Rend. Sem. Mat. Univ. Pol. Torino
Vol. 57, 1 (1999)

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A REMARK ABOUT MINIMAL SURFACES WITH FLAT EMBEDDED ENDS

Abstract. In questo lavoro si prova un teorema di ostruzione all'esistenza di superficie minime complete nello spazio Euclideo aventi soltanto code piatte lisce. Nella dimostrazione confluiscono tecniche di geometria algebrica (spinori su supeficie di Riemann compatte) e differenziale (herisson e formula di monotonicità). Come corollario si ottiene che una superficie minimale di genere due avente tre code piatte lisce e tipo spinoriale pari non è immergibile minimalmente in $\mathbb{R}^3$.

1. Introduction

In this paper we prove a non-existence result for certain complete minimal surfaces having embedded flat ends and bounded curvature. The obstructions have been found by means of an algebraic herisson (see (11) and [14]). In particular we show (see Theorem 5.2) the non-existence of untwisted genus 2 minimal surfaces having 3 embedded flat ends.

We systematically use the theory developed by Kusner and Schmitt (cf. [5]). A short account of it is given in § 1. The minimal surfaces are studied by means of holomorphic spinors on compact Riemann surfaces. This fits very well with D. Mumford's previous work (cf. [8]), moreover some new hidden geometry appears (see Remark 5.2). The results of § 1, except Proposition 5.2, are due to the previous mentioned authors.

In § 2 we recall the spin representation of a minimal surface and prove our result. The main tool is a well known singularity (or monotonicity) formula (cf. [4] and [3]). Theorem 5.2 works out a heuristic argument of Kusner and Schmitt (cf. [5] § 18) in the easiest non trivial case.

2. Spin bundles on a compact Riemann surface

Let X be a compact connected Riemann surface of genus g. References for the basic facts we need can be found in the first chapter of [1] or in [9]. Let $\mathcal{F}$ be an abelian sheaf on X and denote by $H^i(X,\mathcal{F})$, $i=0,1$, its coomology groups, $H^0(X,\mathcal{F})$ is the space of global sections of $\mathcal{F}$. Set $h^i(\mathcal{F}) = \dim\left(H^i(X,\mathcal{F})\right)$.

Let $\mathcal{O}_X$ and ω_X denote respectively the structure and the canonical line bundle of X. Let $D=\sum_{i=1}^{d} p_i$, $d=\deg(D)$, be an effective divisor with *distinct* points. Fix a spin bundle of X, i.e. a line bundle L on X such that $L^2=\omega_X$ (cf. [5] and [8]). The isomorphism $\phi : L\otimes L \to \omega_X$ defines a spin structure of X. We say that L is even (odd) if $h^0(L)$ is even (odd). Consider the line bundle $L(D) = L\otimes \mathcal{O}_X(D)$, $L(D)^2=\omega_X(2D)$. We identify $L(D)$ with the sheaf of the meromorphic sections of L having simple poles at D. The Riemann-Roch and Serre-duality

*Partially supported by 40% project Geometria Algebrica M.U.R.S.T., by G.N.S.A.G.A. (C.N.R.) Italy and by European project Sci. "Geom. of Alg. variety".

theorems give:

$$h^0(L(D)) - h^1(L(D)) = d \quad \text{and} \quad h^1(L(D)) = h^0(L(-D))\,.$$

Here $L(-D)$ is the sheaf of the holomorphic sections of L vanishing at D. The sheaf inclusion $L(-D) \subset L(D)$ defines the quotient sheaf $\mathcal{L} = L(D)/L(-D)$. Set $V = H^0(X, \mathcal{L})$, $\dim(V) = h^0(\mathcal{L}) = 2d$.

Fix coordinates $\{U_i, z_i\}$ centred in p_i, $z_i(p_i) = 0$, we assume that $p_i \in U_j$ if and only if $i = j$. We trivialise the previous line bundles on U_i. If s and t are sections of $L(D)$ defined on U_i, we write:

$$s \equiv \left\{ a_{i,-1} z_i^{-1} + a_{i,0} + \sum_{n>0} a_{i,n} z_i^n \right\} \zeta_i\,; \qquad t \equiv \left\{ b_{i,-1} z_i^{-1} + b_{i,0} + \sum_{n>0} b_{i,n} z_i^n \right\} \zeta_i \tag{1}$$

where $\zeta_i^2 = \phi(\zeta_i \otimes \zeta_i) = dz_i$ is provided by the spin structure. The truncated expansions:

$$s' = \left(a_{i,-1} z_i^{-1} + a_{i,0}\right) \zeta_i\,, \qquad t' = \left(b_{i,-1} z_i^{-1} + b_{i,0}\right) \zeta_i$$

define elements of $\Gamma(U_i, \mathcal{L})$. Multiplication provides $st \in \Gamma(U_i, \omega_X(2D))$, a meromorphic differential on X with (double) pole at D. The residues:

$$a_{i,-1} b_{i,0} + b_{i,-1} a_{i,0} = \mathrm{Res}_{p_i}(st) \tag{2}$$

are well defined. Accordingly we set:

$$(s,t)_{p_i} = a_{i,-1} b_{i,0}\,. \tag{3}$$

The form (3) defined on $\Gamma(U_i, L(D))$ is *intrinsic*. To see this (cf. [5]) introduce the meromorphic function $h = \frac{s}{t}$, $h(p_i) = \frac{a_{i,-1}}{b_{i,-1}}$ as $b_{i,-1} \neq 0$. We have:

$$(s,t)_{p_i} = \frac{1}{2} h(p_i) \mathrm{Res}_{p_i}(t^2) \text{ if } b_{i,-1} \neq 0 \text{ and } (s,t)_{p_i} = \mathrm{Res}_{p_i}(st) \text{ if } b_{i,-1} = 0\,.$$

The symmetric part of (3) is (2) and both vanish on $\Gamma(U_i, L(D - 2p_i))$. Then

$$B(s,t) = \sum_i (s,t)_{p_i} = \sum_i a_{i,-1} b_{i,0}\,. \tag{4}$$

defines a bilinear form on V. The symmetric and the anti-symmetric part of B are respectively:

$$Q(s,t) = \frac{1}{2}(B(s,t) + B(t,s)) = \sum_i \mathrm{Res}_{p_i}(st) \tag{5}$$

$$\Omega(s,t) = \frac{1}{2}(B(s,t) - B(t,s))\,. \tag{6}$$

We have three maximal isotropic space of Q:

I) $V_1 = \{a_{i,-1} = 0\}_{i=1,\dots,d} = \{\text{images of local holomorphic sections}\}$.

II) $V_2 = \{\text{image of global section of } L(D)\}$.

III) $V_0 = \{s \in V,\ a_{i,0} = 0,\ i = 1, \dots, d\} = \{s \in V \ :\ B(t,s) = 0 \text{ for any } t \text{ of } V_1\}$.

Observe that V_1 and (hence) V_0 are *intrinsic*. The exact sequence

$$0 \to H^0(X, L(-D)) \to H^0(X, L(D)) \xrightarrow{\tau} L(D)/L(-D)$$

identifies $H^0(X, L(D))/H^0(X, L(-D))$ and $V_2 = \text{image}(\tau)$. By Riemann-Roch $\dim(V_2) = d$. The isotropy of V_2 follows from the global residues theorem. We identify $V_1 \cap V_2 = V(D)$ with $H^0(X, L)/H^0(X, L(-D))$. Set $S(D) = V_0 \cap V_2$ and

$$K(D) = \tau^{-1}(S(D)) = \left\{ s \in H^0(X, L(D) \ : \ s \equiv \left(a_{i,-1} z_i^{-1} + \sum_{n>0} a_{i,n} z_i^n \right) \zeta_i) \text{ near } p_i \right\} .$$

Note that $K(D) \cap H^0(X, L) = H^0(X, L(-D))$. This gives an isomorphism $K(D)/H^0(X, L(-D)) \cong S(D)$.

REMARK 5.1. If $h^0(L(-D)) = 0$, e.g. if $d > g - 1$, the spaces $S(D)$ and $V(D)$ can be identified respectively with $K(D)$ and $H^0(X, L)$. In particular, following Mumford (cf. [8]), we identify $H^0(X, L)$ with $V(D)$ which is the intersection of two maximal isotropic spaces: $V_1 \cap V_2 = V(D)$. From this it follows that the parity of $h^0(L)$ is invariant under deformation. The forms B and Ω were introduced in [5].

Next we consider Ω. The restriction of Ω to any Q-isotropic space equals B, in particular $B = \Omega : V_2 \times V_2 \to \mathbb{C}$:

$$\Omega(s, t) = \sum_i a_{i,-1} b_{i,0} = -\sum_i b_{i,-1} a_{i,0} .$$

Define, composing with τ, the anti-symmetric form Ω' on $H^0(X, L(D))$. Set

$$\begin{aligned} \ker(\Omega) &= \{ s \in V_2 \ : \ \Omega(s, t) = 0 \text{ for any } t \in V_2 \} \\ \ker(\Omega') &= \left\{ s \in H^0(X, L(D)) \ : \ \Omega'(s, t) = 0 \text{ for any } t \in H^0(X, L(D)) \right\} \end{aligned} \tag{7}$$

We may identify $\ker(\Omega)$ and $\ker(\Omega')/H^0(X, L(-D))$.

PROPOSITION 5.1 (THEOREM 15 OF [5]). *We have:*

$i)$ $h^0(L(D)) = \dim(\ker(\Omega')) \bmod 2$ *and* $d = \dim(\ker(\Omega)) \bmod 2$*;*

$ii)$ $\ker(\Omega) = V(D) \oplus S(D)$ *and* $\ker(\Omega') = K(D) + H^0(X, L)$*;*

$iii)$ $\dim(S(D)) + \dim(V(D)) = d \bmod 2$.

Proof. $i)$ Ω and Ω' are anti-symmetric. $ii)$ Clearly $V(D) = V_2 \cap V_1$ and $S(D) = V_2 \cap V_0$ are contained $\ker(\Omega)$. Conversely let $s = s_0 + s_1 \in \ker(\Omega)$, $s_i \in V_i$, $i = 0, 1$. One has $B(s_1, v) = 0 = B(v, s_0)$ for any $v \in V$. Take $t \in V_2$ then $B(s, t) = \Omega(s, t) = 0$:

$$0 = B(s_0 + s_1, t) = B(s_0, t) + B(s_1, t) = B(s_0, t) + B(t, s_0) = Q(t, s_0) .$$

Hence $Q(t, s_0) = 0$ for all $t \in V_2$. Since V_2 is maximal isotropic $s_0 \in V_2$, in the same way $s_1 \in V_2$. $iii)$ It follows from $i)$ and $ii)$.

□

REMARK 5.2. If $h^0(L) = 0$ and d is odd, Proposition 5.1 implies the existence of $s \in H^0(L(D))$ such that $s \equiv \left\{a_{i,-1}z_i^{-1} + \sum_{n>0} a_{i,n}z_i^n\right\}\zeta_i$ near all the points p_i.

REMARK 5.3. One has $\dim(S(D)) \leq \dim(V_1) = d$ and the Clifford inequality $2\dim V_2 \cap V_1 \leq 2h^0(L) \leq g-1$ (cf. [9]).

We can prove the following:

PROPOSITION 5.2. *If $D = \sum p_i$ and $\dim K(D) = d = \deg(D)$ then $d \leq g+1$. Assume $d = g+1$ and $g > 1$ or $d = g$ and $g > 3$. Then X is hyperelliptic and the points p_i of D are Weierstrass points of X. If $d = g+1$ and $g > 1$ then L is isomorphic to $\mathcal{O}_X(D-2p_i)$. If $d = g$ and $g > 3$ L is isomorphic to $\mathcal{O}_X(D-B)$ where B is a Weierstrass point distinct from the p_i.*

Proof. If $d \geq g$ we have $h^0(L(-D)) = 0$ and hence $d = \dim S(D) = \dim K(D)$. This holds if and only if $\Omega' = 0$, i.e. $H^0(X, L(D)) = K(D)$ and then $\ker(\Omega) = S(D)$. Now $ii)$ of Proposition 5.1 implies $V(D) = H^0(X,L)/H^0(X,L(-D)) = 0$. It follows that $h^0(L) = 0$.

From Riemann Roch theorem we get therefore $h^0(L(p_i)) = h^0(L(-p_i)) + 1 = 1$ for all i. The sheaf inclusion $L(p_i) \subset L(D)$ provides sections

$$\sigma_i \in H^0(X, L(p_i)) \subset K(D),\ \sigma_i \neq 0\,.$$

If $i \neq j$ the σ_i are holomorphic on p_j, then $\sigma_i(p_j) = 0$ ($\sigma_i \in K(D)$). Therefore the zero divisor, (σ_i), of σ_i contains $D - p_i$, hence $g = \deg(L(p_i)) \geq d-1$: $\mathbf{d} \leq \mathbf{g}+\mathbf{1}$. We write:

$$(\sigma_i) = D - p_i + B_i\,, \tag{8}$$

where B_i is an effective divisor of degree $g+1-d$. For any i and j, $j \neq i$. We obtain: $D - 2p_i + B_i \equiv D - 2p_j + B_j$, $\equiv$ denotes the linear equivalence. We get

$$B_i + 2p_j \equiv B_j + 2p_i\,.$$

If $\mathbf{d} = \mathbf{g}+\mathbf{1}$ we have $B_i = B_j = 0$ and $2p_j \equiv 2p_i$: X is hyperelliptic and the p_i are Weierstrass points. The (8) shows that L is isomorphic to $\mathcal{O}_X(D-2p_i)$.

Assume $\mathbf{d} = \mathbf{g}$ and $\mathbf{g} > \mathbf{3}$. The B_i are points of X. The first 3 relations:

$$B_1 + 2p_2 \equiv B_2 + 2p_1; \quad B_1 + 2p_3 \equiv B_3 + 2p_1; \quad B_3 + 2p_2 \equiv B_2 + 2p_3$$

give 3 "trigonal" series on X. If X were non-hyperelliptic only two of such distinct series can exist (one if $g > 4$ see [1]). Then two, say the firsts, of the above equations define the same linear series:

$$B_1 + 2p_2 \equiv B_2 + 2p_1 \equiv B_1 + 2p_3 \equiv B_3 + 2p_1\,.$$

We obtain $2p_2 \equiv 2p_3$: X is hyperelliptic, which gives a contradiction.

Now X is hyperelliptic and let φ be its hyperelliptic involution. Assume that p_1 is not a Weierstrass point, i.e. $\varphi(p_1) \neq p_1$. Set $B_1 = B$. Any degree 3 linear series on X has a fixed point. Since $p_1 \neq p_i$ for $i \neq 1$ and $B + 2p_i \equiv B_i + 2p_1$ it follows that $B = p_1$. From (8) we obtain

$$L(p_1) \equiv (\sigma_i) \equiv D - p_1 + B = D\,,$$

then $L \equiv \mathcal{O}_X(D - p_1) \equiv \mathcal{O}_X\big(\sum_{i>1} p_i\big)$. This is impossible: L has not global sections. It follows that all the p_i are Weierstrass points. From $D - 2p_1 + B \equiv L$ we get $\omega_X \equiv \mathcal{O}_X(2D - 4p_1 + 2B)$, then B is a Weierstrass point: $L = \mathcal{O}_X(D-B)$, $2B \equiv 2p_i$, and $B \neq p_i$ for all i.

□

3. Spinors and minimal surfaces

Let X be a genus g compact connected surface $D = \sum_i p_i$ a degree d divisor with distinct points. Let $F : X - D \to \mathbb{R}^3$ be a complete minimal immersion having bounded curvature and embedded ends. Then (cf. [11], [5] and [13]) there are a spin structure L on X, and sections of s and t of $L(D)$ such that

$$F(q) = \mathrm{Re} \int_{[p,q]} \left(s^2 - t^2, i\left(s^2 + t^2\right), 2st\right) + C \tag{9}$$

where p is a fixed point and C is a constant vector. The (9) is the spin representation of the minimal surface $F(X - D)$. We set

$$\left(s^2 - t^2, i\left(s^2 + t^2\right), 2st\right) = (\omega_1, \omega_2, \omega_3)\,.$$

Starting with two sections of $L(D)$, s and t, formula (9) is a well defined immersion if and only if the following conditions hold:

A) $\mathrm{Re} \int_\gamma (\omega_1, \omega_2, \omega_3) = 0$, for any $\gamma \in H_1(X - D, \mathbb{Z})$ (period);

B) $\{s = t = 0\} = \emptyset$ (immersion).

DEFINITION 5.1. *The immersion $F : X - D \to \mathbb{R}^3$ is* untwisted *if L is even and* twisted *if L is odd (see [5] about its topological meaning). We say that $F(X - D)$ has embedded flat ends if $Res_{p_i}(\omega_j) = 0$ for any $i = 1, \ldots, d$ and $j = 1, 2, 3$.*

Let $\Pi = \mathrm{span}(s, t)$ be the plane space generated by s and t. We have:

PROPOSITION 5.3 ([5] THEOREM 13). *If in (9) $Y = F(X - D)$ is complete then*

i) Y has flat embedded ends if and only if $\Pi \subset K(D)$,

ii) if Y has flat embedded ends then $\Pi \cap H^0(X, L) = \{0\}$.

Proof. i) Up to a rotation we may assume s holomorphic at the point $p_i \in D$. Locally $s = \{a_{i,0} + \ldots\}\zeta_i$ and $t = \{b_{i,-1} z_i^{-1} + b_{i,0} \ldots\}\zeta_i$. Now $b_{i,-1} \neq 0$ otherwise F extends on p_i and $F(X - D)$ is not complete. We now have:

$$\begin{aligned} i\mathrm{Res}_{p_i}\left(t^2 + s^2\right) = 0 = \mathrm{Res}_{p_i}\left(t^2 - s^2\right) = 0 \quad &\Rightarrow \quad b_{i,-1}b_{i,0} = 0 \\ \mathrm{Res}_{p_i}(st) = 0 \quad &\Rightarrow \quad b_{i,-1}a_{i,0} = 0\,. \end{aligned}$$

Therefore we obtain $b_{i,0} = a_{i,0} = 0$ and then that s and t belong to $K(D)$. The converse is clear. ii) Since t has a pole at p_i $\Pi \cap H^0(X, L) \neq \{0\}$ only if $s \in H^0(X, L) \cap K(D) = H^0(X, L(-D))$. The s should vanish to any point of D, but then $\omega_3 = st$ would be a holomorphic differential. The period condition A implies (cf. [9]) $\omega_3 = st = 0$, $s = 0$ a contradiction. $\square$

REMARK 5.4. (not used it in the sequel). A class of very important bounded curvature minimal surfaces are the one with horizontal embedded ends. In fact the actually embedded minimal surfaces have parallel ends (c.f. [2]). Assume that in (9) s, t give the representation of a surface with horizontal embedded ends. This means that $D = D_1 \cup D_2$, $D_1 \subset h^{-1}(0)$ and $D_2 \subset h^{-1}(\infty)$, $h = \frac{s}{t}$. That is $s \in H^0(X, L(D_2))$ and $t \in H^0(X, L(D_1))$. From

$i\mathrm{Res}_{p_i}(t^2+s^2)=0$ and $\mathrm{Res}_{p_i}(t^2-s^2)=0$ we see that $\mathrm{Res}_{p_i}(t^2)=0$, $p_i \in D_1$: $t \in K(D_1)$ and similarly $s \in K(D_2)$.

From now on $F : X - D \to \mathbb{R}^3$ will denote a complete minimal immersion with *flat embedded ends*, where X is a genus g compact and connected Riemann surface and $D = \sum_{i=1}^{d} p_i$, $\deg(D) = d$.

Let $\rho : X \to Y$ be a non constant holomorphic map and $B \subset X$ be the branch divisor, without multiplicity, of ρ. Define $H : Y - \rho(D) \to \mathbb{R}^3$ by

$$H(q) = \sum_{\{p:\rho(p)=q\}} F(p)\,. \tag{10}$$

The above summation is taken with multiplicity, H is the trace of F by ρ. Set $T = \rho(D)$ and $E = \rho^{-1}(T)$. There is also a well defined trace map for holomorphic differential forms

$$\mathrm{Tr}(\rho) : H^0\left(X - E, \omega_{X-E}\right) \to H^0\left(Y - T, \omega_{Y-T}\right)\,.$$

This is defined as follows. Take $q \notin \rho(B) \cap T$ and let U be an open simply connected set coordinated by $y : U \to \mathbb{C}$. We assume that $\rho^{-1}(U) = \cup_{\{p:\rho(p)=q\}} W_p$ where the restriction $\rho|_{W_p} = \rho_p : W_p \to U$ are biholomorphisms. The compositions

$$z_p = y \cdot \rho_p : W_p \to U$$

are coordinate maps of X. If $\Theta \in H^0(X - E, \omega_{X-E})$ and $\Theta = \{a_p(z_p)dz_p\}$ on $\rho^{-1}(U)$ we put

$$\mathrm{Tr}(\rho)(\Theta) = \sum_{\{p:\rho(p)=q\}} a_p(z_p)dy\,.$$

Then one extends $\mathrm{Tr}(\rho)(\Theta)$ on $Y - T$ by taking care of the multiplicity.

LEMMA 5.1. *If the ends points P_i are branches of ρ, i.e. $D \subset B$, then H is constant.*

Proof. (Compare with [14]). Set $B = p_1 + \cdots + p_b$, where $D = \sum_{i=1}^{d} p_i$, $d \le b$. Up to a translation we have

$$F(p) = \mathrm{Re}\int_{[Q,p]} (\omega_1, \omega_2, \omega_3)$$

where $Q \notin D$. Set

$$\Omega_j = \mathrm{Tr}(\rho)(\omega_j)\,.$$

We shall show that the Ω_j extends to abelian an differential of Y. We assume this for a moment and prove the lemma. Set $L = \rho(Q)$, we obtain:

$$H(s) - H(L) = \mathrm{Re}\int_{[L,s]} (\Omega_1, \Omega_2, \Omega_3)\,.$$

Since the right side is well defined then the left one turns out to be zero. In fact any non trivial holomorphic differential has a non-zero real period (cf. [9]). It follows that H is constant.

We prove now that the Ω_j extend holomorphically. Take p_i in D and set $\rho(p_i) = S$. We may choose coordinates $z = z_i$ of X, $z(p_i) = 0$, and y of Y, $y(S) = 0$, such that ρ locally is written: $y = z^n$, $n > 1$. Expanding near p_i we get

$$\omega_j \equiv \left\{c_j z^{-2} + g_j(\mathrm{z})\right\} dz$$

where $g_j(z)$ is a holomorphic function. It is elementary that

$$\operatorname{Tr}\left(z^n\right)\left\{z^{-2}dz\right\} = -\operatorname{Tr}\left(z^n d\frac{1}{z}\right) = 0 \quad (n > 1)\,.$$

All the above terms appear in the defining sums of $\operatorname{Tr}(\rho)(\omega_j) = \Omega_j$. This implies that Ω_j extends holomorphically at p_i.

□

We still assume that the ends of F, $F : X - D \to \mathbb{R}^3$, are in the *branch* locus of $\rho : D \subset B$. Consider the disjoint union

(11) $$B = D \cup B' \cup B''$$

where $B' = \{P \in B - D :\ P \text{ is of total ramification for } \rho\}$. Set $k = \deg B'$ (if $B' = \emptyset$ $k = 0$). We have:

THEOREM 5.1. *With the previous assumption $d > k$.*

Proof. Assume by contradiction that $k \geq d$. Set $B' = \{Q_1, \ldots, Q_k\}$. Up to a translation take $F(p) = \operatorname{Re}\int_{[Q_1,p]}(\omega_1, \omega_2, \omega_3)$. Then $H(q) = \operatorname{Re}\sum_{\{p \xrightarrow{\rho} q\}} \int_{[Q_1,p]}(\omega_1, \omega_2, \omega_3)$ is constant. Since the Q_i have total ramification $H(\rho(Q_i)) = nF(Q_i)$, hence

$$nF(Q_i) = H(\rho(Q_i)) = H(\rho(Q_1)) = nF(Q_1) = 0\,.$$

We see that $F(Q_i) = 0$, $i = 1, \ldots, k$ and that $F(X - D)$ has a point of multiplicity $k \geq d$ at the origin. On the other hand it is well known that a minimal surface with d embedded ends cannot have a k-ple point if $k \geq d$ (see [4] or [3] for two quite different proofs). This provides the contradiction.

□

COROLLARY 5.1. *Let X be a hyperelliptic curve and $F : X - D \to \mathbb{R}^3$ be a complete immersed minimal surface with flat embedded ends. Assume that the ends points $P_i \in D$ are Weierstrass points of X. Then* $\deg(D) > g + 1$.

Proof. Let ρ be the hyperelliptic $2:1$ covering of the Riemann sphere, B is the set of hyperelliptic points $\deg(B) = 2g + 2$, and $B' = B - D$.

□

COROLLARY 5.2. *Let $D = P_1 + \cdots + P_d$, assume* $\dim K(D) = d = g+1$ *or* $\dim K(D) = d = g$ *and* $g > 3$. *Then there are not complete minimal immersions in $\mathbb{R}^3$ of $X - D$ with flat embedded ends.*

Proof. Use Proposition 5.2 and then Corollary 5.1.

□

THEOREM 5.2. *Minimal untwisted immersions of genus 2 having 3 flat ends do not exist.*

Proof. Arguing by contradiction we would have $h^0(L(D)) = 3$, $h^0(L) = 0$ (L is even) and that $\dim(K(D))$ is odd (by Proposition 5.1 *i*)). From Proposition 5.3 we would obtain $\dim(K(D)) \geq 2$ and then $\dim K(D) = 3$. This contradicts Corollary 5.2.

□

Let $F : X - D \to \mathbb{R}^3$ be as before. The flat ends are in the branch of $h = \frac{s}{t}$, the extended Gauss map of $F(X - D)$. Therefore Theorem 5.1 applies and stronger restrictions should hold (see [14]). We give an example of this. We recall (cf. [12]) that F *has vertical flux* if $t^2 = \omega$ and $s^2 = h^2\omega$ have not *complex* periods. It means that there are meromorphic functions L_1 and L_2 on X such that $dL_1 = \omega$ and $dL_2 = h^2\omega$.

PROPOSITION 5.4. *If F has vertical flux, then h has not points of total ramification.*

Proof. Define the Lopez-Ros [6] deformation of $F = F_1$: $F_\lambda : X - D \to \mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$, $\lambda \in \mathbb{R}$, $\lambda > 0$:

$$F_\lambda = \left(\lambda L_1 - \frac{1}{\lambda}\overline{L_2}, 2\mathrm{Re}\int_{[p,q]} h\omega\right).$$

The spin representation of F_λ is $\left(\sqrt{\lambda}s, \frac{1}{\sqrt{\lambda}}t\right)$. Note that $F_\lambda(X - D)$ has flat ends at D. If h had a total branch point then, by a result of Nayatami (cf. [10] th. 2), the dimension of the bounded Jacobi fields of F_λ, $\lambda \gg 0$, would be 3. Therefore (see [7]) F_λ cannot have only flat ends.

□

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AMS Subject Classification: 53A10.

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Lavoro pervenuto in redazione il 10.3.1998.