_n \to \infty$ and constants $c > 0$, $\varepsilon > 0$ such that $Q(t_n + ct_n^{-p}) \geqslant Q(t_n) + \varepsilon$ for all $n \geqslant n_0(\varepsilon, c)$. Inverting $Q$ (see Lemma A.11(i)), we get that $f\big(Q(t_n) + \varepsilon\big) \leqslant t_n + ct_n^{-p}$, which combined with $t_n < f\big(Q(t_n) + \varepsilon/2\big)$ (see Lemma A.11(ii)) gives that, for each $n \geqslant n_0(\varepsilon, c)$,

$$f_p(s_n;\varepsilon/2) \leqslant c \quad \text{with} \quad s_n := Q(t_n) + \varepsilon/2.$$

Since $s_n \to \infty$, the latter violates the assumption $f \in A\Pi^p_\infty$, concluding the proof of the first part of Lemma A.6.

To prove (A.5), we note that, for any natural $M \geqslant 2$ and any $s \geqslant 2M$,

$$f(s+1)^{p+1} - f(s)^{p+1} \geqslant f_p(s;1) \geqslant I(M) := \inf_{s \geqslant M} f_p(s;1),$$

and, therefore,

$$f(s)^{p+1} - f(M)^{p+1} \geqslant (s - M - 1)I(M).$$

Hence $\liminf_{s\to\infty} f(s)s^{-1/(p+1)} \geqslant I(M)^{1/(p+1)}$. Since $I(M) \to \infty$ (as $M \to \infty$) by the assumption, the latter implies that $f(s)s^{-1/(p+1)} \to \infty$ as $s \to \infty$, which in turn yields (A.5). Lemma A.6 is proved. □

*Proof of Lemma A.7.* Assume first that (A.6) holds true, i.e., for each $\varepsilon > 0$ there is $t_0 = t_0(\varepsilon)$ such that $Q_p(t;\varepsilon) \geqslant 1/\varepsilon$ for each $t \geqslant t_0$. By Lemma A.11(i), the latter is equivalent to $f\big(Q(t) + 1/\varepsilon\big) \leqslant t + \varepsilon t^{-p}$. Substituting $t := f(s) \to \infty$ into this inequality and then applying $Q(f(s)) \geqslant s$ (see Lemma A.11(iii)), we obtain $f_p(s;1/\varepsilon) \leqslant \varepsilon$ for each $\varepsilon > 0$ and each $s \geqslant s_0(\varepsilon)$, i.e., $f \in A\Pi^p_0$.

To prove the inverse implication, assume for a moment that there are a sequence $t_n \to \infty$ and constants $\delta > 0$, $c > 0$ such that $Q_p(t_n;\delta) < c$ for each $n \geqslant n_0(c,\delta)$. Here, inverting $Q$ (see Lemma A.11(i),(ii)) and denoting $s_n := Q(t_n) \to \infty$, we obtain that $f_p(s_n;c) > \delta$ for each $n \geqslant n_0(c,\delta)$, contradicting the assumption $f \in A\Pi^p_0$. This completes the proof of the first part of Lemma A.7.

We will prove (A.7) under the weaker condition by assuming (A.6) for some $c > 0$. (The forthcoming arguments are applied to prove assertion (ii) of Lemma A.8 as well). Write $Q^{(c)}(t) := Q(c^{1/(p+1)}t)$ and observe that

$$\lim_{t\to\infty} Q_p^{(c)}(t;1) = \lim_{t\to\infty} Q_p(t;c) = \infty,$$

i.e., limit (A.6) for $c > 0$ is reduced to that for $c = 1$. With the abbreviation $R(k) := [k^p]$, we obtain that, for fixed natural $M \geqslant 3$ and any natural $t \geqslant 2M$,

$$\begin{aligned} &Q^{(c)}(t) - Q^{(c)}(M) \\ &\quad = \sum_{k=M}^{t-1} \sum_{l=0}^{R(k)-1} \left( Q^{(c)}\Big(k + \frac{l+1}{R(k)}\Big) - Q^{(c)}\Big(k + \frac{l}{R(k)}\Big)\right) \\ &\quad \geqslant \sum_{k=M}^{t-1} \sum_{l=0}^{R(k)-1} Q_p^{(c)}\Big(k + \frac{l}{R(k)}; 1\Big) \geqslant \inf_{\tau\geqslant M} Q_p^{(c)}(\tau;1) \sum_{k=M}^{t-1} R(k) \\ &\quad = \inf_{\tau\geqslant M} Q_p^{(c)}(\tau;1) \frac{t^{p+1}}{p+1}\,(1+\mathrm{o}(1)) \end{aligned}$$

as $t \to \infty$. Here, by (A.6), the infimum tends to infinity as $M \to \infty$, therefore, (A.7) is fulfilled. This completes the proof of Lemma A.7. □

*Proof of Lemma A.8.* (i) We first prove that if, for each $c > 0$, the function $f_p(s;c)$ is asymptotically bounded away from zero as $s \to \infty$, then $Q_p(t;\delta) = \mathrm{O}(1)$ as $t \to \infty$, for some $\delta > 0$. Assume otherwise that, for each $\delta > 0$, there exists a sequence $t_n \to \infty$ such that $Q_p(t_n;\delta) \geqslant 2M$ for any $M > 0$ and any $n \geqslant n_0(M)$. Here, inverting $Q$ similarly as in the proof of Lemma A.6, we obtain that $f_p(s_n;M) \leqslant \delta$ with $s_n := Q(t_n) + M \to \infty$ as $n \to \infty$. Therefore, $\liminf_{s\to\infty} f_p(s;M) \leqslant \delta$. Since $\delta > 0$ is arbitrary, we obtain the contradiction proving the desired implication. We next observe that, if $Q_p(t;\delta) = \mathrm{O}(1)$ for some $\delta > 0$, then

$$Q_p(t;k\delta) = \mathrm{O}(1) \quad \text{as} \quad t \to \infty, \quad \text{for any} \quad k \in \mathbb{N}. \tag{A.9}$$

This implication is easily proved by induction in $k$. We omit the details.

With the abbreviation

$$M := 2 \limsup_{s\to\infty} f_p(s;1) > 0,$$

we finally show that the function $Q_p(t;M)$ is asymptotically bounded away from zero as $t \to \infty$. For this, fix an arbitrary sequence $t_n \to \infty$, and define a sequence $\{s_n\}$ by $f(s_n+) \geqslant t_n \geqslant f(s_n)$ $(n \in \mathbb{N})$. Write $\tau_n := f(s_n)$. Combining Lemmas A.10 and A.11(iii), we have that $Q(t_n) - Q(\tau_n) = \mathrm{o}(1)$ and, consequently,

$$Q_p(t_n;M) \geqslant Q_p(\tau_n;M) + \mathrm{o}(1) \quad \text{as} \quad n \to \infty. \tag{A.10}$$

On the other hand, from the definition of $M$, it follows that $f(s_n)+Mf(s_n)^{-p} > f(s_n+1)$ for any $n \geqslant n_0$. Applying $Q$ to both sides of this inequality and then using Lemmas A.10 and A.11(iii), we obtain that $Q_p(\tau_n; M) \geqslant 1/2$ for $n \geqslant n_0$. The latter combined with (A.10) implies that the sequence $Q_p(t_n; M)$ is asymptotically bounded away from zero, as claimed. This and (A.9) conclude the proof of part (i).

(ii) The assertion is shown by the same arguments as in the proof of limits (A.5) and (A.7). We omit the details.

(iii) If $a(s)$ or $s-a(s)$ are bounded from above for any large $s$, then the bounds in (iii) simply follow from part (ii) and condition (4.3).

For simplicity we abbreviate $a := a(s)$, and assume that both $a$ and $s-a$ tend to infinity as $s \to \infty$. By combining assumption (4.3) and the limit $f(s) \asymp s^{1/(p+1)}$, we obtain that, for $s \geqslant s_0$,

$$
\begin{aligned}
f(s) - f(a) &\leqslant \sum_{0 \leqslant k \leqslant s-a} (f(a+k+1) - f(a+k)) \\
&\leqslant \text{const} \sum_{0 \leqslant k \leqslant s-a} (a+k)^{-p/(p+1)} \\
&\leqslant \text{const}\, a^{-p/(p+1)} + \text{const} \int_0^{s-a} (a+k)^{-p/(p+1)}\, \mathrm{d}k
\end{aligned}
$$

and

$$
\begin{aligned}
f(s) - f(a) &\geqslant \sum_{0 \leqslant k \leqslant s-a-1} (f(a+k+1) - f(a+k)) \\
&\geqslant \text{const} \sum_{0 \leqslant k \leqslant s-a-1} (a+k)^{-p/(p+1)} \\
&\geqslant \text{const}\, (s-a-1) s^{-p/(p+1)},
\end{aligned}
$$

as claimed. Lemma A.8 is proved. □

**Remark A.12** (A relationship with classical regular variation)**.** (i) Consider the case $p = 0$. Obviously, for $\beta = \infty$ or $\beta = 0$, $f$ is in $A\Pi^0_\beta$ if and only if $g := \exp \circ f \circ \log \in RV_\beta$. Therefore, for $p = 0$, Lemmas A.6 and A.7 follow from the well-known results for the class $RV_\beta$ with $\beta = \infty$ and $\beta = 0$, respectively (Bingham et al., 1987).

The class $\mathrm{O}A\Pi^0$ links to the exponential type distributions $1 - \mathrm{e}^{-Q}$, with $Q(t) \asymp t$ as $t \to \infty$. Moreover, if $f \in \mathrm{O}A\Pi^0$, then $g := \exp \circ f \circ \log$ is in $\mathrm{O}RV$, the class of O-regularly varying functions studied, e.g., in (Bingham et al., 1987, Section 2).

(ii) In the case of $p \geqslant 0$, if $f \in \mathrm{O}A\Pi^p$, then $f \circ \log$ is asymptotically balanced or, equivalently, the maximum $\xi_{1,V}$ of i.i.d. sample $\xi_V$ is stochastically compact (Bingham et al., 1987, Sections 3.11 and 8.13.12).

### A.3. Class $PI_{<2}$

Recall that the class $PI_{<2}$ consists of functions $f := Q^{\leftarrow}$ such that

$$\liminf_{s\to\infty} \frac{f(cs)}{f(s)} > 1 \quad \text{for some} \quad 1 < c < 2;$$

cf. (4.5). This is the subclass of positive increase class $PI$ considered, e.g., in (Bingham et al., 1987, Section 2.1.2).

**Lemma A.13.** *$f \in PI_{<2}$ if and only if*

$$\limsup_{t\to\infty} Q(C_0 t)/Q(t) < 2 \quad \textit{for some} \quad C_0 > 1. \tag{A.11}$$

*In this case,*

$$\limsup_{t\to\infty} \frac{\log Q(t)}{\log t} \leqslant \text{const} := \frac{\log 2}{\log C_0}. \tag{A.12}$$

*Proof.* Assume in contrary to (A.11) that, for any $C > 1$, there is a sequence $t_n = t_n(C) \to \infty$ such that $Q(Ct_n) \geqslant (2-\varepsilon)Q(t_n)$ for any $\varepsilon > 0$ and any $n \geqslant n_0(\varepsilon)$. Similarly as in the proof of Lemma A.6, inverting $Q$ (see Lemma A.11(i),(ii)) and writing $s_n := Q(t_n) + \varepsilon \to \infty$, we obtain that

$$\limsup_{n} f(\delta s_n)/f(s_n) \leqslant C \quad \text{for any} \quad 1 < \delta < 2 \quad \text{and any} \quad C > 1$$

or, equivalently, $\lim_n f(\delta s_n)/f(s_n) = 1$, contradicting the assumption $f \in PI_{<2}$. According to these arguments, the inclusion $f \in PI_{<2}$ implies (A.11).

To show the inverse implication, we suppose otherwise that $f := Q^{\leftarrow} \notin PI_{<2}$, i.e., for each $1 < c < 2$, there exists a sequence $s_n = s_n(c) \to \infty$ such that $f(cs_n) \leqslant (1+\varepsilon) f(s_n)$ for any $\varepsilon > 0$ and any $n \geqslant n_0(\varepsilon)$. In this inequality, we invert $f$ (see Lemma A.11(i)) to obtain $cs_n \leqslant Q\big((1+\varepsilon)f(s_n)\big)$. On the other hand, by Lemma A.11(iii),

$$cs_n \geqslant cQ\big(f(s_n)-\big) \geqslant cQ\big((1-\varepsilon)f(s_n)\big).$$

Summarizing these estimates and using the abbreviations $t_n := (1-\varepsilon)f(s_n)$ and $\delta := (1+\varepsilon)/(1-\varepsilon)$, we have that $Q(\delta t_n) \geqslant cQ(t_n)$. Since $1 < c < 2$ is an arbitrary constant but close to 2, the latter implies the limit $\limsup_t Q(\delta t)/Q(t) \geqslant 2$ for each $\delta > 1$, contradicting assumption (A.11). This concludes the first part of the lemma.

Let us show (A.12). By (A.11), there exist numbers $C > 1$ and $t_0 = t_0(C)$ such that $Q(Ct)/Q(t) \leqslant 2$ for all $t \geqslant t_0$. Applying this estimate, we obtain that, for any $n \in \mathbb{N}$ and any $t \in \big[C^n t_0; C^{n+1} t_0\big)$,

$$\begin{aligned} Q(t) &= \frac{Q(t)}{Q(C^n t_0)} \cdot \frac{Q(C^n t_0)}{Q(C^{n-1} t_0)} \cdots \frac{Q(Ct_0)}{Q(t_0)} \cdot Q(t_0) \\ &\leqslant Q(t_0) 2^{n+1} \leqslant \text{const}\, t^{(\log 2)/\log C}, \end{aligned}$$

i.e., (A.12) is done. Lemma A.13 is proved. ◻

### *A.4. Comparison of the classes $A\Pi^p_\infty$, $A\Pi$ and $PI_{<2}$. Examples*

In view of limit theorems for eigenvalues (see Theorems 6.2 and 6.9), we need to compare the classes $A\Pi^p_\infty$ (4.1), $SA\Pi^2_\infty$ (5.4), $A\Pi$ (6.3) and $PI_{<2}$ (4.5) of functions $f := Q^{\leftarrow}$.

**Lemma A.14.** (i) *For any $p \geqslant 0$, there exist examples $f_1 \in PI_{<2}\backslash A\Pi^p_\infty$ and $f_2 \in A\Pi^p_\infty\backslash PI_{<2}$. Consequently, there is $f_2 \in SA\Pi^2_\infty\backslash PI_{<2}$.*

(ii) *There exist examples $f_3 \in PI_{<2}\backslash A\Pi$ and $f_4 \in A\Pi\backslash PI_{<2}$ with an auxiliary function $a_4 \geqslant 1$.*

(iii) *For any $p > 0$, there exists an example $f_5 \in \big(A\Pi^p_\infty \cap PI_{<2}\big)\backslash A\Pi$ and, therefore, there is $f_5 \in \big(SA\Pi^2_\infty \cap PI_{<2}\big)\backslash A\Pi$.*

(iv) *For $p \geqslant 0$, if $f \in A\Pi$ with an auxiliary function $a : \mathbb{R}_+ \to \mathbb{R}_+$ such that $s^p a(s) \to \infty$ as $s \to \infty$, then $f \in A\Pi^p_\infty$.*

(v) *For $0 \leqslant \rho < \infty$, if $\mathrm{e}^f$ is in $RV_\rho$, then $f(s + \log s) - f(s) \to 0$ as $s \to \infty$.*

*Proof.* (i) Write $Q(t) := [t]$, $t \geqslant 0$; i.e., $1 - \mathrm{e}^{-Q}$ is the geometric distribution. Let us show that $f_1 := Q^{\leftarrow} \in PI_{<2}\backslash A\Pi^p_\infty$. Indeed, since $Q(n) - Q(n-) = 1$ for all $n \in \mathbb{N}$, Lemma A.10 implies that $f_1 \notin A\Pi^p_\infty$ for any $p \geqslant 0$. However, since $Q$ satisfies (A.11), we have that $f_1 \in PI_{<2}$, as claimed.

Consider the function $f_2(s) := \int_1^s b(t)\,\mathrm{d}t$, where $b(t) := n$ if $2^{2n} < t \leqslant 2^{2n+1}$, and $b(t) := 2^n$ if $2^{2n+1} < t \leqslant 2^{2n+2}$ for all $n \in \mathbb{N} \cup \{0\}$. Let us show that $f_2 \in A\Pi^0_\infty\backslash PI_{<2}$. Obviously $f_2 \in A\Pi^0_\infty$. With $s_n := 2^{2n+1}$ and $1/2 < \delta < 1$, we see that

$$\frac{f_2(\delta s_n)}{f_2(s_n)} = 1 - \frac{\int_{\delta s_n}^{s_n} b(t)\,\mathrm{d}t}{\int_1^{s_n} b(t)\,\mathrm{d}t},$$

where $\int_{\delta s_n}^{s_n} b(t)\,\mathrm{d}t = \mathrm{const}\, 4^n n$ and

$$\int_1^{s_n} b(t)\,\mathrm{d}t = \sum_{l=0}^{n-1}\left(\int_{s_l/2}^{s_l} l\,\mathrm{d}t + \int_{s_l}^{2s_l} 2^l\,\mathrm{d}t\right) + \int_{s_n/2}^{s_n} n\,\mathrm{d}t$$

$$= \sum_{l=0}^{n-1}\left(l \cdot 4^l + 2 \cdot 8^l\right) + n \cdot 4^n = \frac{2}{7} \cdot 8^n(1 + \mathrm{o}(1))$$

as $n \to \infty$. Summarizing, we find that $f_2(\delta s_n)/f_2(s_n) \to 1$ (as $n \to \infty$) for each $1/2 < \delta < 1$, i.e., $f_2 \notin PI_{<2}$. Since $A\Pi^1_\infty \subset SA\Pi^2_\infty$, we also obtain that $f_2 \in SA\Pi^2_\infty\backslash PI_{<2}$.

(ii) As in part (i) above, let $f_3$ be the inverse of the cumulative hazard function of the geometric distribution. Since $f_3$ is not in $A\Pi$ (Resnick, 1987, Corollary 1.6), we obtain that $f_3 \in PI_{<2}\backslash A\Pi$.

To prove the existence of $f_4 \in A\Pi\backslash PI_{<2}$, it suffices (via Lemmas A.1 and A.13) to find a continuous function $a : [1;\infty) \to [1;\infty)$, with derivative $a'(t) \to 0$ as $t \uparrow \infty$, such that the function $Q(t) := \int_1^t 1/a(s)\,\mathrm{d}s$ does not satisfy (A.11). For this, we abbreviate $t_n := (\log n)^n$, $m_n := (\log n)^2$,

$$\varepsilon_n := \frac{1}{m_n}\left(\frac{t_{n+1}}{t_n} - 1\right) - \frac{1}{t_n} \quad \text{and} \quad b_n := 1 + t_{n+1}\varepsilon_n,$$

and consider the functions

$$a(t) := \begin{cases} t/t_n & \text{if } \ t_n < t \leqslant t_{n+1} - m_n, \\ -\varepsilon_n t + b_n & \text{if } \ t_{n+1} - m_n < t \leqslant t_{n+1}; \quad \text{for } n \geqslant 2, \end{cases} \tag{A.13}$$

and $Q(t) := \int_1^t 1/a(s)\,\mathrm{d}s$. Obviously, $a \geqslant 1$ is continuous in $[1;\infty)$ and $a'(t) \to 0$ as $t \uparrow \infty$. Therefore, $f_4 := Q^{\leftarrow} \in A\Pi$ with the auxiliary function $a$. Let us show that, for each $c > 1$,

$$\frac{Q(ct_n)}{Q(t_n)} = 1 + \frac{\int_{t_n}^{ct_n} 1/a(t)\,\mathrm{d}t}{\int_1^{t_n} 1/a(t)\,\mathrm{d}t} \to \infty \quad \text{as} \quad n \to \infty, \tag{A.14}$$

so that $f_4 \notin PI_{<2}$. Indeed, from (A.13) we see that, for $n \geqslant n_0(c)$,

$$\int_{t_n}^{ct_n} \frac{\mathrm{d}t}{a(t)} = \int_{t_n}^{ct_n} \frac{t_n}{t}\,\mathrm{d}t = t_n \log c. \tag{A.15}$$

To estimate the integral $\int_1^{t_n}$ in (A.14), we again use (A.13) and the bound $a \geqslant 1$. Thus,

$$\begin{aligned} \int_{t_2}^{t_n} \frac{\mathrm{d}t}{a(t)} &\leqslant \sum_{k=2}^{n-1} \left( t_k \int_{t_k}^{t_{k+1}-m_k} \frac{\mathrm{d}t}{t} + \int_{t_{k+1}-m_k}^{t_{k+1}} 1\,\mathrm{d}t \right) \\ &\leqslant \sum_{k=2}^{n-1} \big( t_k \log(t_{k+1}/t_k) + m_k \big) \leqslant t_n (\log n)^{-1/2} \end{aligned} \tag{A.16}$$

for any $n \geqslant n_0$. Now (A.15) and (A.16) imply (A.14), as claimed.

(iii) Consider the example $Q(t) = t + \sin t$ $(t \geqslant 0)$ given by Von Mises. Obviously, $Q$ satisfies (A.4) and (A.11), consequently, $f_5 := Q^{\leftarrow}$ is in $A\Pi_\infty^p \cap PI_{<2}$ for any $p > 0$. However, $f_5 \notin A\Pi$. We observe that the function $f_6(s) := s + \sin s$ $(s \geqslant 0)$ is also in $\big(A\Pi_\infty^p \cap PI_{<2}\big) \setminus A\Pi$ for any $p > 0$. (This is verified by straightforward calculations). Consequently, since $A\Pi_\infty^1 \subset SA\Pi_\infty^2$, the functions $f_5$ and $f_6$ are in $\big(SA\Pi_\infty^2 \cap PI_{<2}\big) \setminus A\Pi$.

(iv) The assertion follows from the definition of $A\Pi$ (6.3) and $A\Pi_\infty^p$ (4.1).

(v) The assertion follows from Lemma A.3(ii). Lemma A.14 is proved. □

We finally provide two examples of distributions which represent RV classes considered in Sections 3–6.

**Example A.15.** For $\alpha > 0$, let $Q_\alpha(t) = t^\alpha$ $(t \geqslant 0)$, i.e., $1 - \mathrm{e}^{-Q_\alpha}$ is Weibull distribution. Clearly $f_\alpha(s) := Q_\alpha^{\leftarrow}(s) = s^{1/\alpha}$ for $s \geqslant 0$. By straightforward calculations, we obtain that if $\alpha < p+1$ (resp., $\alpha > p+1$ or $\alpha = p+1$), then $f_\alpha \in A\Pi_\infty^p$ (resp., $f_\alpha \in A\Pi_0^p$ or $f_\alpha \in \mathrm{O}A\Pi^p$). Also, for $\alpha < 3$, $f_\alpha \in SA\Pi_\infty^2$. Finally, for any $\alpha > 0$, $f_\alpha \in PI_{<2}$, $\exp \circ f_\alpha \in RV_\infty$ and $f_\alpha$ is in $A\Pi$ with the auxiliary function $a(t) := \alpha^{-1} t^{1-\alpha}$. The latter means that the distribution $1 - \mathrm{e}^{-Q_\alpha}$ is in the domain of attraction of the max-stable Gumbel law; cf. Section 6.1.

**Example A.16.** Given $\gamma > 0$ and $\rho > 0$, let $Q_{\gamma,\rho}(t) = e^{\rho^{-1}t^{\gamma}}$ ($t \geqslant t_0$), i.e., the fractional double exponential distribution. Then $f_{\gamma,\rho}(s) = (\rho \log s)^{1/\gamma}$ for $s \geqslant s_0$. Obviously, if $0 < \gamma < 1$ (resp., $\gamma > 1$ or $\gamma = 1$ ), then $\exp \circ f_{\gamma,\rho}$ is in $RV_\infty$ (resp., $RV_0$ or $RV_\rho$); cf. Section 6.

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